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Subclasses of analytic functions involving a family of integral operators

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Abstract. In the present paper, we introduce and investigate some new subclasses of analytic functions associated with a family of generalized Srivastava-Attiya operator. Such results as subordination and superordination properties, inclusion relationships, integral-preserving properties and convolution properties are proved. Several sandwich-type results are also derived.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : \ z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Let $\mathcal{H}(\mathbb{U})$ be the linear space of all analytic functions in $\mathbb{U}$. For a positive integer number n and $a \in \mathbb{C}$, we let

$$\mathcal{H}[a, n] := \left\{\mathfrak{f} \in \mathcal{H}(\mathbb{U}) : \quad \mathfrak{f}(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \right\}.$$

Let $f, \ g \in \mathcal{A}$, where f is given by (1.1) and g is defined by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k.$$

Then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) := z^p + \sum_{k=2}^{\infty} a_k b_k z^k =: (g * f)(z).$$

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = g\big(\omega(z)\big) \quad (z \in \mathbb{U}).$$

Indeed, it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

In the following we recall a general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined by (cf., e.g., [18, p. 121 et sep.])

$$\Phi(z, s, a) := \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s}$$

$$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \ \text{ when } \ |z| < 1;\ \Re(s) > 1 \ \text{ when } \ |z| = 1),$$

where, as usual,

$$\mathbb{Z}_0^- := \mathbb{Z} \setminus \mathbb{N} \quad (\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\};\quad \mathbb{N} := \{1, 2, 3, \ldots\}).$$

Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ can be found in the recent investigations by (for example) Choi and Srivastava [3], Ferreira and López [5], Garg et al. [6], Lin et al. [7], Luo and Srivastava [10], Wen and Liu [19], Wen and Yang [20] and others.

Recently, Srivastava and Attiya [17] (see also Răducanu and Srivastava [14], Liu [9], Prajapat and Goyal [13]) introduced and investigated the linear operator:

$$\mathcal{J}_{s,\, b}(f) :\ \mathcal{A} \longrightarrow \mathcal{A}$$

defined, in terms of the Hadamard product (or convolution), by

$$\mathcal{J}_{s,\, b} f(z) := G_{s,\, b}(z) * f(z) \quad (z \in \mathbb{U};\ b \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C};\ f \in \mathcal{A}), \tag{1.2}$$

where, for convenience,

$$G_{s,\, b}(z) := (1 + b)^s [\Phi(z, s, b) - b^{-s}] \quad (z \in \mathbb{U}). \tag{1.3}$$

It is easy to observe from (1.2) and (1.3) that

$$\mathcal{J}_{s,\ b}f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+b}{k+b}\right)^{s} a_k z^k.$$

Motivated essentially by the above-mentioned Srivastava-Attiya operator, Al-Shaqsi and Darus [1] (see also Darus and Al-Shaqsi [4]) introduced and investigated the following integral operator:

$$\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+b}{k+b}\right)^{s} \frac{\lambda!(k+\mu-2)!}{(\mu-2)!(k+\lambda-1)!} a_k z^k \quad (z \in \mathbb{U}), \tag{1.4}$$

where (and throughout this paper unless otherwise mentioned) the parameters $s,\ b,\ \lambda$ and μ are constrained as follows:

$$s \in \mathbb{C}; \quad b \in \mathbb{C} \setminus \mathbb{Z}_0^- \quad \lambda > -1 \quad \text{and} \quad \mu > 0.$$

We note that $\mathcal{J}_{s,b}^{1,2}$ is the Srivastava-Attiya operator, $\mathcal{J}_{0,b}^{\lambda,\mu}$ is the well-known Choi-Saigo- Srivastava operator (see [2]).

It is easily verified from (1.4) that

$$z\left(\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f\right)'(z) = \mu \mathcal{J}_{s,\ b}^{\lambda,\ \mu+1} f(z) - (\mu-1)\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z), \tag{1.5}$$

$$z\left(\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} f\right)'(z) = (\lambda+1)\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z) - \lambda \mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} f(z), \tag{1.6}$$

and

$$z\left(\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu} f\right)'(z) = (b+1)\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z) - b\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu} f(z). \tag{1.7}$$

By making use of the subordination between analytic functions and the operator $\mathcal{J}_{s,\ b}^{\lambda,\ \mu}$, we now introduce the following subclasses of analytic functions.

