

# On the Hausdorff dimension of the graph of a Weierstrass type function

Loredana Biacino

**Abstract.** In this note a theorem to compare the box dimension of Weierstrass type functions and their Hausdorff dimension is established. Moreover a method to determine the Hausdorff dimension of the graphs of functions such as the Weierstrass or the Mandelbrot functions is given.

**Mathematics Subject Classification (2010):** Primary 26A15, Secondary 26A16.

**Keywords:** Hölder continuous functions, box dimension, Hausdorff dimension, Weierstrass-type functions.

## 1. Introduction

In this paper the Hausdorff and the box dimension of the graphs of some Weierstrass type functions are compared. Recall that the *Hausdorff dimension* of a set  $E \subseteq R^n$  is defined in terms of the *k-dimensional Hausdorff measure* of  $E$ , denoted by  $H^k(E)$  and given by

$$H^k(E) = \lim_{\delta \rightarrow 0} \inf \left\{ \sum_i |E_i|^k, E \subseteq \bigcup E_i, |E_i| < \delta \right\}, \quad (1.1)$$

where  $|E_i|$  denotes the diameter of  $E_i$  and the infimum is over all (countable)  $\delta$ -covers  $E_i$  of  $E$  (see Falconer [2] and [3]). It is given by:

$$H - \dim E = \inf \{k > 0 : H^k(E) = 0\}. \quad (1.2)$$

There are other classes of covers leading to the Hausdorff dimension; in particular it is possible to consider in definition (1.1) instead of all covers of  $E$ , the covers obtained by the family of half-open  $d$ -adic cubes in  $R^n$ , that is cubes of the form:

$$\{x \in R^n, h_i d^{-m} \leq x_i < (h_i + 1)d^{-m}, \text{ for } i = 1, 2, \dots, n\}$$

where  $h_i$  and  $m$  are arbitrary integers. Then if the minimum in (1.1) is restricted to the class of these particular covers, one obtains the *net measure* of  $E$ , denoted by  $N^k(E)$ . It is evident that  $H^k(E) \leq N^k(E)$ , but it is also

possible to prove that there exists a constant  $A > 0$ , only depending on the dimension of the space,  $n$ , and on  $d$ , such that  $N^k(E) \leq AH^k(E)$  for every  $E \subseteq R^n$  (see Mattila [4], 5.2 and Falconer [2], 5.1 for binary cubes). One of the most immediate modification of the Hausdorff dimension is given in terms of the upper and lower box dimension of a set, defined in the following way. Let  $N_\delta(E)$  be the smallest number of sets of diameter at most  $\delta$  which cover  $E$ . Then the following numbers:

$$\underline{\dim}_B E = \underline{\lim}_{\delta \rightarrow 0} \frac{\log N_\delta(E)}{-\log \delta} \quad (1.3)$$

$$\overline{\dim}_B E = \overline{\lim}_{\delta \rightarrow 0} \frac{\log N_\delta(E)}{-\log \delta} \quad (1.4)$$

are called respectively the *lower and upper box dimensions* or *lower and upper Minkowski dimensions* of  $E$ , and, if they agree, their common value is the *box dimension* of  $E$ , denoted by  $\dim_B E$  or  $\Delta(E)$ . It is possible to prove that in (1.3) or in (1.4),  $N_\delta(E)$  can be substituted by the number of  $\delta$ -mesh cubes meeting  $E$ , that is the cubes of the form:

$$\{x \in R^n : h_i \delta \leq x_i < (h_i + 1)\delta, i = 1, 2, \dots, n\}$$

where  $\delta > 0$  and  $h_1, \dots, h_n$  are integers. In general it is:

$$H - \dim(E) \leq \underline{\dim}_B E \leq \overline{\dim}_B E.$$

If  $f : [a, b] \rightarrow R$  is a continuous function and if

$$G = \{(a, b) \in R^2 : a \leq x \leq b, y = f(x)\}$$

is its graph, then  $H - \dim G \geq 1$  (see Falconer [2], lemma 1.8); moreover, if  $f$  is  $\alpha$ -Hölder continuous then:  $\overline{\dim}_B G \leq 2 - \alpha$  (see Falconer [2], Theorem 8.1).

Some general discussion about the Hausdorff dimension of the graph of a Hölder continuous function can be found in [5]. In this paper we will consider Weierstrass type functions:

$$f(x) = \sum_{n \in N} \frac{\varphi(b_n x)}{b_n^\delta},$$

where  $\varphi : R \rightarrow R$  is periodic (with period 1), Lipschitz continuous and  $b_n$  is a sequence for which there exists  $B > 1$  such that  $b_1 \geq B, b_{n+1} \geq B b_n$  for every  $n \in N$ . Obviously it is not restrictive to suppose  $\varphi(x) \geq 0$  for every  $x \in R$ , since if this is not the case, it is possible to consider  $\psi(x) = \varphi(x) + m$ , where  $m = \min_{x \in [0,1]} \varphi(x)$  and observe that  $\psi(x) \geq 0$  for every  $x \in R$  and

$$\sum_{n \in N} \frac{\psi(b_n x)}{b_n^\delta} = f(x) + \sum_{n \in N} \frac{m}{b_n^\delta},$$

that is the corresponding function to  $\psi$  differs from  $f$  by a constant.

