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Uniform weighted approximation by positive linear operators

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Abstract. We characterize the functions defined on a weighted space, which are uniformly approximated by a sequence of positive linear operators and we obtain the range of the weights which can be used for uniform approximation. We, also, obtain an estimation of the remainder in terms of the usual modulus of continuity. We give particular results for the Szász-Mirakjan and Baskakov operators.

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1. Introduction

Let $I \subseteq \mathbb{R}$ be a noncompact interval and let $\rho\colon I \to [1,\infty)$ be an increasing and differentiable function called weight. Let $B_\rho(I)$ be the space of all functions $f\colon I \to \mathbb{R}$ such that $|f(x)| \leq M \cdot \rho(x)$, for every $x \in I$, where $M > 0$ is a constant depending on f and ρ, but independent of x. The space $B_\rho(I)$ is called weighted space and it is a Banach space endowed with the ρ-norm

$$\|f\|_\rho = \sup_{x\in I} \frac{|f(x)|}{\rho(x)}.$$

Let $C_\rho(I) = C(I) \cap B_\rho(I)$ be the subspace of $B_\rho(I)$ containing continuous functions.

Let $(A_n)_{n\geq 1}$ be a sequence of positive linear operators defined on the weighted space $C_\rho(I)$. It is known (see [13]) that A_n maps $C_\rho(I)$ onto $B_\rho(I)$ if and only if $A_n\rho \in B_\rho(I)$.

In the paper [7], the authors present some ideas related to the approximation of functions in weighted spaces and enounced some unsolved problems in weighted approximation theory. Three such problemms are:

1. Let $\mathcal{F}$ be a linear subspace of $\mathbb{R}^I$ and $A_n\colon \mathcal{F} \to C(I)$ a sequence of positive linear operators. For which weights ρ, does A_n map $C_\rho(I) \cap \mathcal{F}$ onto $C_\rho(I)$

with uniformly bounded norms?
2. For which functions $f \in C_\rho(I)$ do we have $\|A_n - f\|_\rho \to 0$, as $n \to \infty$?
3. Which moduli of smoothness are appropriate for weighted approximation?

Some ideas to solve these problems and some partial solutions were given in the article already mentioned. In this paper, we give some answers to these three problems. For a given sequence of positive linear operators, A_n, we characterize those functions f belonging to $C_\rho(I)$ such that $\|A_n - f\|_\rho \to 0$ and obtain all the weights ρ for which this uniform convergence in the ρ-norm is true. We, also, obtain an estimation of the remainder $A_n f - f$, in terms of the modulus of continuity of the function f. As applications, we give some results related to the Szász-Mirakjan and Baskakov operators.

We will use the following modulus of continuity

$$\omega_\varphi(f,\delta) = \sup_{\substack{t,x\in I\\ |\varphi(t)-\varphi(x)|\le\delta}} |f(t)-f(x)|,$$

for all $f \in B(I)$, where $\varphi\colon I \to J$, $(J \subset \mathbb{R})$, is a differentiable bijective function, with $\varphi'(x) > 0$, for all $x \in I$. This modulus is a particular case of the general modulus

$$\omega_d(f,\delta) = \sup\left\{\, |f(t)-f(x)| \,:\, t,x \in X,\ d(t,x) \le \delta \,\right\},$$

where f is a bounded function defined on X and (X,d) is a compact metric space. For details related to this general modulus of continuity see [8], [15] and [20]. The particular modulus $\omega_\varphi(f,\delta)$ is obtained for the metric $d(t,x) = |\varphi(t)-\varphi(x)|$ and has the following properties (see [17], for example)

Proposition 1.1. *Let $f \in B(I)$ and $\delta > 0$.*

(i) $\omega_\varphi(f,\delta) = \omega(f\circ\varphi^{-1},\delta)$, where ω is the usual modulus of continuity.

(ii) Let $(\delta_n)_{n\ge 1}$ be sequence of positive real numbers converging to 0. Then $f\circ\varphi^{-1}$ is uniformly continuous on J if and only if $\omega_\varphi(f,\delta_n) \to 0$.

