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Some new properties of Generalized Bernstein polynomials

Donatella Occorsio

Abstract. Let $B_m(f)$ be the Bernstein polynomial of degree m. The Generalized Bernstein polynomials

$$B_{m,\lambda}(f,x) = \sum_{i=1}^{\infty} (-1)^{i+1} \binom{\lambda}{i} B_m^i(f;x), \lambda \in \mathbb{R}^+$$

were introduced in [13]. In the present paper some of their properties are revisited and some applications are presented. Indeed, the stability and the convergence of a quadrature rule on equally spaced knots is studied and a class of curves depending on the shape parameter λ, including both Bézier and Lagrange curves, is introduced.

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1. Introduction

The operator $B_{m,\lambda}$, introduced and studied in [13], is defined as

$$B_{m,\lambda} = \sum_{i=1}^{\infty} (-1)^{i+1} \binom{\lambda}{i} B_m^i, \quad \lambda \in \mathbb{R}^+,$$

where $B_m^i = B_m(B_m^{i-1})$, and B_m is the Bernstein operator. $B_{m,\lambda}$ is a linear operator, not always positive, that maps bounded functions into polynomials of degree at most m. The sequence $\{B_{m,\lambda}(f)\}_m$ has the property of improving the order of convergence when the smoothness of the function increases (see [11, 14]). For instance, assuming $f \in C^{(2[\lambda])}([0,1]), \lambda \geq 1$, we have $|f - B_{m,\lambda}(f)| = O\left(\frac{1}{m^\lambda}\right)$. In this sense, the sequence $\{B_{m,\lambda}(f)\}_m$ produces a significant enhancement with respect to the behavior of the ordinary Bernstein sequence.

Moreover the sequence $\{B_{m,\lambda}(f)\}_m$ includes both Bernstein polynomials ($\lambda = 1$) and, as limit case, the Lagrange interpolating polynomial on equally spaced knots ($\lambda \to \infty$). In spite of these mentioned properties, the expression derived in [13] in the monomial basis $(1, x, \ldots, x^m)$ is no easy for the computation and, in addition, produces instability in the polynomial evaluation.

In the present paper we first express $B_{m,\lambda}(f)$ as the Bernstein polynomial of a function g, suitable related to f. Therefore, the evaluation of $B_m(g)$ can be performed by de Casteljau scheme, which is a stable algorithm. Moreover, using $B_{m,\lambda}(f) = B_m(g)$, we can revisit some proofs, like, for instance, the property of mapping bounded functions into polynomials. In order to exploit the above mentioned "good" properties, we consider two applications. The first is the approximation of integrals $\int_0^1 f(x)dx$, obtained by replacing the function f with $B_{m,\lambda}(f)$. By this way, it is derived a simple quadrature rule that we prove to be stable and convergent and whose order of accuracy as faster decays as smoother is the integrand function f. Such kind of formulas can be of interest since there are not so many polynomial quadrature rules involving equally spaced points and having a "good" behavior of the error.

The second application deals with the employment of $\{B_{m,\lambda}\}_\lambda$ in CAGD (Computer Aided Geometric Design), by considering a possible generalization of the well-known Bézier curves. Given a control polygon

$$\mathbf{P} = [\mathbf{P}_0, \ldots, \mathbf{P}_m], \ \mathbf{P}_j \in \mathbb{R}^2,$$

we call the curves of parametric equations

$$\mathbf{B}_{m,\lambda}[\mathbf{P}_0, \ldots, \mathbf{P}_m](t) = \sum_{j=0}^{m} p_{m,j}^{(\lambda)}(t)\mathbf{P}_j, \quad 0 \le t \le 1,$$

Generalized Bézier curves. Curves in this class change continuously their shape, "bridging" the Bézier curve $\mathbf{B}_m[\mathbf{P}_0, \ldots, \mathbf{P}_m]$ to the Lagrange interpolating curve $\mathbf{L}_m[\mathbf{P}_0, \ldots, \mathbf{P}_m]$. Some generalization in this sense where introduced and studied in [2], [3], [4], [15] (see also [9], [16]).

The outline of this paper is as follows. Section 2 contains the new vector expression and some properties deducible from this. In Section 3 are stated the announced applications, equipped with some numerical and graphical tests. Finally, Section 4 will contain the proofs of the main results.