In the main theorems of this paper (Theorem 3.2, Remark 3.3, Theorem 3.4) rather general hypotheses for a class of Weierstrass type functions will be established in order the Hausdorff dimension of the graph of  $f$  to be equal to the box dimension when this achieves its maximum. To obtain this result

some technical ideas (in Lemma 2.2) have been borrowed from an old paper by Besicovitch and Ursell (see [1]). Interesting suggestions have been found also in [6].

## 2. Three lemmas

In order to establish the main theorem some lemmas are needed:

**Lemma 2.1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be an  $\alpha$ -Hölder continuous function, ( $0 < \alpha \leq 1$ ),  $d$  a natural number,  $d > 1$ , let  $\{Q_i\}$  be a finite cover of  $G$  constituted by meshes, and let  $Q$  be a  $\frac{1}{d^r}$ -mesh ( $r \in \mathbb{N}$ ) from the cover  $\{Q_i\}$ . Let  $\{Q^j\}_{j \in A}$  be a cover of  $G \cap Q$  constituted by  $\frac{1}{d^k}$ -meshes with fixed  $k > r$ . Then there exists a constant  $C > 0$  depending only on  $f$  such that*

$$\sum_{j \in A} |Q^j|^{2-\alpha} \leq Cm(F), \quad (2.1)$$

where  $F$  is the projection of  $G \cap Q$  on the  $x$ -axis and  $m$  is the unidimensional Lebesgue measure

*Proof.*  $F$  is a Lebesgue measurable set, since it is the projection of a Borel set. It is  $m(F) = \lim_{s \rightarrow \infty} m(A_s)$ , where  $m$  is the Lebesgue measure and  $\{A_s\}_{s \in \mathbb{N}}$  is a sequence of open sets, decreasing with respect to the inclusion relation. Let us cover  $A_s$  by intervals  $I_n$  that are linear  $\frac{1}{d^k}$ -meshes. The oscillation of  $f$  in every one of these intervals is less than or equal to  $L(\frac{1}{d^k})^\alpha$ , where  $L$  is the Hölder coefficient of  $f$ . Therefore the part of  $G$  whose projection is enclosed in  $A_s$  is covered by  $\frac{1}{d^k}$ -square meshes whose number is at most  $\frac{m(A_s)L|I_n|^\alpha}{|I_n|^2}$ ; let us call these meshes by  $Q_{s,n}$ ; then, if  $s$  is fixed, it is:

$$\sum_n |Q_{s,n}|^{2-\alpha} \leq m(A_s)L(\sqrt{2})^{2-\alpha}.$$

Since this inequality holds for every  $s \in \mathbb{N}$ , we have, keeping in mind that  $\bigcap_{s \in \mathbb{N}} \{Q_{s,n}\}$  is a cover  $\{Q^j\}_{j \in A}$  of  $G \cap Q$  constituted by  $\frac{1}{d^k}$ -meshes:

$$\sum_{j \in A} |Q^j|^{2-\alpha} \leq L(\sqrt{2})^{2-\alpha} \lim_{s \rightarrow \infty} m(A_s),$$

whence (2.1) and the lemma is proved.  $\square$

In the next lemma, very near to the ideas of a well known paper by Besicovitch and Ursell (see [1], where however a particular case is considered), and in the sequel we will consider a periodic function  $\varphi : \mathbb{R} \rightarrow [0, P]$  with period 1, nonnegative, continuous and piecewise differentiable; assume that  $\varphi'_-(x)$  and  $\varphi'_+(x)$  are finite and different from 0 for every  $x \in \mathbb{R}$ . Then there exist two constants  $c > 0$  and  $c_1 > 0$  such that:

$$c \leq |\varphi'(x)| \leq c_1 \quad (2.2)$$

for every  $x \in \mathbb{R}$  such that  $\varphi$  is differentiable in  $x$ .

If  $\varphi$  satisfies all the previous conditions we will refer to it as a *smooth function*.

