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On a class of pseudo-parallel submanifolds in Kenmotsu space forms

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Abstract. In this article we prove that pseudo-parallel normal anti-invariant submanifolds in Kenmotsu space forms are always semi-parallel.

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1. Introduction

In 2008, [2], F. Dillen, J. Van der Veken and L. Vrancken proved that Lagrange pseudo-parallel submanifolds of complex space forms are always semi-parallel.

In this paper we prove that a n-dimensional pseudo-parallel and normal anti-invariant submanifold M in a $(2n+1)$-dimensional Kenmotsu space form $\widetilde{M}(c)$ is always semi-parallel. We also prove that this is not generally true for pseudo-parallel Legendre submanifolds in Sasaki space forms.

Now, we remember some necessary useful notions and results for our next considerations.
Let $\widetilde{M}$ be a C^∞–differentiable, $(2n+1)$–dimensional almost contact manifold with the almost contact metric structure (F, ξ, η, g), where F is a $(1,1)$ tensor field, η is a 1-form, g is a Riemannian metric on $\widetilde{M}$, ξ is the Reeb vector field, all these tensors satisfying the following conditions :

$$F^2 = -I + \eta \otimes \xi; \quad \eta(\xi) = 1; \quad g(FX, FY) = g(X, Y) - \eta(X)\eta(Y) \quad (1.1)$$

for all X, Y in $\chi(\widetilde{M})$.

Let M be a submanifold of $\widetilde{M}$. We consider ∇ the Levi-Civita connection induced by $\widetilde{\nabla}$ on M, $\nabla^\perp$ the connection in the normal bundle $T^\perp(M)$, h the second fundamental form on M and $A_{\vec{n}}$ the Weingarten operator. The well-known Gauss–Weingarten formulas on M are:

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y); \qquad \widetilde{\nabla}_X \vec{n} = -A_{\vec{n}} X + \nabla^{\perp}_X \vec{n} \tag{1.2}$$

for X, Y in $\chi(M)$ and $\vec{n}$ in $\chi^{\perp}(M)$.

We consider the Sasaki form Ω on $\widetilde{M}$, given by $\Omega(X,Y) = g(X,FY)$. Also, denote by N_F the Nijenhius tensor of F. It is known that $\widetilde{M}$ is a Sasaki manifold if and only if

$$d\eta = \Omega; \qquad N^{(1)} = N_F + 2d\eta \otimes \xi = 0$$

or equivalently

$$(\widetilde{\nabla}_X F)Y = g(X,Y)\xi - \eta(Y)X. \tag{1.3}$$

An almost normal contact manifold $\widetilde{M}$ is a Kenmotsu manifold if and only if

$$d\eta = 0; \quad d\Omega = 2\eta \wedge \Omega.$$

It is also known that, similar to the characterization (1.3) of Sasaki manifolds, $\widetilde{M}$ is a Kenmotsu manifold if and only if

$$(\widetilde{\nabla}_X F)Y = -\eta(Y)FX - g(X,FY)\xi \tag{1.4}$$

for all X, Y in $\chi(\widetilde{M})$.

From [3] and [5], we have the following expressions of the curvature tensor in Sasaki and Kenomotsu space forms :

$$\begin{aligned}\widetilde{R}(X,Y)Z = {} & \frac{c+3(-1)^{i+1}}{4}[g(Y,Z)X - g(X,Z)Y] + \frac{c-(-1)^{i+1}}{4}[\eta(X)\eta(Z)Y \\ & - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \Omega(X,Z)FY \\ & - \Omega(Y,Z)FX + 2\Omega(X,Y)FZ],\end{aligned} \tag{1.5}$$

where $i = -1$ for Sasakian case and $i = 1$ for Kenmotsu case.

In the case of a $(2n+1)$-dimensional contact manifold $\widetilde{M}$, the contact distribution $\mathcal{D} = \ker \eta$ is totally non integrabile and the maximal dimension of its integral submanifolds M (called *the integral submanifolds of the contact manifold* $\widetilde{M}$) is n. A maximal integral submanifold M of a contact manifold $\widetilde{M}$ is a *Legendre* submanifold. Moreover, it is well known that an integral submanifold M of a contact manifold $\widetilde{M}$ is characterized by any of

(i) $\eta = 0, \quad d\eta = 0$;
(ii) $FX \in \chi^{\perp}(M)$ for all X in $\chi(M)$.