Definition 1.1. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$ if it satisfies the subordination condition

$$(1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} + \alpha \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1} f(z)}{z} \prec \phi(z) \quad (z \in \mathbb{U};\ \alpha \in \mathbb{C};\ \phi \in \mathcal{P}). \tag{1.8}$$

Definition 1.2. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{G}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$ if it satisfies the subordination condition

$$(1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} f(z)}{z} + \alpha \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} \prec \phi(z) \quad (z \in \mathbb{U};\ \alpha \in \mathbb{C};\ \phi \in \mathcal{P}). \tag{1.9}$$

Definition 1.3. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{H}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$ if it satisfies the subordination condition

$$(1-\alpha)\frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu} f(z)}{z} + \alpha \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} \prec \phi(z) \quad (z \in \mathbb{U};\ \alpha \in \mathbb{C};\ \phi \in \mathcal{P}). \tag{1.10}$$

In the present paper, we aim at proving some subordination and superordination properties, inclusion relationships, integral-preserving properties and convolution properties associated with the operator $\mathcal{J}_{s,\ b}^{\lambda,\ \mu}$. Several sandwich-type results involving this operator are also derived.

2. Preliminary results

In order to prove our main results, we need the following lemmas.

Lemma 2.1. ([11]) *Let the function Ω be analytic and convex (univalent) in $\mathbb{U}$ with $\Omega(0) = 1$. Suppose also that the function Θ given by*

$$\Theta(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots$$

is analytic in $\mathbb{U}$. If

$$\Theta(z) + \frac{z\Theta'(z)}{\zeta} \prec \Omega(z) \quad (\Re(\zeta) > 0;\ \zeta \neq 0;\ z \in \mathbb{U}), \tag{2.1}$$

then

$$\Theta(z) \prec \chi(z) = \frac{\zeta}{n} z^{-\frac{\zeta}{n}} \int_0^z t^{\frac{\zeta}{n}-1} h(t)dt \prec \Omega(z) \quad (z \in \mathbb{U}),$$

and χ is the best dominant of (2.1).

Denote by Q the set of all functions f that are analytic and injective on $\overline{\mathbb{U}} - E(f)$, where

$$E(f) = \left\{ \varepsilon \in \partial\mathbb{U} : \lim_{z \to \varepsilon} f(z) = \infty \right\},$$

and such that $f'(\varepsilon) \neq 0$ for $\varepsilon \in \partial\mathbb{U} - E(f)$.

Lemma 2.2. ([12]) *Let q be convex univalent in $\mathbb{U}$ and $\kappa \in \mathbb{C}$. Further assume that $\Re(\overline{\kappa}) > 0$. If*

$$p \in \mathcal{H}[q(0), 1] \cap Q,$$

and $p + \kappa z p'$ is univalent in $\mathbb{U}$, then

$$q(z) + \kappa z q'(z) \prec p(z) + \kappa z p'(z)$$

implies $q \prec p$, and q is the best subdominant.

Lemma 2.3. ([15]) *Let q be a convex univalent function in $\mathbb{U}$ and let $\sigma,\ \eta \in \mathbb{C}$ with*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0,\ -\Re\left(\frac{\sigma}{\eta}\right)\right\}.$$

If p is analytic in $\mathbb{U}$ and

$$\sigma p(z) + \eta z p'(z) \prec \sigma q(z) + \eta z q'(z),$$

then $p \prec q$, and q is the best dominant.

Lemma 2.4. ([16]) *Let the function Υ be analytic in $\mathbb{U}$ with*

$$\Upsilon(0) = 1 \quad \text{and} \quad \Re(\Upsilon(z)) > \frac{1}{2} \quad (z \in \mathbb{U}).$$

Then, for any function Ψ analytic in $\mathbb{U}$, $(\Upsilon * \Psi)(\mathbb{U})$ is contained in the convex hull of $\Psi(\mathbb{U})$.

3. Properties of the function class $\mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$

We begin by proving our first subordination property given by Theorem 3.1 below.