**Lemma 2.2.** *Consider, for  $0 < \alpha < 1$ , the following function:*

$$f(x) = \sum_{n \in N} \frac{\varphi(b_n x)}{b_n^\alpha}, \quad (2.3)$$

where  $\varphi$  is a smooth function and where  $(b_n)_{n \in N}$  is such that there exists  $B > 1, B \in N$  for which  $b_{n+1} \geq Bb_n$  for every  $n \in N$  and

$$\lim_{n \rightarrow \infty} \frac{\log b_{n+1}}{\log b_n} = 1. \quad (2.4)$$

Let  $\{Q_i\}$  be a cover of  $G$  constituted by  $\frac{1}{B^u}$ -meshes ( $u \in N$ ), and let  $Q$  be a  $\frac{1}{B^r}$ -mesh from  $\{Q_i\}$ . Let  $\{Q^j\}_{j \in A}$  be a cover of  $G \cap Q$  constituted by  $\frac{1}{B^k}$ -meshes with fixed  $k > r$ . Then, if  $B^{1-\alpha} > 1 + \frac{c_1}{c}$ , where  $c$  and  $c_1$  are as in (2.2), there exists a constant  $\lambda > 0$  such that, for enough large  $r$ :

$$\sum_{j \in A} |Q^j|^{2-\alpha} \leq \lambda |Q|^{2-\alpha}. \quad (2.5)$$

*Proof.* By (2.1), in order to prove (2.5) we have to determine an upper bound for the measure of the set  $F = \{x \in R : (x, f(x)) \in Q\}$ . To this end observe that, if we consider for every  $s \in N$  the function  $f_s(x) = \sum_{n \leq s} \frac{\varphi(b_n x)}{b_n^\alpha}$  and if  $\varphi(x) \leq 1$  for every  $x \in R$  as is not restrictive to suppose, then:

$$|f(x) - f_{k+\nu-1}| \leq \sum_{n \geq k+\nu} \frac{\varphi(b_n x)}{b_n^\alpha} \leq \frac{B^\alpha}{b_{k+\nu}^\alpha (B^\alpha - 1)},$$

where, given  $r \in N$  and  $k > r$ ,  $\nu$  has been chosen in such a way that:

$$\frac{1}{b_{k+\nu}^\alpha} \leq \frac{1}{B^r} < \frac{1}{b_{k+\nu-1}^\alpha}.$$

Therefore:

$$|f(x) - f_{k+\nu-1}| \leq \frac{B^\alpha}{B^r (B^\alpha - 1)}. \quad (2.6)$$

Consider the strip  $S$  obtained prolonging  $Q$  downwards a distance  $\frac{B^\alpha}{B^r (B^\alpha - 1)}$ . By (2.6) if  $(x, f(x)) \in Q$  then  $(x, f_{k+\nu-1}(x)) \in S$  and, since  $f_{k+\nu-1}(x) \leq f_{k+\nu}(x) \leq f(x)$ , also  $(x, f_{k+\nu}(x)) \in S$ . Therefore:

$$F \subseteq F_{k+\nu-1} \cap F_{k+\nu},$$

where  $F_s = \{x \in R : (x, f_s(x)) \in S\}$  for every  $s \in N$ . By hypotheses  $\varphi$  is strictly increasing or decreasing in a finite number of intervals of  $[0, 1]$ : let  $M \in N$  be their number.

Now in every interval  $I$  in which  $f'_s(x)$  is either positive or negative, it is, for every  $x \in I$ :

$$f'_s(x) = \sum_{n \leq s} b_n^{1-\alpha} \varphi'(b_n x) = b_s^{1-\alpha} \sum_{n \leq s} \left(\frac{b_n}{b_s}\right)^{1-\alpha} \varphi'(b_n x),$$

whence, by (2.2), there exists  $c_2 = c - \frac{c_1}{B^{1-\alpha}-1} > 0$  by hypothesis, such that:

$$|f'_s(x)| \geq c_2 b_s^{1-\alpha} \quad (2.7)$$

and therefore, for every  $s \in N$ , the sign of  $f'_s(x)$  in  $I$  is the same of  $\varphi'(b_s x)$ . Consequently in every interval of unitary length, the function  $f_{k+\nu-1}$  is strictly increasing or decreasing in at most  $M([b_{k+\nu-1}] + 1)$  intervals, where  $[b_{k+\nu-1}]$  denotes the integer part of  $b_{k+\nu-1}$ ; by Lagrange theorem and by (2.7) the length of an interval  $I$  where the oscillation of  $f_{k+\nu-1}$  is not greater than  $\frac{1}{B^r}(1 + \frac{B^\alpha}{B^{\alpha-1}}) = \frac{C_\alpha}{B^r}$  is given by:

$$|I| \leq \frac{C_\alpha}{B^r c_2 b_{k+\nu-1}^{1-\alpha}}.$$

Consider now the intervals  $J$  enclosed in the previous ones in which the function  $f_{k+\nu}$  has an oscillation less or equal to  $\frac{C_\alpha}{B^r}$ . As before we have:

$$|J| \leq \frac{C_\alpha}{B^r c_2 b_{k+\nu}^{1-\alpha}}.$$

For every previous interval  $I$  there are

$$M([b_{k+\nu}] + 1)|I| \leq \frac{MC_\alpha 2b_{k+\nu}}{B^r c_2 b_{k+\nu-1}^{1-\alpha}},$$