Another properties valid on these submanifolds in the case of Sasaki manifolds and useful for our considerations are given in [7] by

Proposition 1.1. *Let M be an integral submanifold of a $(2n+1)$-dimensional Sasaki manifold $\widetilde{M}$, $n \geq 1$. Then:*

(i) $A_{\xi} = 0$;
(ii) $A_{FX}Y = A_{FY}X$;
(iii) $A_{FY}X = -[Fh(X,Y)]^T$;
(iv) $\nabla^{\perp}_X(FY) = g(X,Y)\xi + F\nabla_X Y + [Fh(X,Y)]^{\perp}$;

(v) $\nabla^{\perp}_X \xi = -FX$ *for all* X, Y *in* $\chi(M)$.

In the case of Kenmotsu manifolds, N. Papaghiuc, [6], introduced the following

Definition 1.2. *A submanifold* M *of a Kenmotsu manifold* $\widetilde{M}$ *is a normal semi-invariant submanifold if* ξ *is normal to* M *and* M *has two distributions* D *and* $D^{\perp}$, *called the invariant, respectively, the anti-invariant distribution of* M *so that*

(i) $T_xM = D_x \oplus D_x^{\perp} \oplus < \xi_x >$;
(ii) D_x, $D_x^{\perp}$, $< \xi_x >$ *are othogonal;*
(iii) $FD_x \subseteq D_x$; $FD_x^{\perp} \subseteq T_x^{\perp}$,

for all $x \in M$.
If $D = 0$ *then* M *is a normal anti-invariant submanifold of* $\widetilde{M}$ *and if* $D^{\perp} = 0$ *then* M *is a invariant submanifold of* $\widetilde{M}$.

Also, from [6], we have the following result

Proposition 1.3. *If* M *is a normal anti-invariant submanifold of a Kenmotsu manifold* $\widetilde{M}$, *then*

(i) $A_{FX}Y = A_{FY}X$, *for all* $X, Y \in D^{\perp}$;
(ii) $A_{\xi}Z = -Z$ *and* $\nabla^{\perp}_Z \xi = 0$, *for all* $Z \in \chi(M)$.

2. Pseudo-parallel submanifolds in Kenmotsu and Sasaki space forms

Proposition 2.1. *If* M *is a* m*-dimensional, normal anti-invariant submanifold of a* $(2n+1)$*-dimensional Kenmotsu manifold* $\widetilde{M}(c)$, *then* $m \leq n$.

Proof. For $x \in M$ we have $T_x\widetilde{M} = T_xM \oplus T_x^{\perp}M$ and $\dim FT_xM = \dim T_xM = m$. Moreover, because M is normal anti-invariant we have $FT_xM \subseteq T_x^{\perp}M$; $FT_xM \perp < \xi_x >$ and then

$$\dim T_x^{\perp}M \geq \dim FT_xM + \dim < \xi_x > = m + 1.$$

Now,

$$2m \leq m + \dim T_x^{\perp}M - 1 = \dim T_xM + \dim T_x^{\perp}M - 1 = \dim T_x\widetilde{M} - 1 = 2n$$

and then $m \leq n$. □

Recall that a submanifold M of the Riemannian manifold $\widetilde{M}$ is *semi-parallel* if

$$(\widetilde{R} \cdot h)(X, Y, V, W) = 0 \tag{2.1}$$

where

$$\begin{aligned}(\widetilde{R} \cdot h)(X, Y, V, W) &= R^{\perp}(X, Y)h(V, W) - h(R(X, Y)V, W) \\ &- h(V, R(X, Y)W)\end{aligned}$$

for all X, Y, Z, W in $\chi(M)$. Here R is the curvature tensor of M and $R^{\perp}$ is the normal component of the curvature tensor $\widetilde{R}$ of $\widetilde{M}$ on M.
M is *pseudo-parallel* if

$$(\widetilde{R}\cdot h)(X,Y,V,W)+\Phi\cdot Q(g,h)(X,Y,V,W)=0, \tag{2.2}$$

where Φ is a differential function on $\widetilde{M}$ and

$$Q(g,h)(X,Y,V,W)=h((X\wedge Y)V,W)+h(V,(X\wedge Y)W),$$
$$(X\wedge Y)V=g(Y,V)X-g(X,V)Y$$

for all X, Y, V, W in $\chi(M)$.