Theorem 3.1. *Let $f \in \mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$ with $\Re(\alpha) > 0$. Then*

$$\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} \prec \frac{\mu}{\alpha} z^{-\frac{\mu}{\alpha}} \int_0^z t^{\frac{\mu}{\alpha}-1}\phi(t)dt \prec \phi(z) \quad (z \in \mathbb{U}). \tag{3.1}$$

Proof. Let $f \in \mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$ and suppose that

$$h(z) := \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} \quad (z \in \mathbb{U}). \tag{3.2}$$

Then h is analytic in $\mathbb{U}$. Combining (1.5), (1.8) and (3.2), we easily find that

$$h(z) + \frac{\alpha}{\mu} z h'(z) = (1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} + \alpha \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1} f(z)}{z} \prec \phi(z) \quad (z \in \mathbb{U}). \tag{3.3}$$

Therefore, an application of Lemma 2.1 for $n = 1$ to (3.3) yields the assertion of Theorem 3.1. □

By virtue of Theorem 3.1, we easily get the following inclusion relationship.

Corollary 3.2. *Let $\Re(\alpha) > 0$. Then $\mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi) \subset \mathcal{F}_{s,\ b}^{\lambda,\ \mu}(0;\phi)$.*

Theorem 3.3. *Let $\alpha_2 > \alpha_1 \geqq 0$. Then $\mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\alpha_2;\phi) \subset \mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\alpha_1;\phi)$.*

Proof. Suppose that $f \in \mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\alpha_2;\phi)$. It follows that

$$(1-\alpha_2)\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}}{z} + \alpha_2 \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1}}{z} \prec \phi(z) \quad (z \in \mathbb{U}). \tag{3.4}$$

Since

$$0 \leqq \frac{\alpha_1}{\alpha_2} < 1$$

and the function ϕ is convex and univalent in $\mathbb{U}$, we deduce from (3.1) and (3.4) that

$$\begin{aligned}
&(1-\alpha_1)\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} + \alpha_1 \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1} f(z)}{z} \\
&\quad = \frac{\alpha_1}{\alpha_2}\left[(1-\alpha_1)\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} + \alpha_1 \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1} f(z)}{z}\right] + \left(1 - \frac{\alpha_1}{\alpha_2}\right)\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} \\
&\quad \prec \phi(z) \quad (z \in \mathbb{U}),
\end{aligned}$$

which implies that $f \in \mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\alpha_1;\phi)$. The proof of Theorem 3.3 is evidently completed. □

Theorem 3.4. *Let $f \in \mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$. If the integral operator F is defined by*

$$F(z) := \frac{\nu+1}{z^{\nu}}\int_0^z t^{\nu-1}f(t)dt \quad (z \in \mathbb{U};\ \nu > -1), \tag{3.5}$$

then

$$\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}F(z)}{z} \prec \phi(z) \quad (z \in \mathbb{U}). \tag{3.6}$$

Proof. Let $f \in \mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\alpha;\ \phi)$. Suppose also that

$$G(z) := \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}F(z)}{z} \quad (z \in \mathbb{U}). \tag{3.7}$$

From (3.5), we deduce that

$$z\left(\mathcal{J}_{s,\ b}^{\lambda,\ \mu}F\right)'(z) + \nu\mathcal{J}_{s,\ b}^{\lambda,\ \mu}F(z) = (\nu+1)\mathcal{J}_{s,\ b}^{\lambda,\ \mu}f(z). \tag{3.8}$$

Combining (3.1), (3.7) and (3.8), we easily get

$$G(z) + \frac{1}{\nu+1}zG'(z) = \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}f(z)}{z} \prec \phi(z) \quad (z \in \mathbb{U}). \tag{3.9}$$

Thus, by Lemma 2.1 and (3.9), we conclude that the assertion (3.6) of Theorem 3.4 holds. □

Theorem 3.5. *Let $f \in \mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$ and $g \in \mathcal{A}$ with $\Re\left(\frac{g(z)}{z}\right) > \frac{1}{2}$. Then*

$$(f * g)(z) \in \mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi).$$

Proof. Let $f \in \mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\eta;\phi)$ and $g \in \mathcal{A}$ with $\Re\left(\frac{g(z)}{z}\right) > \frac{1}{2}$. Suppose also that

$$H(z) := (1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}f(z)}{z} + \alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1}f(z)}{z} \prec \phi(z) \quad (z \in \mathbb{U}). \tag{3.10}$$