(iii) $|f(t)-f(x)| \le \left(1+\dfrac{|\varphi(t)-\varphi(x)|}{\delta}\right)\omega_\varphi(f,\delta)$, for every $t,x\in I$.

2. Main result

Theorem 2.1. *Let $A_n : C_\rho(I) \to B_\rho(I)$ be positive linear operators reproducing constant functions and satisfying the conditions*

$$\sup_{x\in I} A_n(|\varphi(t)-\varphi(x)|,x) = a_n \to 0, \quad (n\to\infty) \tag{2.1}$$

$$\sup_{x\in I} \frac{A_n(|\rho(t)-\rho(x)|,x)}{\rho(x)} = b_n \to 0. \quad (n\to\infty) \tag{2.2}$$

If $A_n(f,x)$ is continuously differentiable and there is a constant $K(f,\rho,n)$ such that

$$\frac{|(A_n f)'(x)|}{\varphi'(x)} \le K(f,\rho,n)\cdot\rho(x), \quad \textit{for every } x\in I, \tag{2.3}$$

and ρ and φ are such that there exists a constant $\alpha > 0$

$$\frac{\rho'(x)}{\varphi'(x)} \leq \alpha \cdot \rho(x), \quad \textit{for every } x \in I, \tag{2.4}$$

then, the following statements are equivalent

(*i*) $\|A_n f - f\|_\rho \to 0$ *as* $n \to \infty$.

(*ii*) $\dfrac{f}{\rho} \circ \varphi^{-1}$ *is uniformly continuous on* J.

Furthermore, we have

$$\|A_n f - f\|_\rho \leq b_n \cdot \|f\|_\rho + 2 \cdot \omega_\varphi\left(\frac{f}{\rho}, a_n\right), \quad \textit{for every } n \geq 1. \tag{2.5}$$

Proof. Let us prove that (ii) implies (i). Using the inequality (ii) of Proposition 1.1, we obtain for a function $f \in C_\rho(I)$

$$\begin{aligned}|f(t) - f(x)| &\leq \frac{|f(t)|}{\rho(t)} \cdot |\rho(t) - \rho(x)| + \rho(x) \cdot \left|\frac{f(t)}{\rho(t)} - \frac{f(x)}{\rho(x)}\right| \\ &\leq \|f\|_\rho \cdot |\rho(t) - \rho(x)| + \rho(x) \cdot \left(1 + \frac{|\varphi(t) - \varphi(x)|}{\delta}\right) \omega_\varphi\left(\frac{f}{\rho}, \delta\right).\end{aligned}$$

Applying the positive linear operators A_n to the last inequality, we obtain

$$\begin{aligned}\frac{|A_n(f,x) - f(x)|}{\rho(x)} \leq \|f\|_\rho &\cdot \frac{A_n(|\rho(t) - \rho(x)|, x)}{\rho(x)} \\ &+ \left(1 + \frac{A_n(|\varphi(t) - \varphi(x)|, x)}{\delta_n}\right) \omega_\varphi\left(\frac{f}{\rho}, \delta_n\right),\end{aligned}$$

which proves the relation (2.5). Because $a_n \to 0$ and $\frac{f}{\rho} \circ \varphi^{-1}$ is uniformly continuous on J we deduce that $\omega_\varphi\left(\frac{f}{\rho}, a_n\right) \to 0$ as $n \to \infty$. Because $b_n \to 0$ we obtain that $\|A_n f - f\|_\rho \to 0$.

Now, let us prove that (i) implies (ii). Because of the relation

$$\begin{aligned}\omega_\varphi\left(\frac{f}{\rho}, \delta_n\right) &\leq \omega_\varphi\left(\frac{f - A_n f}{\rho}, \delta_n\right) + \omega_\varphi\left(\frac{A_n f}{\rho}, \delta_n\right) \\ &\leq \|f - A_n f\|_\rho + \omega_\varphi\left(\frac{A_n f}{\rho}, \delta_n\right)\end{aligned}$$

it remains to prove that $\omega_\varphi\left(\frac{A_n f}{\rho}, \delta_n\right) \to 0$.