Let $\widetilde{M}(c)$ be a Kenmotsu space form with $\dim\widetilde{M}(c)=2n+1$ and M be a n-dimensional normal anti-invariant submanifold. We consider $\{X_1,...,X_n\}$ a local orthonormal basis in $\chi(M)$ and $\{\xi, FX_1,...,FX_n\}$ a local orthonormal basis in $\chi^{\perp}(M)$.
Because M is normal anti-invariant manifold and taking into account (1.1) and (1.4) we have:

$$g(FX,FY)=g(X,Y);\ \widetilde{\nabla}_X(FY)=F\widetilde{\nabla}_XY;\ F\widetilde{R}(X,Y)Z=\widetilde{R}(X,Y)FZ \tag{2.3}$$

for all X, Y, Z in $\chi(M)$. Because $Fh(X,Y)$ belongs to $\chi(M)$ and taking into account (1.2) and (2.3), we obtain

$$\nabla^{\perp}_X(FY)=F\nabla_XY;\qquad -A_{FY}X=Fh(X,Y). \tag{2.4}$$

From (1.1) and Proposition 1.3 we have

$$h(X,Y)=FA_{FY}X-g(X,Y)\xi=FA_{FX}Y-g(X,Y)\xi. \tag{2.5}$$

We define the 3-form $C(X,Y,Z)=g(h(X,Y),FZ)$ for all X, Y, Z in $\chi(M)$. From the symmetry of h and taking into account Proposition 1.3 and (2.5), it follows that C is a totally symmetric 3-form.

From (1.5), the Codazzi equation and the fact that M is normal and anti-invariant, we have

$$\widetilde{R}(X,Y)Z=\frac{c-3}{4}[g(Y,Z)X-g(X,Z)Y] \tag{2.6}$$

and

$$R(X,Y)Z=\frac{c-3}{4}[g(Y,Z)X-g(X,Z)Y]+A_{h(Y,Z)}X-A_{h(X,Z)}Y.$$

But from (2.5) and Proposition 1.3, we obtain

$$A_{h(X,Z)}Y=A_{FY}A_{FX}Z+g(X,Z)Y$$

and then

$$R(X,Y)Z=\frac{c+1}{4}[g(Y,Z)X-g(X,Z)Y]+[A_{FX},A_{FY}]Z. \tag{2.7}$$

Moreover, from (2.4) we have:

$$R^{\perp}(X,Y)FZ=FR(X,Y)Z \tag{2.8}$$

for all X, Y, Z in $\chi(M)$.

Now, we give the main result of this article.

Theorem 2.2. *Any n-dimensional pseudo-parallel normal anti-invariant submanifold M of a $(2n+1)$-dimensional Kenmotsu space form $\widetilde{M}(c)$, with $n \geq 1$, is semi-parallel.*

Proof. We have

$$\begin{aligned} g((\widetilde{R}\cdot h)(X,Y,V,W),FZ) &= g(R^{\perp}(X,Y)h(V,W),FZ) \\ &- g(h(R(X,Y)V,W),FZ) \\ &- g(h(V,R(X,Y)W),FZ) \end{aligned}$$

for X,Y,V,W in $\chi(M)$. Denote by

$$T_1 = g(R^{\perp}(X,Y)h(V,W),FZ) \quad T_2 = g(h(R(X,Y)V,W),FZ)$$

$$T_3 = g(h(V,R(X,Y)W),FZ).$$

Because the 3-form C is totally symmetric, it follows that T_2 is symmetric in Z and W. From (2.5), Proposition 1.3, (2.7), (1.1) and (2.8), we obtain:

$$\begin{aligned} T_1 &= g(R^{\perp}(X,Y)h(V,W),FZ) = g(R^{\perp}(X,Y)FA_{FV}W,FZ) \\ &= \frac{c+1}{4}[g(X,Y)g(h(Y,W),FV) - g(Y,Z)g(h(X,W),FV)] \\ &+ g([A_{FX},A_{FY}]A_{FV}W,Z), \end{aligned}$$

$$\begin{aligned} T_3 &= g(h(V,R(X,Y)W),FZ) = g(h(V,Z),FR(X,Y)W) \\ &= \frac{c+1}{4}[g(Y,W)g(h(X,Z),FV) - g(h(Y,Z),FV)g(X,W)] \\ &+ g(A_{FV}Z,[A_{FX},A_{FY}]W). \end{aligned}$$