It follows from (3.10) that

$$(1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}(f*g)(z)}{z} + \alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1}(f*g)(z)}{z} = H(z) * \frac{g(z)}{z} \quad (z \in \mathbb{U}). \tag{3.11}$$

Since the function ϕ is convex and univalent in $\mathbb{U}$, by virtue of (3.10), (3.11) and Lemma 2.2, we conclude that

$$(1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}(f*g)(z)}{z} + \alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1}(f*g)(z)}{z} \prec \phi(z) \quad (z \in \mathbb{U}), \tag{3.12}$$

which implies that the assertion of Theorem 3.5 holds. □

Theorem 3.6. *Let q_1 be univalent in $\mathbb{U}$ and $\Re(\alpha) > 0$. Suppose also that q_1 satisfies*

$$\Re\left(1 + \frac{zq_1''(z)}{q_1'(z)}\right) > \max\left\{0, -\Re\left(\frac{\mu}{\alpha}\right)\right\}. \tag{3.13}$$

If $f \in \mathcal{A}$ satisfies the subordination

$$(1-\alpha)\frac{\mathcal{J}_{s,\,b}^{\lambda,\,\mu} f(z)}{z} + \alpha \frac{\mathcal{J}_{s,\,b}^{\lambda,\,\mu+1} f(z)}{z} \prec q_1(z) + \frac{\alpha}{\mu} z q_1'(z), \tag{3.14}$$

then

$$\frac{\mathcal{J}_{s,\,b}^{\lambda,\,\mu} f(z)}{z} \prec q_1'(z),$$

and q_1 is the best dominant.

Proof. Let the function h be defined by (3.2). We know that (3.3) holds. Combining (3.3) and (3.14), we find that

$$h(z) + \frac{\alpha}{\mu} z h'(z) \prec q_1(z) + \frac{\alpha}{\mu} z q_1'(z). \tag{3.15}$$

By Lemma 2.3 and (3.15), we readily get the assertion of Theorem 3.6. □

If f is subordinate to $\mathcal{F}$, then $\mathcal{F}$ is superordinate to f. We now derive the following superordination result for the class $\mathcal{F}_{s,\,b}^{\lambda,\,\mu}(\alpha;\phi)$.

Theorem 3.7. *Let q_2 be convex univalent in $\mathbb{U}$, $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Also let*

$$\frac{\mathcal{J}_{s,\,b}^{\lambda,\,\mu} f(z)}{z} \in \mathcal{H}[q_2(0), 1] \cap Q$$

and

$$(1-\alpha)\frac{\mathcal{J}_{s,\,b}^{\lambda,\,\mu} f(z)}{z} + \alpha \frac{\mathcal{J}_{s,\,b}^{\lambda,\,\mu+1} f(z)}{z}$$

be univalent in $\mathbb{U}$. If

$$q_2(z) + \frac{\alpha}{\mu} z q_2'(z) \prec (1-\alpha)\frac{\mathcal{J}_{s,\,b}^{\lambda,\,\mu} f(z)}{z} + \alpha \frac{\mathcal{J}_{s,\,b}^{\lambda,\,\mu+1} f(z)}{z},$$

then

$$q_2(z) \prec \frac{\mathcal{J}_{s,\,b}^{\lambda,\,\mu} f(z)}{z},$$

and q_2 is the best subdominant.

Proof. Let the function h be defined by (3.2). Then

$$q_2(z) + \frac{\alpha}{\mu} z q_2'(z) \prec (1-\alpha)\frac{\mathcal{J}_{s,\,b}^{\lambda,\,\mu} f(z)}{z} + \alpha \frac{\mathcal{J}_{s,\,b}^{\lambda,\,\mu+1} f(z)}{z} = h(z) + \frac{\alpha}{\mu} z h'(z).$$

An application of Lemma 2.4 yields the desired assertion of Theorem 3.7. □

Combining the above results of subordination and superordination, we easily get the following "sandwich-type result".