Applying the Cauchy mean value theorem, there is c between $x \in I$ and $t \in I$, such that

$$\varphi'(c)\left[\frac{A_n(f,t)}{\rho(t)} - \frac{A_n(f,x)}{\rho(x)}\right] = \left(\frac{A_n f}{\rho}\right)'(c) \cdot [\varphi(t) - \varphi(x)].$$

We have

$$\omega_\varphi\left(\frac{A_n f}{\rho}, \delta_n\right) = \sup_{\substack{t,x \in I \\ |\varphi(t) - \varphi(x)| \leq \delta_n}} \left|\frac{A_n(f,t)}{\rho(t)} - \frac{A_n(f,x)}{\rho(x)}\right| \leq \left\|\frac{1}{\varphi'} \cdot \left(\frac{A_n f}{\rho}\right)'\right\| \cdot \delta_n,$$

which implies $\omega_\varphi\left(\frac{A_nf}{\rho},\delta_n\right)\to 0$, for a suitable choice of the sequence $\delta_n\to 0$, if

$$\left\|\frac{1}{\varphi'}\cdot\left(\frac{A_nf}{\rho}\right)'\right\|=\sup_{x\in I}\left|\frac{1}{\varphi'(x)}\cdot\left(\frac{A_nf}{\rho}\right)'(x)\right|<\infty.$$

But, for every $f\in C_\rho(I)$ and for every $n\geq 1$

$$\begin{aligned}\left\|\frac{1}{\varphi'}\cdot\left(\frac{A_nf}{\rho}\right)'\right\| &= \left\|\frac{1}{\varphi'}\cdot\left(\frac{(A_nf)'}{\rho}-\frac{A_nf\cdot\rho'}{\rho^2}\right)\right\| \\ &\leq \left\|\frac{(A_nf)'}{\rho\cdot\varphi'}\right\|+\|f\|_\rho\cdot\|A_n\rho\|_\rho\cdot\left\|\frac{\rho'}{\rho\cdot\varphi'}\right\|<\infty,\end{aligned}$$

because of the relations (2.3) and (2.4). So, the theorem is proved. □

Remark 2.2. For $\rho(x)=1$, the result of Theorem 2.1 was obtained by Totik [23], by de la Cal and Cárcamo [10] and by myself [18].

Remark 2.3. The function φ is close connected with the given sequence of positive linear operators. It can be obtained in the following manner: we choose the function θ such that

$$\theta'(x)\sqrt{A_n((t-x)^2,x)}\leq K_n,$$

where K_n is a constant not depending on x, and such that θ verifies the conditions (2.3) and (2.4). Then, by the argument of the implication $(i)\Rightarrow(ii)$ from the Theorem 2.1, we obtain that $\frac{f}{\rho}\circ\theta^{-1}$ is uniformly continuous. But, in most of the cases, θ^{-1} has a complicate form and the relation (2.1) is difficult to prove. To overcome this, we consider φ such that $\theta\circ\varphi^{-1}$ is uniformly continuous. So, we get that $\frac{f}{\rho}\circ\varphi^{-1}$ is a uniformly continuous function.

Remark 2.4. The relation (2.4) gives us the connection between the function φ and the weight ρ. We must have

$$\rho(x)\leq Me^{\alpha\cdot\varphi(x)},\qquad\text{for every } x\in I, \tag{2.6}$$

where $M,\alpha>0$ are constants independent of x. So, we have obtained the range of the weights ρ, for which Theorem 2.1 is valid.

Remark 2.5. The maximal class of weights is $\rho(x)=e^{\alpha\varphi(x)}$. In order to prove the result of the Theorem for this weight, we prove first the inequality $A_n(\rho,x)\leq C_\alpha\rho(x)$, for every $x\in I$ and for every $\alpha>0$, where $C_\alpha>0$ is a constant independent of x. Then, we prove the relation

$$\lim_{n\to\infty}\sup_{x\in I}A_n(|\varphi(t)-\varphi(x)|^2,x)=0. \tag{2.7}$$

Using the Cauchy-Schwarz inequality for positive linear operators we get that the sequence

$$a_n=\sup_{x\in I}A_n(|\varphi(t)-\varphi(x)|,x)\leq\sup_{x\in I}\sqrt{A_n(|\varphi(t)-\varphi(x)|^2,x)}$$