Also,

$$T_1 - T_3 = T_4 + T_5$$

where

$$\begin{aligned} T_4 &= \frac{c+1}{4}[g(h(Y,W),FV)g(X,Y) - g(h(X,W),FV)g(Y,Z) \\ &- g(h(X,Z),FV)g(Y,W) + g(h(Y,Z),FV)g(X,W)] \end{aligned}$$

is symmetric in W and Z and

$$T_5 = g([A_{FX},A_{FY}]A_{FV}W,Z) - g(A_{FV}Z,[A_{FX},A_{FY}]W).$$

On the other hand, from the symmetry of h we have

$$g(A_{FV}Z,[A_{FX},A_{FY}]W) = -g([A_{FX},A_{FY}]A_{FV}Z,W).$$

From this we deduce that

$$T_5 = g([A_{FX},A_{FY}]A_{FV}Z,W)) + g([A_{FX},A_{FY}]A_{FV}W,Z)$$

is symmetric in W and Z and $g((\widetilde{R} \cdot h)(X,Y,V,W), FZ)$ is symmetric in W and Z. Because M is pseudo-parallel it follows that $g(Q(g,h)(X,Y,V,W), FZ)$ is symmetric in W and Z or equivalently

$$g(Y,W)g(h(V,X), FZ) - g(X,W)g(h(V,Y), FZ)$$
$$= g(Y,Z)g(h(V,X), FW) - g(X,Z)g(h(V,Y), FW).$$

Taking $X = W = V$, $Z, Y \perp X$ in this relation, we obtain

$$-g(X,X)g(h(Y,Z), FX) = g(Y,Z)g(h(X,X), FX). \tag{2.9}$$

Let x be in M and $S = \{V \in T_pM | g(V,V) = 1\}$ – the unit sphere and $f : S \to \mathcal{F}(M)$, where $f(V) = g(h(V,V), FV)$ for all V in S. Because f is a continue function on S, it results that f attains its maximum in a vector field X_0, tangent to the submanifold in x.

Let $\{e_1, ..., e_{n-1}, X_0\}$ be a local orthonormal basis in $\chi(M)$. Taking $Y = Z = X_0$ and $X = e_i$ in (2.9), we have:

$$g(h(X_0, X_0), Fe_i) = -f(e_i), \quad i = 1, ..., n-1.$$

and for $Y = Z = e_i$ and $X = X_0$

$$g(h(e_i, e_i), FX_0) = -f(X_0), \quad i = 1, ..., n-1.$$

$\{\xi, Fe_1, ..., Fe_{n-1}, FX_0\}$ is a local orthonormal basis in $\chi^{\perp}(M)$ and

$$h(e_i, e_i) = -f(X_0)FX_0 - \xi - \sum_{j=1}^{n-1} f(e_j)Fe_j$$

$$h(X_0, X_0) = f(X_0)FX_0 - \xi - \sum_{j=1}^{n-1} f(e_j)Fe_j.$$

From these last two equalities we obtain

$$h(e_i, e_i) = h(X_0, X_0) - 2f(X_0)FX_0, \quad h(X_0, X_0) = f(X_0)FX_0 - \xi. \tag{2.10}$$

and $f(e_i) = 0$ for $i = 1...n-1$. From (2.10) we have $g(h(X_0, X_0), FV) = 0$ for all $V \perp X_0$, V in S. Moreover, (2.10) and (2.5) implies that

$$FA_{FX_0}X_0 = f(X_0)FX_0 \quad \text{or} \quad -A_{FX_0}X_0 = -f(X_0)X_0$$

and then

$$A_{FX_0}X_0 = \lambda_1 X_0; \quad \lambda_1 = f(X_0). \tag{2.11}$$

Putting $X = X_0$ and $Y \perp X_0$ in (2.9), we obtain

$$-g(X_0, X_0)g(h(Y,Z), FX_0) = g(Y,Z)g(h(X_0, X_0), FX_0)$$

and then

$$A_{FX_0}Y = -\lambda_1 Y. \tag{2.12}$$

For $Y \perp X_0$, $X = Y$, $Y = Z = X_0$ in (2.9) we have:

$$-g(Y,Y)g(h(X_0, X_0), FY) = g(X_0, X_0)g(h(Y,Y), FY)$$

or

$$g(h(Y,Y), FY) = 0.$$

Using the totally symmetry of the 3-form C and the last equality, we have

$$g(h(Y,Z),FW)=0$$

for all $Y,Z,W\perp X_0$, Y,Z,W in $\chi(M)$. From (2.11) and (2.12) we have

$$h(X_0,X_0)=\lambda_1 FX_0+\xi, \qquad h(X_0,Y)=-\lambda_1 FY \tag{2.13}$$

for $Y\perp X_0$. Taking $X=X_0$ and $Z,Y\perp X_0$ in (2.9) we have:

$$h(Y,Z)=-\lambda_1 g(Y,Z)FX_0. \tag{2.14}$$

Taking $Z=Y$, $Z\perp X_0$ and Z an unitary vector field in (2.14), we obtain

$$A_{FY}Y=-\lambda_1 X_0. \tag{2.15}$$

If $\lambda_1=0$ then h vanishes at x. We suppose that $\lambda_1\neq 0$. For $n>2$, we consider two othonormal vector fields Y and Z, so that $Y,Z\perp X_0$. Then

$$R(X_0,Y)Y=(\frac{c+1}{4}-2\lambda_1^2)X_0,$$

and

$$R(Y,Z)Z=(\frac{c+1}{4}+\lambda_1^2)Y.$$

Because M is a pseudo-parallel manifold, we have

$$(\widetilde{R}\cdot h)(X_0,Y,Y,Y)+\Phi(x)Q(g,h)(X_0,Y,Y,Y)=0.$$

where

$$(\widetilde{R}\cdot h)(X_0,Y,Y,Y)=3\lambda_1(\frac{c+1}{4}-2\lambda_1^2)FY,$$

$$(Q\cdot h)(X_0,Y,Y,Y)=-2\lambda_1 FY.$$

From these last three equalities we have:

$$\Phi(x)=\frac{3(\frac{c+1}{4}-2\lambda_1^2)}{2}. \tag{2.16}$$

Also, we have:

$$(\widetilde{R}\cdot h)(X_0,Y,Y,Z)+\Phi(x)Q(g,h)(X_0,Y,Y,Z)=0.$$

But

$$(\widetilde{R}\cdot h)(X_0,Y,Y,Z)=\lambda_1(\frac{c+1}{4}-2\lambda_1^2)FZ$$

$$Q(g,h)(X_0,Y,Y,Z)=-\lambda_1 FZ.$$

From these last three equalities we deduce

$$\Phi(x)=\frac{c+1}{4}-2\lambda_1^2. \tag{2.17}$$

From (2.17) and (2.16) we obtain $\Phi(x)=0$, that is M is semi-parallel. □

Now, let M be a Legendre submanifold in a Sasaki space form $\widetilde{M}(c)$. Taking into account (1.3) and the fact that M is a Legendre submanifold, we have

$$F\widetilde{\nabla}_X Y = \widetilde{\nabla}_X(FY) - g(X,Y)\xi$$

$$F\widetilde{R}(X,Y)Z = \widetilde{R}(X,Y)FZ + g(Y,Z)FX - g(X,Z)FY$$

for all X, Y, Z in $\chi(M)$.

Because M is a Legendre submanifold, using (1.2) and (1.3) we obtain:

$$h(X,Y) = FA_{FY}X; \quad \nabla^{\perp}_X FY = F\nabla_X Y + g(X,Y)\xi \tag{2.18}$$

for X, Y, Z in $\chi(M)$. We also obtain that the 3-form C is totally symmetric for Legendre pseudo-parallel submanifolds in Sasaki space forms. Moreover, from (1.5) we have

$$\widetilde{R}(X,Y)Z = \frac{c+3}{4}[g(Y,Z)X - g(X,Z)Y]$$

and

$$R^{\perp}(X,Y)FZ = FR(X,Y)Z - g(Y,Z)FX + g(X,Z)FY$$

for all X, Y, Z in $\chi(M)$.

We define the tensor field

$$\theta(X,Y,Z,V,W) = g(h(X,V),FZ)g(Y,W) - g(h(Y,V),FZ)g(X,W) \tag{2.19}$$

for X, Y, Z, W in $\chi(M)$. Then θ is anti-symmetric in X and Y.

The submanifold M has *axial semi-symmetry* if θ is symmetric in Z and W.

Proposition 2.3. *Let M be a Legendre pseudo-parallel submanifold in the Sasaki space form $\widetilde{M}(c)$ so that M has axial semi-symmetry. Then, for each $x \in M$, there is $X_0 \in T_pM$, X_0 a unit vector field and $\lambda_1 \in \mathcal{F}(M)$ so that:*

$$A_{FX_0}X_0 = \lambda_1 X_0; \qquad c = 1 + 8\lambda_1^2.$$

From Proposition 2.3, we observe that the Sasaki space form $\widetilde{M}(c)$ has Legendre pseudo-parallel submanifolds only if the λ_1 is constant.

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