such intervals and therefore :

$$m(F) \leq \frac{MC_\alpha 2b_{k+\nu}}{B^r c_2 b_{k+\nu-1}^{1-\alpha}} \frac{C_\alpha}{B^r c_2 b_{k+\nu}^{1-\alpha}} \leq \frac{2MC_\alpha^2 b_{k+\nu}^\alpha}{c_2^2 B^{2r} b_{k+\nu-1}^{1-\alpha}}$$

whence, by the choice of  $\nu$  :

$$m(F) \leq \frac{2MC_\alpha^2 b_{k+\nu}^\alpha}{c_2^2 B^{r(2-\alpha)} b_{k+\nu-1}^{\alpha^2+1-\alpha}}. \quad (2.8)$$

By (2.4), for every  $\varepsilon > 0$  it is possible to determine  $k_o$  such that for every  $k > k_o$  it is  $b_{k+\nu} \leq b_{k+\nu-1}^{1+\varepsilon}$ . Now it is possible to determine  $\varepsilon > 0$  in such a way that  $(1+\varepsilon)\alpha \leq \alpha^2 - \alpha + 1$  and therefore, for enough large  $k$ :  $b_{k+\nu}^\alpha \leq b_{k+\nu-1}^{\alpha^2-\alpha+1}$ . By (2.8) we can conclude that there is a positive constant  $\gamma > 0$  such that  $m(F) \leq \frac{\gamma}{B^{r(2-\alpha)}}$  for enough large  $r$  and, by (2.1), the lemma is proven.  $\square$

**Remark 2.3.** It is worth noticing that it is possible to apply Lemma 2.2 even in situations in which the conditions stated there do not hold, for example if  $\varphi$  is such that it is possible to perform on it a geometrical transformation obtaining a smooth function  $\psi$  in such a way that the corresponding functions to  $\varphi$  and  $\psi$  have the same geometrical measure properties. For example consider the function

$$\varphi(x) = \frac{1 + \sin(2\pi x)}{4} \text{ if } -\frac{1}{4} \leq x < \frac{3}{4}, \quad \varphi(x+1) = \varphi(x) \text{ for every } x \in R;$$

the hypotheses of Lemma 2.2 are not satisfied, since there exist points  $x$  such that  $\varphi'(x) = 0$ . Consider now the function:

$$\begin{aligned} \psi(x) &= \frac{1}{2} + 2x - \varphi(x) \text{ if } -\frac{1}{4} \leq x < \frac{1}{4}, \quad \psi(x) = \frac{3}{2} - 2x - \varphi(x) \text{ if } \frac{1}{4} \leq x \leq \frac{3}{4}, \\ \psi(x+1) &= \psi(x) \text{ for every } x \in R. \end{aligned}$$

It is a smooth function since it is nonnegative and continuous in  $R$ , piecewise differentiable and  $2 - \frac{\pi}{2} \leq |\psi'(x)| \leq 2 + \frac{\pi}{2}$  for every  $x$  where  $\psi$  is differentiable.

Consider the function (2.3), where  $b_n = d^n$  with a fixed  $d \in N$ , consider also the function:

$$f_k(x) = \sum_{n \leq k} \frac{\varphi(b_n x)}{b_n^\alpha}.$$

Let  $I$  be an interval where  $\varphi(b_k x)$  is either strictly increasing or strictly decreasing and therefore  $\varphi(b_s x)$  for every  $s \leq k$  is either strictly increasing or strictly decreasing. But then also  $\psi(b_s x)$  for  $s \leq k$  is either strictly increasing or decreasing. Therefore it is easy to check that if  $x'$  and  $x''$  belong to  $I$ , then, substituting  $\varphi$  by  $\psi$  in  $f_k$ , we get:

$$\begin{aligned} f_k(x') - f_k(x'') &= (x' - x'') \sum_{n \leq k} \frac{\pm 2}{d^{n\alpha}} - \frac{\psi(d^k x') - \psi(d^k x'')}{d^{k\alpha}} \\ &\quad - \sum_{n < k} \frac{\psi(d^n x') - \psi(d^n x'')}{d^{n\alpha}} \end{aligned}$$

whence:

$$|f_k(x') - f_k(x'')| \geq |x' - x''| \left\{ d^{k(1-\alpha)} \left( 2 - \frac{\pi}{2} \right) - \frac{2}{d^\alpha - 1} - \left( 2 + \frac{\pi}{2} \right) \frac{d^{k(1-\alpha)}}{d^{1-\alpha} - 1} \right\}$$