is convergent to 0. Using the Geometric-Logarithmic-Arithmetic Mean Inequality (see [19, p. 40])

$$\sqrt{u \cdot v} \leq \frac{u-v}{\ln u - \ln v} < \frac{u+v}{2}, \quad 0 < v < u, \tag{2.8}$$

we obtain

$$\left| e^{\alpha\varphi(t)} - e^{\alpha\varphi(x)} \right| \leq \frac{e^{\alpha\varphi(t)} + e^{\alpha\varphi(x)}}{2} \cdot \alpha \left| \varphi(t) - \varphi(x) \right|, \quad t, x \in I,$$

and

$$\begin{aligned} b_n &= \sup_{x \in I} \frac{A_n(|\rho(t) - \rho(x)|, x)}{\rho(x)} \\ &\leq \sup_{x \in I} \frac{\alpha}{2} \frac{\sqrt{A_n((\rho(t)+\rho(x))^2, x)}}{\rho(x)} \left(A_n(|\varphi(t) - \varphi(x)|^2, x) \right)^{\frac{1}{2}} \\ &\leq \sup_{x \in I} \frac{\alpha}{2} \left(\frac{S_n(\rho^2(t), x)}{\rho^2(x)} + 2\frac{S_n(\rho, x)}{\rho(x)} + 1 \right)^{\frac{1}{2}} \left(A_n(|\varphi(t) - \varphi(x)|^2, x) \right)^{\frac{1}{2}} \\ &\leq \frac{\alpha}{2} \sqrt{C_{2\alpha} + 2C_\alpha + 1} \cdot \sup_{x \in I} \left(A_n(|\varphi(t) - \varphi(x)|^2, x) \right)^{\frac{1}{2}}. \end{aligned}$$

If (2.7) is true, then $(b_n)_{n \in \mathbb{N}}$ converges to 0. To obtain the result of the Theorem 2.1 it remains to prove (2.3).

3. Applications

Lemma 3.1. *For every $\alpha > 0$ and $\rho(x) = e^{\alpha\sqrt{x}}$, the Szász-Mirakjan operators defined by*

$$S_n f(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad x \in [0, \infty),$$

map $C_\rho[0,\infty)$ onto $C_\rho[0,\infty)$.

Proof. Let us notice that $S_n(\rho, x)$ exists for every $x \geq 0$. This is true because

$$S_n(\rho, x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} e^{\alpha \frac{\sqrt{k}}{\sqrt{n}}} \leq e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} e^{\alpha \frac{k}{\sqrt{n}}} = e^{nxe^{\frac{\alpha}{\sqrt{n}}} - nx}.$$

Because $S_n f$ converges uniformly to f on $[0,1]$ (see [1], for example), we have $S_n(\rho, x) \leq C_{1,\alpha} \cdot \rho(x)$, for every $x \in [0,1]$. Let us prove that $S_n(\rho, x) \leq C_{2,\alpha} \cdot \rho(x)$, for every $x \geq 1$.

$$\begin{aligned} \frac{S_n(e^{\alpha\sqrt{t}}, x)}{e^{\alpha\sqrt{x}}} &= e^{-nx} \sum_{k > nx} \frac{(nx)^k}{k!} e^{\alpha\left(\sqrt{\frac{k}{n}} - \sqrt{x}\right)} + e^{-nx} \sum_{k \leq nx} \frac{(nx)^k}{k!} \frac{e^{\alpha\sqrt{\frac{k}{n}}}}{e^{\alpha\sqrt{x}}} \\ &\leq e^{-nx} \sum_{k > nx} \frac{(nx)^k}{k!} e^{\frac{\alpha}{\sqrt{x}}\left(\frac{k}{n} - x\right)} + e^{-nx} \sum_{k \leq nx} \frac{(nx)^k}{k!} \\ &\leq e^{nx\left(e^{\frac{\alpha}{n\sqrt{x}}} - 1\right) - \alpha\frac{x}{\sqrt{x}}} + 1. \end{aligned}$$