Theorem 3.8. *Let q_3 be convex univalent and q_4 be univalent in $\mathbb{U}$, $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Suppose also that q_4 satisfies*

$$\Re\left(1 + \frac{z q_4''(z)}{q_4'(z)}\right) > \max\left\{0, -\Re\left(\frac{\mu}{\alpha}\right)\right\}.$$

If

$$0 \neq \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} \in \mathcal{H}[q_3(0), 1] \cap Q,$$

and

$$(1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} + \alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1} f(z)}{z}$$

is univalent in $\mathbb{U}$, *also*

$$q_3(z) + \frac{\alpha}{\mu} z q_3'(z) \prec (1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} + \alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1} f(z)}{z} \prec q_4(z) + \frac{\alpha}{\mu} z q_4'(z),$$

then

$$q_3(z) \prec \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} \prec q_4(z),$$

and q_3 *and* q_4 *are, respectively, the best subordinant and the best dominant.*

4. Properties of the function classes $\mathcal{G}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$ and $\mathcal{H}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$

By means of (1.6) and (1.7), and by similarly applying the methods used in the proofs of Theorems 3.1–3.8, respectively, we easily get the following properties for the function classes $\mathcal{G}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$ and $\mathcal{H}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$. Here we choose to omit the details involved.

Corollary 4.1. *Let* $f \in \mathcal{G}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$ *with* $\Re(\alpha) > 0$. *Then*

$$\frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} f(z)}{z} \prec \frac{\lambda+1}{\alpha} z^{-\frac{\lambda+1}{\alpha}} \int_0^z t^{\frac{\lambda+1}{\alpha}-1}\phi(t)dt \prec \phi(z) \quad (z \in \mathbb{U}).$$

Corollary 4.2. *Let* $\alpha_2 > \alpha_1 \geqq 0$. *Then* $\mathcal{G}_{s,\ b}^{\lambda,\ \mu}(\alpha_2;\phi) \subset \mathcal{G}_{s,\ b}^{\lambda,\ \mu}(\alpha_1;\phi)$.

Corollary 4.3. *Let* $f \in \mathcal{G}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$. *If the integral operator* F *is defined by* (3.5), *then*

$$\frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} F(z)}{z} \prec \phi(z) \quad (z \in \mathbb{U}).$$

Corollary 4.4. *Let* $f \in \mathcal{G}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$ *and* $g \in \mathcal{A}$ *with* $\Re\left(\frac{g(z)}{z}\right) > \frac{1}{2}$. *Then*

$$(f * g)(z) \in \mathcal{G}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi).$$

Corollary 4.5. *Let* q_5 *be univalent in* $\mathbb{U}$ *and* $\Re(\alpha) > 0$. *Suppose also that* q_5 *satisfies*

$$\Re\left(1 + \frac{z q_5''(z)}{q_5'(z)}\right) > \max\left\{0, -\Re\left(\frac{\lambda+1}{\alpha}\right)\right\}.$$

If $f \in \mathcal{A}$ *satisfies the subordination*

$$(1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} f(z)}{z} + \alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} \prec q_5(z) + \frac{\alpha}{\lambda+1} z q_5'(z),$$

then

$$\frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu}f(z)}{z} \prec q_5'(z),$$

and q_5 is the best dominant.

Corollary 4.6. *Let q_6 be convex univalent in $\mathbb{U}$, $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Also let*

$$\frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu}f(z)}{z} \in \mathcal{H}[q_6(0), 1] \cap Q$$

and

$$(1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu}f(z)}{z} + \alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}f(z)}{z}$$

be univalent in $\mathbb{U}$. If

$$q_6(z) + \frac{\alpha}{\lambda+1}zq_6'(z) \prec (1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu}f(z)}{z} + \alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}f(z)}{z},$$

then

$$q_6(z) \prec \frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu}f(z)}{z},$$

and q_6 is the best subdominant.

Corollary 4.7. *Let q_7 be convex univalent and q_8 be univalent in $\mathbb{U}$, $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Suppose also that q_8 satisfies*

$$\Re\left(1 + \frac{zq_8''(z)}{q_8'(z)}\right) > \max\left\{0, -\Re\left(\frac{\lambda+1}{\alpha}\right)\right\}.$$

If

$$0 \neq \frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu}f(z)}{z} \in \mathcal{H}[q_7(0), 1] \cap Q,$$

and

$$(1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu}f(z)}{z} + \alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}f(z)}{z}$$

is univalent in $\mathbb{U}$, also

$$q_7(z) + \frac{\alpha}{\lambda+1}zq_7'(z) \prec (1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu}f(z)}{z} + \alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}f(z)}{z} \prec q_8(z) + \frac{\alpha}{\lambda+1}zq_8'(z),$$

then

$$q_7(z) \prec \frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu}f(z)}{z} \prec q_8(z),$$

and q_7 and q_8 are, respectively, the best subordinant and the best dominant.