2. The $B_{m,\lambda}(f)$ polynomials

For any continuous function f on the unit interval $[0, 1]$ ($f \in C^0([0, 1])$), let $B_m(f)$ be the $m-$th Bernstein polynomial

$$B_m(f; x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right), \quad p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}. \qquad (2.1)$$

Denoting by $B_m^i(f) = B_m(B_m^{i-1}(f)), B_m^0(f) = f$ the i-th iterate of the Bernstein polynomial, in [13] (see also [1], [8], [12]) the authors introduced and studied the following linear combination of $B_m^i(f)$,

$$B_{m,\lambda}(f,x) = \sum_{i=1}^{\infty}(-1)^{i+1}\binom{\lambda}{i}B_m^i(f;x), \lambda \in \mathbb{R}^+. \tag{2.2}$$

For any fixed λ, $\{B_{m,\lambda}(f)\}_m$, will be called sequence of *generalized Bernstein polynomials of parameter* λ. For $\lambda = 1$, $B_{m,\lambda} = B_m$. The special case $\lambda \in \mathbf{N}$ was studied in [12]. Here we will consider the case $\lambda \geq 1$. An expression of the polynomial $B_{m,\lambda}(f)$ is

$$B_{m,\lambda}(f;x) = \sum_{j=0}^{m} p_{m,j}^{(\lambda)}(x) f\left(\frac{j}{m}\right), \quad 0 \leq x \leq 1, \tag{2.3}$$

where

$$p_{m,j}^{(\lambda)}(x) = \sum_{i=1}^{\infty}\binom{\lambda}{i}(-1)^{i-1}B_m^{i-1}(p_{m,i};x). \tag{2.4}$$

Since by (2.3) the evaluation of $B_{m,\lambda}$ is not feasible, first we derive a vectorial form of the basis $\{p_{m,k}^{(\lambda)}\}_{k=0}^m$, by which for any function f, the polynomial $B_{m,\lambda}(f)$ coincides with the Bernstein polynomial $B_m(g)$, g being a function related to f.

Theorem 2.1. *Assume* $\lambda \geq 1$. *Setting*

$$\mathbf{p}_m^{(\lambda)}(x) = [p_{m,0}^{(\lambda)}(x), p_{m,1}^{(\lambda)}(x), \ldots, p_{m,m}^{(\lambda)}(x)]^T,$$

and

$$\mathbf{p}_m(x) = [p_{m,0}(x), \ldots, p_{m,m}(x)]^T,$$

one has

$$\mathbf{p}_m^{(\lambda)}(x)^T = \mathbf{p}_m(x)^T C_{m,\lambda}, \tag{2.5}$$

where

$$C_{m,\lambda} = A^{-1}[I-(I-A)^{\lambda}] = [I-(I-A)^{\lambda}]A^{-1} \in \mathbb{R}^{(m+1)\times(m+1)}, \tag{2.6}$$

$$(A)_{i,j} = p_{m,j}(t_i), \quad i = 0,1,\ldots,m, j = 1,2,\ldots,m \tag{2.7}$$

$t_i = i/m, i = 0,1,\ldots,m$, *and* I *is the identity matrix of order* $(m+1)$. *Then, for any* $f \in C^0([0,1])$, *setting*

$$\mathbf{f}_m = [f_0, f_1, \ldots, f_m]^T, \quad f_i = f(t_i), \tag{2.8}$$

the polynomial $B_{m,\lambda}(f)$ *can be represented in the following form*

$$B_{m,\lambda}(f;x) = \mathbf{p}_m(x)^T C_{m,\lambda}\mathbf{f}_m. \tag{2.9}$$

In the case $\lambda = k \in \mathbb{N}$, the matrix $C_{m,\lambda}$ is given by

$$\begin{aligned} C_{m,k} &= [I + (I-A) + (I-A)^2 + \ldots (I-A)^{k-1}] \\ &= A^{-1}[I-(I-A)^k] \end{aligned} \tag{2.10}$$

and the polynomial $B_{m,\lambda}(f)$ is directly computed by using a very simple algorithm, as the expression in (2.10) suggests. However, when λ is not an integer, the matrix series in (2.2) can be obtained by an equivalent finite process. To do this, we need the following definition of matrix function on the spectrum (see for instance [10]).

Definition 2.2. *Let B a real matrix of order n and suppose that $\xi_1, \xi_2, \dots, \xi_s$ are the distinct eigenvalues of B of algebraic multiplicity $n_1, n_2, \dots, n_s$, respectively. Let f be defined on the spectrum of B. Then $f(B) := H_n(f; B)$, where $H_n(f)$ is the Hermite interpolating polynomial of degree less than n that satisfies the interpolation conditions*

$$H_n(f^{(j)}; \xi_i) = f^{(j)}(\xi_i), \quad j = 0, 1, \dots, n_i - 1, \quad i = 1, 2, \dots, s.$$

Denote by $\Delta_h f(x) = f(x+h) - f(x)$ the forward difference of the function f and shift $h \in \mathbb{R}$, and let be $\Delta_h^i = \Delta_h^{i-1}(\Delta_h)$. About the eigenvalues of the matrix A we prove:

Proposition 2.3. *The eigenvalues $\{\xi_{m,i}\}_{i=0}^m$ of the matrix A are*

$$\xi_{m,0} = \xi_{m,1} = 1, \quad \xi_{m,i} = \prod_{j=1}^{i-1} \left(1 - \frac{j}{m}\right) = \binom{m}{i} \Delta_{\frac{1}{m}}^i e_i(0), \quad i = 2, \dots, m, \tag{2.11}$$

with $e_k(x) = x^k, k \in \mathbb{N}$. Therefore, denoting by $\{\mu_i\}_{i=1}^m$ the eigenvalues of $C_{m,\lambda}$,

$$\mu_{m,0} = \mu_{m,1} = 1, \quad \mu_{m,i} = \frac{1 - (1 - \xi_{m,i})^\lambda}{\xi_{m,i}}, \quad i \geq 2. \tag{2.12}$$

For any set of knots $x_1, x_2, \dots, x_i$, the so-called *divided differences* of a given function f are defined recursively by

$$[x_1; f] = f(x_1),$$

$$[x_1, ..., x_k; f] = \frac{[x_2, ..., x_k; f] - [x_1, ..., x_{k-1}; f]}{x_k - x_1}, \quad \text{if } x_k \neq x_{k-1}$$

and, if $f^{(i-1)}(x_1)$ exists,

$$[x_1, x_2, \dots, x_i; f] = \frac{f^{(i-1)}(x_1)}{(i-1)!}, \quad \text{if } x_1 = x_2 = \dots = x_i, \quad i \geq 2.$$

Then, by using Proposition 2.3 and Definition 2.2, we can deduce

Corollary 2.4. *Assume $\lambda \geq 1$. Setting*

$$\sigma(x) = [1 - (1-x)^\lambda] x^{-1},$$

we have

$$C_{m,\lambda} = I\sigma(\xi_{m,0}) + \sum_{j=1}^m [\xi_{m,0}, \xi_{m,1}, \dots, \xi_{m,j}; \sigma] \prod_{k=0}^{j-1} (A - \xi_{m,k} I) =: \rho(A). \tag{2.13}$$

Therefore

$$B_{m,\lambda}(f; x) = \mathbf{p}_m(x)^T \rho(A) \mathbf{f}_m, \tag{2.14}$$

Remark 2.5. By the previous result it follows that $B_{m,\lambda}(f)$ can be considered as the $m-$th Bernstein polynomial of the function g such that

$$g_k := g(t_k) = [\rho(A)\mathbf{f}_m]_k, \quad k = 0, 1, \ldots, m,$$

i.e.

$$B_m(g; x) = B_{m,\lambda}(f; x) = \mathbf{p}_m(x)^T \mathbf{g}_m,$$

where

$$\mathbf{g}_m := [g_0, g_1, \ldots, g_m]^T. \tag{2.15}$$

As a consequence, we can now compute the polynomial $B_{m,\lambda}(f)$ by using the de Casteljau recursive scheme.

Remark 2.6. Let us denote by $L_m(f)$ the Lagrange polynomial interpolating f at the equally spaced knots $t_j, \quad j = 0, 1, \ldots, m$, i.e.

$$L_m(f; x) = \sum_{j=0}^{m} l_{m,j}(x) f(t_j) = \mathbf{l}_m(x)^T \mathbf{f}_m,$$

where

$$l_{m,j}(x) = \prod_{j \neq i=1}^{m} \frac{(x - t_i)}{(t_j - t_i)}, \quad \mathbf{l}_m(x) = [l_{m,0}(x), l_{m,1}(x), \ldots, l_{m,m}(x)]^T,$$

and $\mathbf{f}_m$ is defined in (2.8). By (2.9) and using $\mathbf{p}_m(x)^T A^{-1} = \mathbf{l}_m(x)^T$ [15], it follows

$$B_{m,\lambda}(f; x) = L_m(h; x) = \mathbf{l}_m(x)^T \mathbf{h}_m \tag{2.16}$$

where

$$\mathbf{h}_m := [h_0, h_1, \ldots, h_m]^T = C_{m,\lambda} \mathbf{f}_m, \quad h_i = h(t_i), \quad i = 0, 1, \ldots, m, \tag{2.17}$$

i.e. $B_{m,\lambda}(f)$ is also the Lagrange polynomial interpolating the function h at the equally spaced knots $t_j, \quad j = 0, 1, \ldots, m$.

As consequence of (2.16), it is very easy to revisit the proof of the next result obtained in [13]:

For any m,

$$\lim_{\lambda \to \infty} B_{m,\lambda}(f; x) = L_m(f; x), \quad \forall f \in C^0([0, 1]), \tag{2.18}$$

uniformly in $x \in [0, 1]$. Indeed, it immediately follows by (2.16), (2.17) and

$$\lim_{\lambda \to \infty} C_{m,\lambda} = A^{-1}. \tag{2.19}$$

Relation (2.18) allows to say that the sequence $\{B_{m,\lambda}\}_\lambda$ links continuously the Bernstein operator to the Lagrange one.