Let  $d$  be large enough that  $2\rho = 2 - \frac{\pi}{2} - \frac{2 + \frac{\pi}{2}}{d^{1-\alpha} - 1} > 0$ ; then fix  $k_o$  in such a way that for every  $k > k_o$  it is  $d^{k(1-\alpha)} \left( 2 - \frac{\pi}{2} - \frac{2 + \frac{\pi}{2}}{d^{1-\alpha} - 1} \right) > \frac{4}{d^\alpha - 1}$ . Then for every  $k > k_o$  it is:

$$|f_k(x') - f_k(x'')| \geq |x' - x''| \rho d^{k(1-\alpha)}.$$

Then a valuation of the length of an interval  $I$  where  $f_k$  has an oscillation not greater than  $\frac{C_\alpha}{B^r}$  (see the proof of Lemma 2.2, where  $B = d \in N$ ) is given by

$$|I| \leq \frac{C_\alpha}{\rho d^r d^{k(1-\alpha)}}.$$

From this point onwards the proof proceeds as the proof of Lemma 2.2. Another example is given by the function

$$\varphi(x) = \frac{1 - \cos(2\pi x)}{2} \quad x \in [0, 1]; \quad \varphi(x+1) = \varphi(x) \quad \text{for every } x \in R;$$

in this case we can consider the related smooth function:

$$\psi(x) = 4x - \varphi(x) \quad \text{if } 0 \leq x \leq \frac{1}{2}; \quad \psi(x) = 4(1-x) - \varphi(x) \quad \text{if } \frac{1}{2} \leq x < 1;$$

$$\psi(x+1) = \psi(x) \quad \text{for every } x \in R.$$

Repeating the procedure developed above it is easily seen that Lemma 2.2 holds.

**Lemma 2.4.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be an  $\alpha$ -Hölder continuous function, ( $0 < \alpha \leq 1$ ), and let  $\{Q_i\}$  be a cover of  $G$  constituted by  $\frac{1}{d^k}$ -meshes, with  $d \in \mathbb{N}$ ,  $d > 1$  and variable  $k \in \mathbb{N}$ . Then there exists a sequence of finite covers  $\{Q_i^n\}_{i=1, \dots, k_n}$  of  $G$  such that, for every  $s \geq 2 - \alpha$  it is:*

$$\sum_{i=1}^{+\infty} |Q_i|^s = \lim_{n \rightarrow \infty} \sum_{i=1, \dots, k_n} |Q_i^n|^s.$$

As a consequence, for such values of  $s$  it is:

$$N^s(G) = \liminf_{\delta \rightarrow 0} \left\{ \sum_i |Q_i|^s : G \subseteq \bigcup_i Q_i, Q_i \text{ finite and } |Q_i| = \frac{1}{d^k} < \delta. \right\} \quad (2.9)$$

*Proof.* If  $\{Q_i\}$  is finite then put  $\{Q_i^n\} = \{Q_i\}$  for every  $n \in \mathbb{N}$ . Otherwise there exist meshes in  $\{Q_i\}$  whose diameter is arbitrarily small and it is possible to execute the following construction.

Let  $\frac{1}{d^1}$  be the greatest edge of the meshes appearing in  $Q_i$  and let  $\{Q_i^1\}_{i=1, \dots, k_1}$  the (finite) cover of  $G$  constituted by  $\frac{1}{d^1}$ -meshes only.

Among all the  $\frac{1}{d^1}$ -meshes considered above, take only those appearing in  $\{Q_i\}$ .

Divide the remaining  $\frac{1}{d^1}$ -meshes in  $\frac{1}{d^{1+1}}$ -meshes and consider only those having a common point with  $G$ . Let  $\{Q_i^2\}_{i=1, \dots, k_2}$  be the cover of  $G$  constituted by the  $\frac{1}{d^1}$ -meshes appearing in  $\{Q_i\}$  and by the  $\frac{1}{d^{1+1}}$ -meshes disjoint from the preceding ones and with at least one common point with  $G$ .

Iterate the procedure: at step  $n$  let  $\{Q_i^n\}_{i=1, \dots, k_n}$  be the finite cover of  $G$  constituted by all the  $\frac{1}{d^1}$ -meshes, the  $\frac{1}{d^{1+1}}$ -meshes,  $\dots$ , the  $\frac{1}{d^{1+n-2}}$ -meshes appearing in  $\{Q_i\}$  and by the  $\frac{1}{d^{1+n-1}}$ -meshes disjoint from the preceding ones and having at least one common point with  $G$ .