Using the inequality $e^t - 1 \le te^t$, we obtain for every $x \ge 1$ and every $n \ge 1$ that

$$\frac{S_n(e^{\alpha\sqrt{t}}, x)}{e^{\alpha\sqrt{x}}} \le e^{nx\frac{\alpha}{n\sqrt{x}}e^{\frac{\alpha}{n\sqrt{x}}} - \frac{\alpha x}{\sqrt{x}}} + 1 \le e^{\alpha\sqrt{x}\frac{\alpha}{n\sqrt{x}}e^{\frac{\alpha}{n\sqrt{x}}}} + 1 \le e^{\alpha^2 e^{\alpha}} + 1.$$

We have proved that

$$\|S_n\rho\|_\rho = \sup_{x\ge 0} \frac{S_n(\rho, x)}{\rho(x)} \le C_\alpha, \tag{3.1}$$

where $C_\alpha > 0$ is a constant dependent of α, but independent of n. □

Corollary 3.2. *For a number $\alpha > 0$ and $\rho(x) = e^{\alpha\sqrt{x}}$, the Szász-Mirakjan operators $S_n\colon C_\rho[0,\infty) \to C_\rho[0,\infty)$ have the property that*

$$\|S_n f - f\|_\rho \to 0, \text{ as } n \to \infty$$

if and only if, the function

$$f(x^2)e^{-\alpha x} \text{ is uniformly continuous on } [0,\infty).$$

Moreover, for $f \in C_\rho[0,\infty)$ we have

$$\|S_n f - f\|_\rho \le \|f\|_\rho \cdot \frac{\alpha C}{\sqrt{n}} + 2\cdot\omega\left(f(t^2)e^{-\alpha t}, \frac{1}{\sqrt{n}}\right), \quad \text{for every } n \ge 1,$$

where $C = \sup_{n\in\mathbb{N}} \frac{1}{2}\sqrt{\|S_n\rho^2\|_{\rho^2} + 2\|S_n\rho\|_\rho + 1}$ is a constant depending only on α.

Proof. We set $\varphi(x) = \sqrt{x}$. The function $\rho(x) = e^{\alpha\sqrt{x}}$ verifies the relation (2.4) with equality.

We have the relations $S_n(1,x) = 1$ and $S_n((t-x)^2, x) = x/n$ (see [1], for example). We prove now the relation (2.7).

$$\begin{aligned}\sup_{x\ge 0} S_n(|\varphi(t)-\varphi(x)|^2, x) &= \sup_{x\ge 0} S_n\left(\frac{|t-x|^2}{\left(\sqrt{t}+\sqrt{x}\right)^2}, x\right)\\ &\le \sup_{x\ge 0}\frac{S_n(|t-x|^2, x)}{x} = \frac{1}{n}.\end{aligned}$$

For a function $f \in C_\rho(I)$ the derivative $(S_n f)'(x)$ fulfills:

$$\begin{aligned}|(S_n f)'(x)| &= \left|\frac{n}{x}\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\left(\frac{k}{n} - x\right) e^{-nx}\frac{(nx)^k}{k!}\right|\\ &\le \frac{n}{x}\|f\|_\rho S_n(\rho(t)|t-x|, x) \le \sqrt{C_{2\alpha}}\,\|f\|_\rho\,\rho(x)\frac{\sqrt{n}}{\sqrt{x}},\end{aligned}$$

because

$$S_n(\rho(t)|t-x|, x) \le \sqrt{S_n(\rho^2(t), x)}\cdot\sqrt{S_n((t-x)^2, x)} \le \sqrt{C_{2\alpha}}\rho(x)\sqrt{\frac{x}{n}}.$$

We obtain

$$\frac{|(S_n f)'(x)|}{\varphi'(x)} \le C_{f,n,\alpha}\cdot\rho(x), \quad \text{for every } x \ge 0,$$

so, the relation (2.3) is proved. □

Remark 3.3. The result from Corollary 3.2 for the limit case, α=0, was obtained in [21], [23], [10] and [18].