In the next Proposition we derive another representation of $B_{m,\lambda}(f)$ by means of the finite difference of the function f at the point 0. This expression generalizes the well-known relation

$$B_m(f; x) = \sum_{k=0}^{m} \binom{m}{k} x^k \Delta_{\frac{1}{m}}^k f(0), \tag{2.20}$$

and it is useful to determine the closed expression of $B_{m,\lambda}(e_k), \quad k=1,2,\dots,$ being $e_k(x)=x^k, k\in\mathbb{N}$.

Theorem 2.5. *Assume $\lambda \geq 1$. Let M be the upper triangular matrix of elements $(M)_{i,j}=\binom{m}{i}\Delta^i_{\frac{1}{m}}e_j(0), \quad i=0,1,\dots,m, j=0,1,\dots,i$, and define*

$$M_{m,\lambda}=M^{-1}[I-(I-M)^\lambda]=[I-(I-M)^\lambda]M^{-1}\in\mathbb{R}^{(m+1)\times(m+1)}. \quad (2.21)$$

For any $f\in C^0([0,1])$, setting

$$\mathbf{d}_m=[f(0), m\Delta_{\frac{1}{m}}f(0),\dots,\binom{m}{k}\Delta^k_{\frac{1}{m}}f(0),...,\Delta^m_{\frac{1}{m}}f(0)]^T,$$

the polynomial $B_{m,\lambda}(f)$ can be represented in the following form

$$B_{m,\lambda}(f;x)=\mathbf{x}^T M_{m,\lambda}\mathbf{d}_m. \quad (2.22)$$

and also

$$B_{m,\lambda}(f;x)=\mathbf{x}^T\rho(M)\mathbf{d}_m, \quad (2.23)$$

where

$$\rho(M)=I\sigma(\xi_{m,0})+\sum_{j=1}^{m}[\xi_{m,0},\xi_{m,1},\dots,\xi_{m,j};\sigma]\prod_{k=0}^{j-1}(M-\xi_{m,k}I). \quad (2.24)$$

$\sigma(x)=[1-(1-x)^\lambda]x^{-1}$ *and* $\xi_{m,i}=\binom{m}{i}\Delta^i_{\frac{1}{m}}e_i(0)$.

Remark 2.6. In view of (2.23), we derive

$$B_{m,\lambda}(e_k;x)=\mathbf{x}^T\tilde{\rho}(A)_k\tilde{\mathbf{d}}_k \quad (2.25)$$

where $\tilde{\rho}(A)_k\in\mathbb{R}^{(m+1)\times k}$ is the matrix formed by the first k columns of $\rho(A)$ and $\tilde{\mathbf{d}}_k\in\mathbb{R}^k$ is the vector formed by the first k components of $\mathbf{d}$.

Remark 2.7. Denoting by $\mathbf{V_m}:=\mathbf{V_m}(t_0,t_1,\dots,t_m)$ the Vandermonde matrix w.r.t the knots $t_0,t_1,\dots,t_m$, i.e. $(\mathbf{V}_m)_{i,j}=t_i^j, \quad i=0,1,\dots,m, j=0,1,\dots,m$, we get

$$M_{m,\lambda}=\mathbf{V}_m^{-1}C_{m,\lambda}\mathbf{V}_m \quad (2.26)$$

which easily follows by combining $\mathbf{V}_m\mathbf{d}_m=\mathbf{f}_m$ and $\mathbf{x}^T=\mathbf{p}_m^T\mathbf{V}_m$.

We conclude this section, giving some details about the computation of polynomials $B_{m,\lambda}(f)$. Since the polynomial $B_{m,\lambda}(f;x)$ is also the Bernstein polynomial of the function $g=C_{m\lambda}\mathbf{f}$, it can be computed by using the de Casteljau algorithm w.r.t. g. The algorithm is numerically stable and requires m^2 long operations, for any $x\in[0,1]$. Since A is a centrosymmetric matrix (i.e. $a_{i,j}=a_{m-i,m-j}, i,j=0,1,2,\dots,m$), we deduce that its construction can be performed in $\left[\frac{m+1}{2}\right]^3$ long operations. Let us distinguish between the case λ integer or not. If $\lambda=k\in\mathbb{N}$, by (2.10), the global cost to construct $C_{m,k}$ is $(k-1)\left[\frac{m+1}{2}\right]^3$. A significant reduction is obtained by choosing $k=2^p$, whereas, by using

$$C_{m,2^p}=C_{m,2^{p-1}}+(I-A)^{2^{p-1}}C_{m,2^{p-1}},$$

the computational effort is almost $m^3 \log_2 k$. (see [15].)
In the general case $\lambda \in \mathbb{R}^+$, we have to use (2.13) and the global cost for compute $C_{m,\lambda}$ increases, requiring almost $(m-2)m^3/2 \sim m^4/2$. Even though the computation of $C_{m,\lambda}$ requires the major computational effort, for fixed values of m and λ its construction can be performed only once.

