Stud. Univ. Babeş-Bolyai Math. 56(2011), No. 4, 3–6

Units in abelian group algebras over indecomposable rings

Peter Danchev

Abstract. Suppose R is a commutative indecomposable unitary ring of prime characteristic p and G is a multiplicative Abelian group such that G_0/G_p is finite. We describe up to isomorphism the unit group $U(RG)$ of the group algebra RG. This extends an earlier result due to Mollov-Nachev (Commun. Algebra, 2006) removing the restriction that G splits.

Mathematics Subject Classification (2010): 16S34, 16U60, 20K21.

Keywords: groups, rings, group rings, indecomposable rings, units, direct decompositions, isomorphisms.

1. Introduction

Throughout the present paper, suppose R is a commutative unitary (i.e., with identity) ring of prime characteristic p and G is an Abelian group written multiplicatively as is customary when exploring group algebras. For such R and G, denote by RG the group algebra of G over R with unit group $U(RG)$ and its normed subgroup $V(RG)$ of units with augmentation 1; note that the decomposition $U(RG) = V(RG) \times U(R)$ holds where $U(R)$ is the unit group of R. As usual, G_0 is the torsion subgroup of G with p-primary component G_p, and $S(RG) = V_p(RG)$ is the p-component of $V(RG)$. Moreover, for any natural number n, ζ_n denotes the primitive nth root of unity and $R[\zeta_n]$ is the free R-module, algebraically generated as a ring by ζ_n, with dimension $[R[\zeta_n] : R]$. As it is well-known, a ring is said to be *indecomposable* if it cannot be decomposed into a direct sum of two or more non-trivial ideals.

The structures of $V(RG)$ and $U(RG)$ have been very intensively studied in the past twenty years (see, e.g., [8]). Some isomorphism description results were obtained in [2] and [11]. The purpose of this work is to improve considerably one of the central results in the latter citation by giving a more direct and conceptual proof (note that some parts of the proof of the corresponding result in [11] are unnecessary complicated). Likewise, our method proposed below gives a new strategy for obtaining other results of this type since it reduces the general case to the p-mixed one.

2. Main results

As noted above, Mollov and Nachev established in ([11], Theorem 5.8) the following assertion.

Theorem (2006). *Let R be a commutative indecomposable ring with identity of prime characteristic p and let G be a splitting Abelian group. Suppose that G_0/G_p is a finite group of exponent n and $n \in U(R)$. Then*

$$U(RG) \cong \coprod_{d/n} \coprod_{\lambda(d)} U(R[\zeta_d]) \times \coprod_{b} G/G_0 \times \coprod_{d/n} \coprod_{\lambda(d)} S(R[\zeta_d](G_p \times G/G_0))$$

where $\lambda(d) = \frac{(G_0/G_p)(d)}{[R[\zeta_d]:R]}$, with $(G_0/G_p)(d)$ the number of elements of G_0/G_p of order d, and $b = \sum_{d/n} \lambda(d)$.

Notice that since $char(R) = p$ is a prime integer, it is self-evident that $exp(G_0/G_p)$ inverts in R, so that the condition $n \in U(R)$ is always fulfilled and hence it is a superfluously stated in the theorem.

The aim that we will pursue is to drop the limitation that G is a splitting group. Specifically, we proceed by proving the following:

Main Theorem. *Suppose R is an indecomposable ring of $char(R) = p$ and G is a group for which G_0/G_p is finite. Then the following isomorphism is true:*

$$U(RG) \cong \coprod_{d/exp(G_0/G_p)} \coprod_{a(d)} [U(R[\zeta_d]) \times [(G/\coprod_{q \neq p} G_q) V_p(R[\zeta_d](G/\coprod_{q \neq p} G_q))]] \tag{$*$}$$

where $a(d) = \frac{|\{g \in G_0/G_p : order(g) = d\}|}{[R[\zeta_d]:R]}$.

In particular:

(1) if G is p-splitting, then

$$U(RG) \cong \coprod_{d/exp(G_0/G_p)} \coprod_{a(d)} [U(R[\zeta_d]) \times V_p(R[\zeta_d](G/\coprod_{q \neq p} G_q))] \times$$
$$\times \coprod_{\sum_{d/exp(G_0/G_p)} a(d)} G/G_0.$$

(2) if G_p is a direct sum of cyclic groups, then

$$U(RG) \cong \coprod_{d/exp(G_0/G_p)} \coprod_{a(d)} [U(R[\zeta_d]) \times (V_p(R[\zeta_d](G/\coprod_{q \neq p} G_q))/(G/\coprod_{q \neq p} G_q)_p)] \times$$
$$\times \coprod_{\sum_{d/exp(G_0/G_p)} a(d)} G/\coprod_{q \neq p} G_q.$$

Moreover, the quotient $V_p(R[\zeta_d](G/\coprod_{q\neq p} G_q))/(G/\coprod_{q\neq p} G_q)_p)$ *is a direct sum of cyclic groups and can be characterized via the Ulm-Kaplansky invariants calculated in* [7].

Proof. First observe that $\coprod_{q\neq p} G_q$ is pure in G_0 as its direct factor and hence it is pure in G because G_0 is pure in G. Since $G_0/G_p \cong \coprod_{q\neq p} G_q$ is finite, it is well known that $\coprod_{q\neq p} G_q = F$ is a direct factor even of G, say $G = F \times M$ for some $M \leq G$. It is obvious that $M \cong G/\coprod_{q\neq p} G_q$ is p-mixed with $M_0 = M_p = G_p$.