Remark 3.4. In [16], it was proved that $S_n(f,x)$ exists for every function f with the property $f(x)=\mathcal{O}(e^{\alpha x\ln x})$, $\alpha>0$ and moreover, S_nf converges uniformly to f on compact subsets of the interval $[0,\infty)$. In [5], Becker studied the global approximation of functions using Szász-Mirakjan operators for the polynomial weight $\rho(x)=1+x^N$, $N\in\mathbb{N}$. Becker, Kucharsky and Nessel [6] studied the global approximation for the exponential weight $\rho(x)=e^{\beta x}$. But because

$$\sup_{x\geq 0}\frac{S_n(e^{\beta t},x)}{e^{\beta x}}=\sup_{x\geq 0}e^{nx(e^{\frac{\beta}{n}}-1)-\beta x}=+\infty,$$

they obtain results only for the space $C(\eta)=\cap_{\beta>\eta}C_\beta$, where C_β is C_ρ for $\rho=e^{\beta x}$. It is also mentioned, that for any $f\in C_\beta$ we have $S_nf\in C_\gamma$, for $\gamma>\beta$ and for $n>\beta/\ln(\gamma/\beta)$. Ditzian [11], also, gives some inverse theorems for exponential spaces. In [2], Amanov obtained that the condition

$$\sup_{x\geq 0}\frac{\rho(x+\sqrt{x})}{\rho(x)}<\infty$$

upon the weight ρ, is necessary and sufficient for the uniform boundedness of the norms of the operators $S_n\colon C_\rho[0,\infty)\to C_\rho[0,\infty)$. He mentions that this condition implies the inequality

$$\rho(x)\leq e^{\alpha\sqrt{1+x}},\quad x\geq 0.$$

He, also, gives a characterization of the functions f which are uniformly approximated by S_nf in the ρ-norm, using a weighted second order modulus of smoothness.

The fact that $\rho(x)=\mathcal{O}(e^{\alpha\sqrt{x}})$ is the maximal class of weights for which S_n maps $C_\rho[0,\infty)$ into $C_\rho[0,\infty)$ can be proved by the following argument: we take $\rho(x)=e^{\alpha\Phi(x)}$, $\alpha>0$, where $\Phi(x)$ is a strictly increasing differentiable function with the properties that

$$\lim_{x\to\infty}\Phi(x)=\infty\quad\text{and}\quad\lim_{x\to\infty}\Phi'(x)\sqrt{x}=\infty,\tag{3.2}$$

and we prove that $\|S_n\rho\|_\rho$ is not finite for all $\alpha>0$. From condition (3.2), by l'Hospital's rule, $\lim_{x\to\infty}\Phi(x)/\sqrt{x}=\infty$, so, there are $M>1/\alpha$ and $x_0\geq 1$ such that $\Phi(x)>M\sqrt{x}$, for $x\geq x_0$. We obtain for $x\geq x_0$

$$\left(e^{\alpha\Phi(x)}\right)''=e^{\alpha\Phi(x)}\left[\alpha\Phi''(x)+[\alpha\Phi'(x)]^2\right]>e^{\alpha\Phi(x)}\left(-\frac{\alpha M}{4x^{\frac{3}{2}}}+\frac{(\alpha M)^2}{4x}\right)>0,$$

so, $e^{\alpha\Phi}$ is convex on $[x_0,\infty)$. We can redefine Φ (if it is necessary), such that $e^{\alpha\Phi}$ is a convex function on $[0,\infty)$. By a result of Cheney and Sharma [9], we deduce that $S_n(\rho,x)\geq\rho(x)$. Suppose that

$$\lim_{x\to\infty}\frac{S_n(\rho,x)}{\rho(x)}=L_\alpha<\infty,\quad\text{for every }\alpha>0.\tag{3.3}$$

Because $S_n\rho \geq \rho$, we obtain $L_\alpha \geq 1$. But, using l'Hospital's rule, we have

$$\begin{aligned} L_\alpha = \lim_{x\to\infty} \frac{S_n(e^{\alpha\Phi(t)},x)}{e^{\alpha\Phi(x)}} &= \lim_{x\to\infty} \frac{(S_n e^{\alpha\Phi})'(x)}{\alpha\Phi'(x)e^{\alpha\Phi(x)}} \\ &\leq \lim_{x\to\infty} \frac{\frac{n}{x}\sqrt{S_n((t-x)^2,x)}}{\alpha\Phi'(x)} \cdot \frac{\sqrt{S_n(e^{2\alpha\Phi(t)},x)}}{e^{\alpha\Phi(x)}} \\ &\leq \lim_{x\to\infty} \frac{1}{\alpha\sqrt{n}\Phi'(x)\sqrt{x}} \cdot \sqrt{\frac{S_n(e^{2\alpha\Phi(t)},x)}{e^{2\alpha\Phi(x)}}} \\ &= 0 \cdot \sqrt{L_{2\alpha}} = 0, \end{aligned}$$

which is a contradiction with $L_\alpha \geq 1$.