3. Two applications

In this section we discuss two different applications.

3.1. A quadrature rule on equally spaced knots

As we have said, quadrature rules involving equally spaced points and having a "good" behavior of the error can be of interest. Indeed, the Newton-Cotes rules present catastrophic instability, since they are based on interpolation processes on equally spaced knots. About the error of composite rules, like Trapezoidal or Simpson rule, they suffer from saturation phenomena, and the error decays like $O(\frac{1}{m^2})$ and $O(\frac{1}{m^4})$, respectively. Here we revisit the following quadrature rule suggested in [12],

$$\int_0^1 f(x)dx = \int_0^1 B_{m,k}(f;x)dx + R_m^k(f) =: \Sigma_m(f) + R_m^k(f), \tag{3.1}$$

where $\lambda = k \in \mathbb{N}$.
Since for any $j = 0, 1, \ldots, m$

$$\int_0^1 p_{m,j}(x)dx = \frac{1}{m+1},$$

by (2.9) and (2.10), we derive

$$\Sigma_m(f) = \frac{1}{m+1} \sum_{j=0}^{m} \left(\sum_{i=0}^{m} (C_{m,k})_{i,j} \right) f(t_j) := \sum_{j=0}^{m} D_j^{(k)} f(t_j). \tag{3.2}$$

Now we prove that the rule is numerically stable and convergent and that for smooth functions the rate of convergence improves as the parameter k increases.

Theorem 3.1. *With the notation used in (3.1)-(3.2),*

$$\sup_m \sum_{j=0}^{m} |D_j^{(k)}| < \infty, \tag{3.3}$$

and for any $f \in C^{2k}([0,1]), k \geq 2, 2k < m$

$$|R_m^k(f)| \leq \frac{\mathcal{C}}{m^k} \left(\|f\|_\infty + \|f^{(2k)}\|_\infty \right), \tag{3.4}$$

where $\mathcal{C}$ is a positive constant independent of f and m.

Now we show the performance of the method by some numerical tests. In the tables for each degree m and for the specified values ok k, we report the values obtained in computing the quadrature sum (3.2) in 16−digits precision, comparing also with the results obtained by using the composite Trapezoidal and Simpson rules. For these rules the value of m represents the number of function evaluations.

Example 3.2.

$$\int_0^1 \frac{\arctan(x)}{(1+x^2)^3} dx$$

In this example the exact value is 0.1713839674246280. Here $f \in C^\infty([0,1])$. The apparent slow convergence depends on the "fast" increasing values of the seminorm $\|f^{(2k)}\|_\infty$. For instance $\|f^{(16)}\|_\infty \sim 1.2 \times 10^{15}$.

m	$k=5$	$k=8$	$k=16$
4	0.17	0.17	0.17
8	0.171	0.171	0.17138
16	0.17138	0.171383	0.171383
32	0.17138396	0.17138395	0.17138396
64	0.171383967	0.1713839674	0.17138396742
128	0.1713839674	0.171383967424	**0.1713839674244628**
256	0.171383967424	**0.171383967424628**	
512	**0.171383967424628**		

m	Trapezoidal	m	Simpson
256	0.17138	64	0.173839
512	0.171383	128	0.17383967
4096	0.17138396	256	0.173839674
16384	0.171383967	512	0.1738396742
131072	0.1713839674	1024	0.17383967424
1048576	0.171383967424	2048	0.1713839674246
4194304	**0.17138396742462**	4096	**0.171383967424628**

Example 3.3.

$$\int_0^1 \frac{(1-x)^{5\pi}}{1+x^3} dx$$

In this example the exact value is 0.0597973223176919. Since the function $f \in C^{15}([0,1])$, in view of the Theorem 3.1, the error behaves like $O\left(\frac{1}{m^7}\right)$. As we can see the machine precision is attained for $m = 1024$, $k = 7$, whereas according to the estimate (3.4) and taking into account the high value of the seminorm $\|f^{(14)}\|_\infty \sim 1.5 \times 10^{15}$, we can expect only 5 exact digits. We remark that the order of convergence improves even though k exceeds the maximum value assuring estimate (3.4).

m	$k = 7$	$k = 16$	$k = 32$
32	0.059	0.059797	0.0597973
64	0.059797	0.059797322	0.0597973223
128	0.059797322	0.059797322317	**0.0597973223176919**
256	0.05979732231	**0.0597973223176919**	
512	0.05979732231769		
1024	**0.0597973223176919**		

m	Trapezoidal	m	Simpson
256	0.059	64	0.05979
1024	0.05979	128	0.0597973
2048	0.059797	512	0.059797322
16384	0.05979732	1024	0.0597973223
131072	0.0597973223	4096	0.059797322317
2097152	0.171383967424	8192	0.05979732231769
3600000	**0.0597973223176**	32768	**0.059797322317691**

As can be observed, the number of function's evaluation required w.r.t. Trapezoidal and Simpson rules is drastically reduced. This aspect can justify the high computational cost needed for the construction of $C_{m,k}$ in (3.2).