Divide the sum  $\sum_{i=1, \dots, k_n} |Q_i^n|^s$  ( $s > 0$ ) in two parts: in the first one put the contributes of all the elements appearing in the starting cover  $\{Q_i\}$ ; in the second part put the contributes of the remaining elements, let they be, for every  $n \in \mathbb{N}$ ,  $\{T_i^n\}$  and let  $h_n$  be their number. By construction, for every  $n \in \mathbb{N}$  the diameter  $|T_i^n|$  is constant with respect to  $i = 1, \dots, h_n$ . Moreover:

$$h_n \leq L \left( \frac{|T_i^n|}{\sqrt{2}} \right)^\alpha \frac{2m(P_n)}{|T_i^n|^2} = L(\sqrt{2})^{2-\alpha} |T_i^n|^{\alpha-2} m(P_n),$$

where  $m(P_n)$  is the linear Lebesgue measure of the projection on the  $x$ -axis of the set  $\bigcup_{i=1, \dots, h_n} T_i^n$ . Therefore:

$$\sum_{i=1, \dots, h_n} |T_i^n|^s \leq L(\sqrt{2})^{2-\alpha} |T_i^n|^{s-(2-\alpha)} m(P_n).$$

Since  $\{Q_i\}$  is a cover of  $G$ , the sequence  $(P_n)$  is decreasing with respect to the inclusion relation and  $\lim_{n \rightarrow \infty} m(P_n) = 0$ . It follows that

$\lim_{n \rightarrow \infty} \sum_{i=1, \dots, h_n} |T_i^n|^s = 0$ , since  $s - (2 - \alpha) \geq 0$ . On the other hand:

$$\lim_{n \rightarrow \infty} \sum_{i=1, \dots, k_n} |Q_i^n|^s = \sum_{i=1}^{+\infty} |Q_i|^s + \lim_{n \rightarrow \infty} \sum_{i=1, \dots, h_n} |T_i^n|^s$$

and Lemma 2.4 is proven.  $\square$

### 3. The main theorems

By the definition of  $H^{2-\alpha}(G)$ , if  $f : [a, b] \rightarrow R$  is  $\alpha$ -Hölder continuous, then

$$H^{2-\alpha}(G) \leq \underline{\lim}_{\delta \rightarrow 0} N_\delta(G) \delta^{2-\alpha}.$$

Lemmas 2.2 and 2.4 allow us to prove that, under the hypotheses of Lemma 2.2, the last inequality can be inverted. Indeed the following theorem holds:

**Theorem 3.1.** *Let*

$$f(x) = \sum_{n \in N} \frac{\varphi(b_n x)}{b_n^\alpha}, \quad (0 < \alpha < 1)$$

where  $\varphi$  is a smooth function and where  $(b_n)_{n \in N}$  is such that there exists  $B > 1$ ,  $B \in N$  for which  $b_{n+1} \geq Bb_n$  for every  $n \in N$  and (2.4) holds. Then, if  $B^{1-\alpha} > 1 + \frac{c_1}{c}$ , where  $c$  and  $c_1$  are as in (2.2), there exists a constant  $\gamma > 0$  such that:

$$\underline{\lim}_{\delta \rightarrow 0} N_\delta(G) \delta^{2-\alpha} \leq \gamma H^{2-\alpha}(G). \quad (3.1)$$

*Proof.* Let  $\{Q_i\}$  be a finite cover of  $G$  constituted by  $\frac{1}{d^k}$ -meshes, with  $k$  variable in  $N$ , such that  $\frac{1}{d^k} < \delta$ . By Lemma 2.2 passing to the g.l.b. we get, for enough small  $\delta$ :

$$\inf_{\delta_1 < \delta} N_{\delta_1}(G) \delta_1^{2-\alpha} \leq \gamma_1 \inf\{\sum |Q_i|^{2-\alpha}, Q_i \text{ finite}, G \subseteq \bigcup Q_i, |Q_i| < \delta\},$$

where  $\delta_1$  is the minimum length of the edges of the elements of  $\{Q_i\}$  and  $\gamma_1$  is a suitable constant. Passing to the limit, by Lemma 2.4, one gets:

$$\underline{\lim}_{\delta \rightarrow 0} N_\delta(G) \delta^{2-\alpha} \leq \gamma_1 N^{2-\alpha}(G);$$

since there exists a constant  $A > 0$  such that  $N^{2-\alpha}(G) \leq AH^{2-\alpha}(G)$  (see [4], 5.2), the thesis follows.  $\square$

**Theorem 3.2.** *Let*

$$f(x) = \sum_{n \in N} \frac{\varphi(b_n x)}{b_n^\alpha},$$

where  $0 < \alpha < 1$ ,  $\varphi$  is a smooth function and where  $(b_n)_{n \in N}$  is such that there exist two numbers  $B > 1, B \in N$  and  $\mu > 0$  for which:  $b_{n+1} \geq Bb_n$  for every  $n \in N$  and  $b_n \geq \mu b_{n+1}$  for every  $n \in N$  (whence (7.7) holds). Then, if  $B$  is enough large, the Hausdorff dimension of  $G$  is maximum, equal to  $2 - \alpha$ . Moreover there exists a constant  $C > 0$  such that, for every interval  $[a, b] \subseteq R$  it is  $H^{2-\alpha}(G) \geq C(b - a)$  if the portion of  $G$  whose projection on the  $x$ -axis is  $[a, b]$  is considered.