Corollary 4.8. *Let $f \in \mathcal{H}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$ with $\Re(\alpha) > 0$. Then*

$$\frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu}f(z)}{z} \prec \frac{b+1}{\alpha}z^{-\frac{b+1}{\alpha}}\int_0^z t^{\frac{b+1}{\alpha}-1}\phi(t)dt \prec \phi(z) \quad (z \in \mathbb{U}).$$

Corollary 4.9. *Let $\alpha_2 > \alpha_1 \geqq 0$. Then $\mathcal{H}_{s,\ b}^{\lambda,\ \mu}(\alpha_2;\phi) \subset \mathcal{H}_{s,\ b}^{\lambda,\ \mu}(\alpha_1;\phi)$.*

Corollary 4.10. *Let $f \in \mathcal{H}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$. If the integral operator F is defined by* (3.5)*, then*

$$\frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu}F(z)}{z} \prec \phi(z) \quad (z \in \mathbb{U}).$$

Corollary 4.11. *Let $f \in \mathcal{H}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$ and $g \in \mathcal{A}$ with $\Re\left(\frac{g(z)}{z}\right) > \frac{1}{2}$. Then*

$$(f * g)(z) \in \mathcal{H}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi).$$

Corollary 4.12. *Let q_9 be univalent in $\mathbb{U}$ and $\Re(\alpha) > 0$. Suppose also that q_9 satisfies*

$$\Re\left(1 + \frac{zq_9''(z)}{q_9'(z)}\right) > \max\left\{0, -\Re\left(\frac{b+1}{\alpha}\right)\right\}.$$

If $f \in \mathcal{A}$ satisfies the subordination

$$(1-\alpha)\frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu}f(z)}{z} + \alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}f(z)}{z} \prec q_9(z) + \frac{\alpha}{b+1}zq_9'(z),$$

then

$$\frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu}f(z)}{z} \prec q_9'(z),$$

and q_9 is the best dominant.

Corollary 4.13. *Let q_{10} be convex univalent in $\mathbb{U}$, $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Also let*

$$\frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu}f(z)}{z} \in \mathcal{H}[q_{10}(0), 1] \cap Q$$

and

$$(1-\alpha)\frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu}f(z)}{z} + \alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}f(z)}{z}$$

be univalent in $\mathbb{U}$. If

$$q_{10}(z) + \frac{\alpha}{b+1}zq_{10}'(z) \prec (1-\alpha)\frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu}f(z)}{z} + \alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}f(z)}{z},$$

then

$$q_{10}(z) \prec \frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu}f(z)}{z},$$

and q_{10} is the best subdominant.

Corollary 4.14. *Let q_{11} be convex univalent and q_{12} be univalent in $\mathbb{U}$, $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Suppose also that q_{12} satisfies*

$$\Re\left(1 + \frac{zq_{12}''(z)}{q_{12}'(z)}\right) > \max\left\{0, -\Re\left(\frac{b+1}{\alpha}\right)\right\}.$$

If

$$0 \neq \frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu}f(z)}{z} \in \mathcal{H}[q_{11}(0), 1] \cap Q,$$

and

$$(1-\alpha)\frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu}f(z)}{z}+\alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}f(z)}{z}$$

is univalent in $\mathbb{U}$, *also*

$$q_{11}(z)+\frac{\alpha}{b+1}zq_{11}'(z)\prec(1-\alpha)\frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu}f(z)}{z}+\alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}f(z)}{z}$$
$$\prec q_{12}(z)+\frac{\alpha}{b+1}zq_{12}'(z),$$

then

$$q_{11}(z)\prec\frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu}f(z)}{z}\prec q_{12}(z),$$

and q_{11} *and* q_{12} *are, respectively, the best subordinant and the best dominant.*

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