Lemma 3.5. *For every $\alpha > 0$ and $\rho(x) = (1+x)^\alpha$, the Baskakov operators defined by*

$$V_n f(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right), \quad x \geq 0,$$

map $C_\rho[0,\infty)$ onto $C_\rho[0,\infty)$.

Proof. In [5], Becker proves that $V_n(1+t^N,x) \leq C_1(1+x^N)$, for every $x \geq 0$ and every $N \in \mathbb{N}$. We deduce that

$$V_n((1+t)^m,x) \leq C_2 \cdot V_n(1+t^m,x) \leq C_3(1+x^m) \leq C_3(1+x)^m,$$

for every $x \geq 0$ and every $m \in \mathbb{N}$. We prove, now, that for $\beta \in [0,1)$ we have $V_n((1+t)^\beta,x) \leq C_4(1+x)^\beta$. Using the inequality $\ln(1+t) \leq t$, for $t > -1$, we obtain

$$\begin{aligned} \frac{V_n((1+t)^\beta,x)}{(1+x)^\beta} &= \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} e^{\beta\left[\ln\left(1+\frac{k}{n}\right)-\ln(1+x)\right]} \\ &= \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} e^{\beta\ln\left(1+\frac{\frac{k}{n}-x}{1+x}\right)} \\ &\leq \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} e^{\beta\frac{\frac{k}{n}-x}{1+x}} \\ &= \left(1-x\left(e^{\frac{\beta}{n(1+x)}}-1\right)\right)^{-n} \cdot e^{\frac{-\beta x}{1+x}}. \end{aligned}$$

The last expression is well-defined for every $x \geq 0$ and every $n \geq 1$, because

$$1-x\left(e^{\frac{\beta}{n(1+x)}}-1\right) \geq 1-\lim_{x\to\infty} x\left(e^{\frac{\beta}{n(1+x)}}-1\right) = 1-\frac{\beta}{n} > 0.$$

Because of the inequality

$$\sup_{x\geq 0}\left[\left(1-x\left(e^{\frac{\beta}{n(1+x)}}-1\right)\right)^{-n} \cdot e^{\frac{-\beta x}{1+x}}\right] \leq \left(1-\frac{\beta}{n}\right)^{-n} \cdot 1 \leq \frac{1}{1-\beta},$$

we deduce that $V_n((1+t)^\beta,x) \leq C_4(1+x)^\beta$, for every $x \geq 0$, where C_4 is a constant not depending on x and n.

For $\alpha > 0$, we choose $m = [2\alpha] \in \mathbb{N}$ and $\beta = 2\alpha - m \in [0, 1)$. Using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} V_n((1+t)^\alpha, x) &= V_n((1+t)^{\frac{m}{2}} \cdot (1+t)^{\frac{\beta}{2}}, x) \\ &\leq \sqrt{V_n((1+t)^m, x) \cdot V_n((1+t)^\beta, x)} \\ &\leq \sqrt{C_3(1+x)^m \cdot C_4(1+x)^\beta} = C_5(1+x)^\alpha, \end{aligned}$$

which proves that $V_n\rho \in C_\rho[0, \infty)$. □

Corollary 3.6. *For a real number $\alpha > 0$ and for $\rho(x) = (1+x)^\alpha$ the Baskakov operators $V_n \colon C_\rho[0, \infty) \to C_\rho[0, \infty)$ have the property that*

$$\|V_n f - f\|_\rho \to 0, \ \text{as } n \to \infty$$

if and only if

$$f(e^x - 1)e^{-\alpha x}, \ \text{is uniformly continuous on } [0, \infty).$$

Moreover, for $f \in C_\rho[0, \infty)$ and for $n \geq 2$, we have

$$\|V_n f - f\|_\rho \leq \|f\|_\rho \cdot \frac{\alpha C}{\sqrt{n-1}} + 2 \cdot \omega \left(f(e^t - 1)e^{-\alpha t}, \frac{1}{\sqrt{n-1}} \right).$$

Proof. Setting $\varphi(x) = \ln(1+x)$, the function $\rho(x) = (1+x)^\alpha$ verifies the relation (2.4) with equality.