*Proof.* It is easy to see that  $f$  is  $\alpha$ -Hölder continuous and therefore the Hausdorff dimension of  $G$  is less than or equal to  $2 - \alpha$ .

To prove the converse inequality we will use Theorem 3.1 of this Section. Indeed consider, for every  $\delta > 0$ , the cover of  $[0, 1]$  constituted by the intervals  $[0, \delta], [\delta, 2\delta], \dots, [p\delta, (p+1)\delta]$ , with  $p = \lceil \frac{1}{\delta} \rceil$ . Let  $k \in N$  be such that:

$$\frac{2}{b_{k+1}} \leq \delta < \frac{2}{b_k} \quad (3.2)$$

and let  $h = \frac{1}{4b_{k+1}}$ . Since  $\delta \geq \frac{2}{b_{k+1}}$ , in every interval of the cover there is an interval whose length is  $\frac{1}{b_{k+1}}$ , let

$$\left[ \frac{j}{b_{k+1}}, \frac{j+1}{b_{k+1}} \right] \subseteq [l\delta, (l+1)\delta]$$

for suitable  $j \in N$ . Therefore, for every  $l = 1, 2, \dots, p$ , the oscillation in  $[l\delta, (l+1)\delta]$  is not less than

$$\left| f\left(\frac{j}{b_{k+1}} + h\right) - f\left(\frac{j}{b_{k+1}}\right) \right| = \left| \sum_{n=1}^{+\infty} \frac{\varphi[b_n(\frac{j}{b_{k+1}} + h)] - \varphi(b_n \frac{j}{b_{k+1}})}{b_n^\alpha} \right|.$$

Assume that  $c$  is the Lipschitz coefficient of  $\varphi$  and, as is not restrictive, that  $\varphi(0) = 0$ ,  $\varphi$  is positive and increasing in  $[0, \frac{1}{4}]$  and therefore  $\varphi(\frac{1}{4}) > 0$ ; we have:

$$\left| f\left(\frac{j}{b_{k+1}} + h\right) - f\left(\frac{j}{b_{k+1}}\right) \right| \geq \frac{|\varphi(\frac{1}{4})|}{b_{k+1}^\alpha} - c \sum_{n=1}^{n=k} \frac{1}{4b_{k+1}} - 2\sum_{n \geq k+2} \frac{1}{b_n^\alpha}. \quad (3.3)$$

Then it is:

$$\begin{aligned} \left| f\left(\frac{j}{b_{k+1}} + h\right) - f\left(\frac{j}{b_{k+1}}\right) \right| &\geq \varphi\left(\frac{1}{4}\right)4^\alpha h^\alpha - \frac{ch^\alpha}{4^{1-\alpha}} \left[ \frac{1}{b_1^\alpha b_{k+1}^{1-\alpha}} + \dots + \frac{1}{b_k^\alpha b_{k+1}^{1-\alpha}} \right] \\ &- \frac{2}{b_{k+2}^\alpha} \left[ 1 + \frac{1}{B^\alpha} + \dots \right] \geq 4^\alpha h^\alpha \left[ \varphi\left(\frac{1}{4}\right) - \frac{c}{4B^{(k+1)(1-\alpha)}(B^\alpha - 1)} - \frac{2}{B^\alpha - 1} \right]. \end{aligned}$$

Therefore, if  $B$  is enough large, there exists a constant  $C_1 > 0$  such that for every  $\delta > 0$  and for every interval  $[l\delta, (l+1)\delta]$  with  $l = 1, \dots, p$ , we have that the oscillation of  $f$  in such an interval is greater than or equal to  $C_1 h^\alpha$  (for the method used here to obtain this inequality see the proof of Zhou and He in Lemma 2.5 of [6], where the particular case of  $\varphi(x) = \sin(x)$  is considered).

Therefore it is  $N_\delta(G) \geq \frac{C_1 h^\alpha}{\delta^{2-\alpha}}$ , whence, by the hypothesis, for every  $\delta > 0$ :

$$N_\delta(G)\delta^{2-\alpha} \geq \frac{C_1 h^\alpha}{\delta^\alpha} \geq C_2 \left(\frac{b_k}{b_{k+1}}\right)^\alpha \geq C_2 \mu^\alpha > 0.$$

It follows that  $\lim_{\delta \rightarrow 0} N_\delta \delta^{2-\alpha} \geq C_2 \mu^\alpha > 0$  and the thesis follows, since this inequality implies, by previous Theorem 3.1, that it is also  $H^{2-\alpha}(G) > 0$ .