3.2. Generalized Bézier curves

Finally we want to show some properties of the parametric curves based on $B_{m,\lambda}$ operator and that in some sense generalize the classical Bézier curves. Such a kind of curves were introduced and studied in [15] in the special case $\lambda \in \mathbb{N}$.

The class of Polya curves represent, for instance, a family of polynomial curves which generalizes Bézier and Lagrange curves (see [2],[3], [4]).

Definition 3.4. *Let* $\mathbf{P} = [\mathbf{P}_0, \ldots, \mathbf{P}_m]^T, \mathbf{P}_j \in \mathbb{R}^2$ *be a given control polygon. Curves of parametric equations*

$$\mathbf{B}_{m,\lambda}[\mathbf{P}_0, \ldots, \mathbf{P}_m](t) = \sum_{j=0}^{m} p_{m,j}^{(\lambda)}(t)\mathbf{P}_j, \quad 0 \leq t \leq 1, \lambda \in \mathbb{R}^+, \tag{3.5}$$

with blending functions $p_{m,j}^{(\lambda)}$ *given in* (2.4)*, will be called GB(*λ*) curves. In particular the curve of equation* (3.5) *reduces to Bézier curve for* $\lambda = 1$

$$\mathbf{B}_m[\mathbf{P}_0, \ldots, \mathbf{P}_m](t) = \sum_{j=0}^{m} p_{m,j}(t)\mathbf{P}_j, \quad 0 \leq t \leq 1, \tag{3.6}$$

while, for $\lambda \to \infty$*,* (3.5) *represents the Lagrange curve of the same control polygon* $\mathbf{P}$.

The flexible parameter λ is used in order to model different shapes w.r.t the same control polygon $\mathbf{P}$, obtaining as extreme cases the Bézier curve and the Lagrange interpolating curve. In this sense λ is a "shape parameter".

It is known (see [7]) that relevant geometric properties of parametric curves descend from corresponding properties of the blending functions $\{p_{m,k}^{(\lambda)}\}$. We now collect some properties satisfied by GB(λ) curves.

- **Coordinate system independence**
  GB(λ) curves will not change if the coordinate system is changed, since
  $$\sum_{j=0}^{m} p_{m,j}^{(\lambda)}(x) = 1.$$
  Indeed, this is proved taking into account that the sum of the elements of each row of $C_{m,\lambda}$ is equal to 1.
- **Smoothness**
  GB(λ) are polynomial curves.
- **Endpoint Interpolation** Indeed,
  $$\mathbf{B}_{m,\lambda}[\mathbf{P}_0,\dots,\mathbf{P}_m](0) = \mathbf{P}_0, \quad \mathbf{B}_{m,\lambda}[\mathbf{P}_0,\dots,\mathbf{P}_m](1) = \mathbf{P}_m,$$
  since $B_{m,\lambda}(f;0) = f(0), \quad B_{m,\lambda}(f;1) = f(1)$ [13].
- **Symmetry**
  Curves are symmetric if they do not change under a reverse reordering of the control points sequence, i.e. if and only if
  $$\mathbf{B}_{m,\lambda}[\mathbf{P}_0,\dots,\mathbf{P}_m](t) = \mathbf{B}_{m,\lambda}[\mathbf{P}_m,\dots,\mathbf{P}_0](1-t),$$
  which holds taking into account
  $$p_{m,j}^{(\lambda)}(x) = p_{m,m-j}^{(\lambda)}(1-x), \quad j = 0,\dots,m. \tag{3.7}$$
- **Preservation of points and lines**
  This is equivalent to $\sum_{j=0}^{m} p_{m,j}^{(\lambda)}(x) = 1, \quad \sum_{k=0}^{m} k p_{m,k}^{(\lambda)}(x) = mx$. The first relation is equivalent to the coordinate system independence, while the second holds in view of [13]
  $$B_{m,\lambda}(e_1;t) = e_1(t), \quad e_1(t) = t.$$
- **Nondegeneracy**
  The curve cannot collapse to a single point, and this is implied from the linear independence of the blending functions $\{p_{m,k}^{(\lambda)}\}$.
- **Numerical stability**
  Since GB(λ) are the Bézier curves of the polygon
  $$\mathbf{T}^{\lambda} := \rho(\mathbf{A})[\mathbf{P_0},\dots,\mathbf{P_m}], \tag{3.8}$$
  the rendering algorithm is essentially the de Casteljau recursive scheme applied to the new control polygon $\mathbf{T}^{\lambda}$.