Next, write $RG = (RM)F$. Since R is indecomposable, it follows from [9] that RM is also indecomposable because there is no prime q which inverts in R such that $M_q \neq 1$. Clearly $exp(F) \in U(R) \subseteq U(RM)$ because $char(R) = char(RM) = p$ and therefore we can apply Theorem 4.4 and Remark 4.5 from [11] to get that $RG \cong \sum_{d/exp(F)} \sum_{a(d)} (RM)[\zeta_d]$, whence $U(RG) \cong \coprod_{d/exp(F)} \coprod_{a(d)} U((RM)[\zeta_d])$. It is straightforward to see that $(RM)[\zeta_d] \cong R[\zeta_d]M$, so that $U((RM)[\zeta_d]) \cong U(R[\zeta_d]M) = V(R[\zeta_d]M) \times U(R[\zeta_d])$. On the other hand, according to [4], [5] or [6], $V(R[\zeta_d]M) = MV_p(R[\zeta_d]M)$ using the fact from [11] that $R[\zeta_d]$ is indecomposable of prime characteristic p as well. This establishes formula $(*)$.

(1) If now G is p-splitting, it is readily seen that it is splitting. Consequently, so is M as its direct factor. Furthermore, it is easily checked that $M \cong G/\coprod_{q\neq p} G_q \cong (G_0 \times G/G_0)/\coprod_{q\neq p} G_q \cong (G_0/\coprod_{q\neq p} G_q) \times (G/G_0) \cong G_p \times (G/G_0)$. Moreover, $M = M_p \times M/M_p \cong G_p \times G/G_0$ because $M/M_p \cong G/\coprod_{q\neq p} G_q/(G/\coprod_{q\neq p} G_q)_0 = G/\coprod_{q\neq p} G_q/G_0/\coprod_{q\neq p} G_q \cong G/G_0$. That is why $MV_p(R[\zeta_d]M) \cong (M/M_p) \times V_p(R[\zeta_d]M) \cong (G/G_0) \times V_p(R[\zeta_d](G/\coprod_{q\neq p} G_q))$ and we are done.

(2) Appealing to [1], $G_p = M_p$ being a direct sum of cyclic groups implies that $V_p(R[\zeta_d]M) = M_p \times T$ for some subgroup T which is a direct sum of cyclic groups. Therefore, $MV_p(R[\zeta_d]M) = M(M_p \times T) = M \times T \cong (G/\coprod_{q\neq p} G_q) \times V_p(R[\zeta_d](G/\coprod_{q\neq p} G_q))/(G/\coprod_{q\neq p} G_q)_p$ since it is easy to see that $M \cap T = M_p \cap T = 1$ because $T = T_p$. This completes the proof. □

Note. In virtue of our lemma from [4], [5] or [6], our result can be generalized to the direct sum (= direct product) of m indecomposable rings, i.e., when the set $id(R)$ of all idempotents of R is finite and contains 2^m elements, that is, $|id(R)| = 2^m$.

That is why, utilizing this approach, the number m in Theorem 5 of [11] can be explicitly computed, namely it is equal exactly to $log_2|id(R)|$.

Remark. The proof of Theorem 2.7 in [10] contains a gap and so it is incomplete. In fact, the authors claimed that they will assume that the splitting group is p-mixed. The reason is that the K-algebras isomorphism $KG \cong KH$ yields that $K(G/\coprod_{q\neq p} G_q) \cong K(H/\coprod_{q\neq p} H_q)$ whenever K is a field of $char(K) = p$. But they need to show that G being splitting ensures that so is $G/\coprod_{q\neq p} G_q$. However, this was already done in [3].

References

[1] Danchev, P.V., *Commutative group algebras of σ-summable abelian groups*, Proc. Amer. Math. Soc., **125**(1997), no. 9, 2559-2564.

[2] Danchev, P.V., *Normed units in abelian group rings*, Glasgow Math. J., **43**(2001), no. 3, 365-373.

[3] Danchev, P.V., *Notes on the isomorphism and splitting problems for commutative modular group algebras*, Cubo Math. J., **9**(2007), no. 1, 39-45.

[4] Danchev, P.V., *Warfield invariants in commutative group rings*, J. Algebra Appl., **8**(2009), no. 6, 829-836.

[5] Danchev, P.V., *Maximal divisible subgroups in p-mixed modular abelian group rings*, Commun. Algebra, **39**(2011), no. 6, 2210-2215.

[6] Danchev, P.V., *Maximal divisible subgroups in modular group rings of p-mixed abelian groups*, Bull. Braz. Math. Soc., **41**(2010), no. 1, 63-72.

[7] Danchev, P.V., *Ulm-Kaplansky invariants in commutative modular group rings*, J. Algebra Number Theory Academia, **1**(2011), no. 2, 127-134.

[8] Karpilovsky, G., *Units of commutative group algebras*, Expo. Math., **8**(1990), 247-287.

[9] May, W.L., *Group algebras over finitely generated rings*, J. Algebra, **39**(1976), 483-511.

[10] May, W.L., Mollov, T.Zh., Nachev, N.A., *Isomorphism of modular group algebras of p-mixed abelian groups*, Commun. Algebra, **38**(2010), 1988-1999.

[11] Mollov, T.Zh., Nachev, N.A., *Unit groups of commutative group rings*, Commun. Algebra, **34**(2006), 3835-3857.

Peter Danchev
13, General Kutuzov Str.
bl. 7, fl. 2, ap. 4
4003 Plovdiv, Bulgaria
e-mail: pvdanchev@yahoo.com