Finally if in the preceding proof we consider that part of  $G$  whose projection on the  $x$ -axis is the interval  $[a, b]$  instead of the interval  $[0, 1]$ , we obtain:

$$H^{2-\alpha}(G) \geq C_2 \mu^\alpha (b - a)$$

whence the thesis.  $\square$

As we have seen in Remark 2.3, Lemma 2.2 and therefore also Theorem 3.2, whose proof is essentially based upon Lemma 2.2, can be proved also under other less restrictive hypotheses. For example we claim that:

**Theorem 3.3.** *If  $d \in \mathbb{N}$  is enough large, the graph of the Weierstrass function*

$$f(x) = \sum_{n \in \mathbb{N}} \frac{\sin(2\pi d^n x)}{d^{n\alpha}}, \quad (0 < \alpha < 1)$$

*has Hausdorff dimension equal to  $2 - \alpha$ . The same conclusion holds, if  $d$  is enough large, for the following function introduced by Mandelbrot (1977):*

$$g(x) = \sum_{n \in \mathbb{N}} \frac{1 - \cos(2\pi d^n x)}{d^{n\alpha}}, \quad (0 < \alpha < 1).$$

*Proof.* Indeed consider the function:

$$\varphi(x) = \frac{1 + \sin(2\pi x)}{4}, \quad -\frac{1}{4} \leq x < \frac{3}{4}, \quad \varphi(x+1) = \varphi(x) \text{ for every } x \in \mathbb{R};$$

as we have seen in Remark 2.3, we can apply Lemma 2.2 and therefore also Theorem 3.2 to this function, obtaining that the Hausdorff dimension of the graph of the function:

$$\sum_{n \in \mathbb{N}} \frac{\varphi(d^n x)}{d^{n\alpha}} = \frac{1}{4} \left( \sum_{n \in \mathbb{N}} \frac{1}{d^{n\alpha}} + \sum_{n \in \mathbb{N}} \frac{\sin(2\pi d^n x)}{d^{n\alpha}} \right)$$

coincides with  $2 - \alpha$  and obviously the same happens for the graph of the function  $f$ . With the same procedure we prove the thesis about Mandelbrot function: in this case we consider the function

$$\varphi(x) = \frac{1 - \cos(2\pi x)}{2}$$

and the related smooth function  $\psi$  given in Remark 2.3. Then Theorem 3.2 is applicable and the thesis is proven.  $\square$

**Theorem 3.4.** *Let*

$$f(x) = \sum_{n \in \mathbb{N}} (-1)^n \frac{\varphi(b_n x)}{b_n^\alpha}, \quad (0 < \alpha < 1)$$

*where  $\varphi$  and  $(b_n)_{n \in \mathbb{N}}$  are as in Theorem 3.2. Then, if  $B$  is enough large, both the box dimension and the Hausdorff dimension of  $G$  are equal to  $2 - \alpha$ .*

*Proof.* As in the proof of Theorem 3.2 above, it is easy to see that  $f$  is  $\alpha$ -Hölder continuous and therefore the Hausdorff dimension of  $G$  is less than or equal to  $2 - \alpha$ . To prove the converse inequality let  $\delta > 0$  and consider the cover of  $[0, 1]$ :  $[0, \delta[, [\delta, 2\delta[, \dots, [p\delta, (p+1)\delta[,$  where  $p = \lfloor \frac{1}{\delta} \rfloor$ . Let  $k \in \mathbb{N}$  be such that (3.2) holds and put  $h = \frac{1}{4b_{k+1}}$ . Then proceed as in the proof of Theorem 3.2, obtaining (3.3). Therefore there exists a constant  $C > 0$  such

that the oscillation of  $f$  in every interval  $[l\delta, (l+1)\delta]$ , ( $0 \leq l \leq p$ ), is not less than  $Ch^\alpha$  and we can conclude as in the proof of Theorem 3.2.  $\square$

## References

- [1] Besicovitch, A.S., Ursell, H.D., *Sets of fractional dimensions (V): on dimensional numbers of some continuous curves*, Journal of the London Math. Soc., **12**(1937), 18-25.
- [2] Falconer, K.J., *The Geometry of Fractal Sets*, Cambridge University Press, 1985.
- [3] Falconer, K.J., *Fractal Geometry*, John Wiley & Sons, 1993.
- [4] Mattila, P., *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, 1995.
- [5] Przytycki, F., Urbanski, M., *On the Hausdorff dimension of some fractal sets*, Studia Math., **93**(1989), 155-186.
- [6] Zhou, S.P., He, G.L., *On a class of Besicovitch functions to have exact box dimension: a necessary and sufficient condition*, Math. Nachr. **278**(2005), no. 6, 730-734.

Loredana Biacino

Dipartimento di Matematica e Applicazioni "R.Caccioppoli"

Via Cinzia, Monte Sant'Angelo, 80126 Napoli

e-mail: [loredana.biacino2@unina.it](mailto:loredana.biacino2@unina.it)