We have the relations $V_n(1, x) = 1$, $V_n(t, x) = x$ and $V_n((t-x)^2, x) = x(1+x)/n$ (see [1], for example). We prove now the relation (2.7). Using the inequality (2.8) we have

$$\begin{aligned} |\varphi(t) - \varphi(x)|^2 &= |\ln(1+t) - \ln(1+x)|^2 \\ &\leq \frac{|t-x|^2}{(1+t)(1+x)} = \left| \sqrt{\frac{1+t}{1+x}} - \sqrt{\frac{1+x}{1+t}} \right|^2 \end{aligned}$$

and using the fact that

$$\begin{aligned} V_n\left(\frac{1}{1+t}, x\right) &= \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \cdot \frac{n}{n+k} \\ &\leq \frac{n}{(n-1)(1+x)} \sum_{k=0}^{\infty} \binom{n+k-2}{k} \frac{x^k}{(1+x)^{n-1+k}} \\ &= \frac{n}{(n-1)(1+x)} \end{aligned}$$

we obtain

$$\begin{aligned} V_n(|\varphi(x) - \varphi(t)|^2, x) &\leq \frac{V_n(1+t, x)}{1+x} - 2V_n(1, x) + (1+x)V_n\left(\frac{1}{1+t}, x\right) \\ &\leq 1 - 2 + \frac{n}{n-1} = \frac{1}{n-1}, \quad \text{for } \ n \geq 2. \end{aligned}$$

The derivative $(V_n f)'(x)$ verifies the relation

$$|(V_nf)'(x)| = \left| \frac{n}{x(1+x)} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\left(x-\frac{k}{n}\right)\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}} \right|$$

$$\leq \frac{n}{x(1+x)} \|f\|_\rho \cdot V_n(\rho(t)|t-x|, x) \leq C_1 \|f\|_\rho \rho(x) \frac{\sqrt{n}}{\sqrt{x(1+x)}},$$

because

$$V_n(\rho(t)|t-x|, x) \leq \sqrt{V_n(\rho^2(t), x)} \cdot \sqrt{V_n((t-x)^2, x)} \leq C_1\rho(x)\sqrt{\frac{x(x+1)}{n}}.$$

We obtain

$$\frac{|(V_nf)'(x)|}{\theta'(x)} \leq C_2\rho(x), \quad \text{for every } x \geq 0,$$

where $\theta(x) = \ln\left(x + \frac{1}{2} + \sqrt{x(1+x)}\right)$. The inequality

$$\frac{\rho'(x)}{\theta'(x)} = \alpha(1+x)^{\alpha-1}\sqrt{x(1+x)} \leq \alpha \cdot \rho(x), \quad x \geq 0,$$

proves the relation (2.4) for the function θ instead of φ. Using the fact that

$$(\theta \circ \varphi^{-1})(x) = \ln\left(e^x - \frac{1}{2} + \sqrt{(e^x-1)e^x}\right)$$

is a uniformly continuous function on $[0, \infty)$ (this is true, because it is a continuous function with the property that $(\theta \circ \varphi^{-1})(x) - x$ has finite limit at infinity) and using Theorem 2.1, the Remarks 2.3 and 2.5, the proof of the corollary is complete. □

Remark 3.7. The result of the Corollary 3.6 for the limit case, $\alpha = 0$, was obtained in [22], [23], [10] and [18].

Remark 3.8. Becker [5] studied the global approximation of functions from the polynomial weighted space and remarked that "polynomial growth is the frame best suited for global results for the Baskakov operators". The reason is that for the exponential weight $\rho(x) = e^{\beta x}$, the series $V_n(\rho, x)$ exists only for $x < (e^{\frac{\beta}{n}} - 1)^{-1}$. Nevertheless, Ditzian [11] gave some inverse results for functions with exponential growth.

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