Moreover, GB(λ) curves satisfy all the properties of the Bézier curves w.r.t. the new control polygon $\mathbf{T}^{\lambda}$.

We conclude proposing two graphical examples showed in Figures 1 and 2. Here, for two given control polygons of 5 and 9 vertices, respectively, the curves GB(λ) are rendered for different shape parameter values.

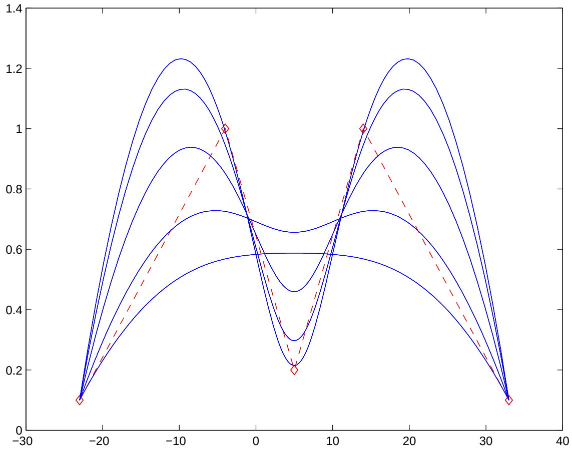

Figure 1. $\lambda = 1, 3, 7.1, 11, 19.9$

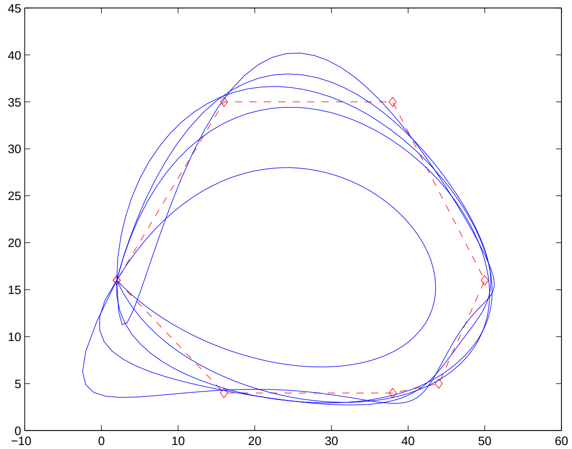

Figure 2. $\lambda = 1, 3, 8.9, 40, 120$

4. The proofs

Proof of Proposition 2.3. It is known that [5]

$$B_m(q_i;x) = \xi_{m,i} q_i(x), \quad m \geq i, \quad q_i \in \mathbb{P}_i. \tag{4.1}$$

i.e., $\xi_{m,i}$ in (2.11) are the eigenvalues of the operator B_m and q_i, $i = 0, \ldots, m$ are the corresponding eigenfunctions. Setting

$$\mathbf{p}_m(x) = [p_{m,0}(x), \ldots, p_{m,m}(x)]^T, \quad \mathbf{q}_m(x) = [q_0(x), \ldots, q_m(x)]^T,$$

$$\gamma_i = [q_i(0), q_i(1/m), \ldots, q_i(1)]^T,$$

$$\Gamma = [\gamma_0, \gamma_1, \ldots, \gamma_m], \quad \Psi = diag[\xi_{m,0}, \xi_{m,1}, \ldots, \xi_{m,m}],$$

(4.1) can be rewritten as

$$\mathbf{p}_m(x)^T\gamma_i = q_i(x)\xi_{m,i}, \quad i = 0, \ldots, m,$$

that is

$$\mathbf{p}_m(x)^T\Gamma = \mathbf{q}_m(x)^T\Psi,$$

and evaluating at $x = t_0, t_1, \ldots, t_m$, it follows

$$A\Gamma = \Gamma\Psi,$$

where A is the matrix in (2.7). Since $\{q_i\}_{i=0}^m$ is a basis for the space $\mathbb{P}_m$, Γ is nonsingular, and by

$$A = \Gamma\Psi\Gamma^{-1}, \tag{4.2}$$

the proposition follows. □

In order to prove Theorem 2.1, we need the following

Theorem 4.1. [10, Th. 3, p.328] *Let B a real matrix of order n and suppose that $z_1, z_2, \ldots, z_s$ are the distinct eigenvalues of B of algebraic multiplicity $n_1, n_2, \ldots, n_s$, respectively. Let the function $f(z)$ have a Taylor series about $z_0 \in \mathbb{R}$*

$$f(z) = \sum_{\nu=0}^{\infty} \alpha_\nu (z - z_0)^\nu$$

with radius of convergence r. Then the function $f(B)$ is defined and is given by

$$f(B) = \sum_{\nu=0}^{\infty} \alpha_\nu (B - z_0 I)^\nu$$

if and only if the distinct eigenvalues of A satisfy one of the following conditions:

1. $|z_k - z_0| < r$;
2. $|z_k - z_0| = r$ *and the series for $f^{(n_k-1)}(z)$ is convergent at the point $z = z_k, \quad 1 \le k \le s$.*

Proof of Theorem 2.1. We recall the following representation given in [15]

$$B_m^i(f; x) = \mathbf{p}_m^T A^{i-1}\mathbf{f}_m. \tag{4.3}$$

Therefore (2.3) becomes

$$B_{m,\lambda}(f; x) = \mathbf{p}_m(x)^T \sum_{i=1}^{\infty} (-1)^{i+1} \binom{\lambda}{i} A^{i-1}\mathbf{f}_m. \tag{4.4}$$

Denoting by $\{\varphi_{m,i} := 1 - \xi_{m,i}\}_{i=0}^m$ the eigenvalues of $I - A$, by Proposition 2.3 it follows $0 \le \varphi_{m,i} < 1, \quad i = 1, 2, \ldots, m$ and

$$\sum_{i=0}^{\infty} (-1)^i \binom{\lambda}{i} A^i = (I - A)^\lambda, \quad \lambda \ge 1. \tag{4.5}$$

The proposition is completely proved combining last relation with (4.4). □

Proof of Theorem 2.5. First we prove

$$B_m^i(f; x) = \mathbf{x}^T M^{i-1}\mathbf{d}_m. \tag{4.6}$$

For $i = 1$ (4.6) holds, in view of (2.20). Assume that (4.6) holds for i. By using (2.20)

$$
\begin{aligned}
B_m^{i+1}(f;x) = B_m(B_m^i(f);x) &= \sum_{l=0}^{m} B_m(e_l;x) \sum_{k=0}^{m} M_{l,k}^{i-1} \binom{m}{k} \Delta_{\frac{1}{m}}^{k} f(0) \\
&= \sum_{l=0}^{m} \sum_{j=0}^{m} x^j \binom{m}{j} \Delta_{\frac{1}{m}}^{j} e_l(0) \sum_{k=0}^{m} M_{l,k}^{i-1} \binom{m}{k} \Delta_{\frac{1}{m}}^{k} f(0) \\
&= \sum_{j=0}^{m} x^j \sum_{k=0}^{m} \binom{m}{k} \Delta_{\frac{1}{m}}^{k} f(0) \sum_{l=0}^{m} M_{i,l} M_{l,k}^{i-1} = \mathbf{x}^T M^i \mathbf{d}_m.
\end{aligned}
$$

By induction (4.6) is true for every i. Following the same arguments used in the proof of Theorem 2.1, under the assumption $\lambda \geq 1$, we get

$$
B_{m,\lambda}(f,x) = \mathbf{x}^T \sum_{i=1}^{\infty} (-1)^{i+1} \binom{\lambda}{i} M^{i-1} \mathbf{d}_m = \mathbf{x}^T M^{-1} [I - (I-M)^{\lambda}] \mathbf{d}_m. \quad \square
$$

Proof of Theorem 3.1. In order to prove (3.3), we start from

$$
\sum_{j=0}^{m} |D_j^{(k)}| = \frac{1}{m+1} \sum_{j=0}^{m} \left| \sum_{i=0}^{m} (C_{m,k})_{i,j} \right| \leq \max_{0 \leq i \leq m} \sum_{j=0}^{m} |(C_{m,k})_{i,j}| = \|C_{m,k}\|_\infty
$$

and by (2.10),

$$
\begin{aligned}
\sum_{i=0}^{m} |D_j^{(k)}| &\leq \|C_{m,k}\|_\infty \leq \|I\|_\infty + \|I-A\|_\infty + \|I-A\|_\infty^2 + \cdots + \|I-A\|_\infty^{k-1} \\
&\leq 1 + 2 + \cdots + 2^{k-1} = 2^k - 1, \text{ since } \|A\|_\infty = 1.
\end{aligned}
$$

To prove (3.4), we use [12]

$$
\|f - B_{m,k}(f)\|_\infty \leq m^{-k} \sum_{\nu=0}^{2k} b_\nu \|f^{(\nu)}\|_\infty
$$

(see also [17]) where b_ν are positive constants independent of f. Therefore, since [6, p.310, Lemma 2.1]

$$
\sum_{\nu=0}^{2k} b_\nu \|f^{(\nu)}\|_\infty \leq \mathcal{C}(\|f\|_\infty + \|f^{(2k)}\|_\infty),
$$

(3.4) follows. $\square$

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Donatella Occorsio
Dipartimento di Matematica ed Informatica
Università degli Studi della Basilicata
Via dell'Ateneo Lucano 10, 85100 Potenza, Italy
e-mail: `donatella.occorsio@unibas.it`