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Differential subordinations obtained by using Al-Oboudi and Ruscheweyh operators

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Abstract. We introduce the operator $\mathcal{D}^n_{\lambda\delta}f$ using the Al-Oboudi and Ruscheweyh operators and we investigate several differential subordinations that generalize previous results.

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1. Introduction

Let $\mathcal{H}(U)$ denote the class of analytic functions in the unit disc

$$U = \{z \in \mathbb{C} \ : \ |z| < 1\}.$$

For $a \in \mathbb{C}$ and $m \in \mathbb{N} = \{1, 2, \ldots\}$, let

$$\mathcal{H}[a, m] = \{f \in \mathcal{H}(U) \ : \ f(z) = a + a_m z^m + \cdots, \ z \in U\}$$

and

$$\mathcal{A}_m = \left\{f \in \mathcal{H}(U) \ : \ f(z) = z + a_{m+1} z^{m+1} + \cdots, \ z \in U\right\},$$

with $\mathcal{A}_1 = \mathcal{A}$.

Let f and g be members of $\mathcal{H}(U)$. The function f is said to be subordinate to g if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1, \ z \in U$, such that $f(z) = g(w(z)), \ z \in U$. In this case, we write $f \prec g$ or $f(z) \prec g(z), \ z \in U$. If the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Let $\Psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the second-order differential subordination

$$\Psi(p(z), \, zp'(z), \, z^2 p''(z); \, z) \prec h(z), \quad z \in U, \tag{1.1}$$

then p is called a solution of the differential subordination. A univalent function q is called a dominant of the solution of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant of (1.1).

In order to prove our main results we shall need the following lemmas.

Lemma 1.1 ([2, p. 71]). *Let h be a convex function with $h(0) = a$ and let $\gamma \in \mathbb{C}^*$ be a complex number such that $Re\gamma \geq 0$. If $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z), \quad z \in U$$

then

$$p(z) \prec q(z) \prec h(z), \quad z \in U,$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t) t^{\gamma/n - 1} dt, \quad z \in U.$$

The function q is convex and the best dominant.

Lemma 1.2 ([3, p. 419]). *Let r be a convex function in U and let*

$$h(z) = r(z) + n\alpha zr'(z), \quad z \in U,$$

where $\alpha > 0$ and $n \in \mathbb{N}$. If

$$p(z) = r(0) + p_n z^n + p_{n+1} z^{n+1} + \cdots, \quad z \in U$$

is holomorphic in U and

$$p(z) + \alpha zp'(z) \prec h(z), \quad z \in U,$$

then

$$p(z) \prec r(z), \quad z \in U,$$

and this result is sharp.

Definition 1.3 ([1, p. 1429]). *For a function $f \in \mathcal{A}$, $\delta \geq 0$ and $n \in \mathbb{N} \cup \{0\}$, the Al-Oboudi differential operator $D_\delta^n f$ is defined by*

$$\begin{aligned} D^0 f(z) &= f(z), \\ D_\delta^1 f(z) &= (1 - \delta) f(z) + \delta z f'(z) = D_\delta f(z), \\ D_\delta^n f(z) &= D_\delta \left(D_\delta^{n-1} f(z) \right), \quad z \in U. \end{aligned} \tag{1.2}$$

Remark 1.4. D_δ^n is a linear operator and for $f \in \mathcal{A}$,

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j,$$

we have

$$D_\delta^n f(z) = z + \sum_{j=2}^{\infty} [1 + (j-1)\delta]^n a_j z^j, \quad z \in U \tag{1.3}$$

and

$$\left(D_\delta^{n+1} f(z) \right)' = \left(D_\delta^n f(z) \right)' + \delta z \left(D_\delta^n f(z) \right)'', \quad z \in U. \tag{1.4}$$

Also, when $\delta = 1$, we obtain the Sălăgean differential operator ([6, p. 363]).

Definition 1.5 ([5, p. 110]). *For a function $f \in \mathcal{A}$ and $n \in \mathbb{N} \cup \{0\}$, the Ruscheweyh differential operator $R^n f$ is defined by*

$$R^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z) = \frac{z}{n!}[z^{n-1} f(z)]^{(n)}, \quad z \in U, \tag{1.5}$$

where $*$ stands for the Hadamard product or convolution.

Remark 1.6. If $f \in \mathcal{A}$,

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j,$$

we have

$$R^0 f(z) = f(z),$$

$$R^1 f(z) = z f'(z),$$

$$(n+1) R^{n+1} f(z) = n R^n f(z) + z \left(R^n f(z)\right)', \tag{1.6}$$

$$R^n f(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^{n} a_j z^j, \quad z \in U. \tag{1.7}$$

Definition 1.7. *Let $n \in \mathbb{N} \cup \{0\}$, $\delta \geq 0$ and $\lambda \geq 0$ with $\delta \neq (\lambda - 1)/\lambda$. For $f \in \mathcal{A}$, let $\mathcal{D}_{\lambda\delta}^n f$ denote the operator defined by $\mathcal{D}_{\lambda\delta}^n : \mathcal{A} \to \mathcal{A}$,*

$$\mathcal{D}_{\lambda\delta}^n f(z) = \frac{1}{1-\lambda+\lambda\delta}[(1-\lambda) D_\delta^n f(z) + \lambda\delta R^n f(z)], \quad z \in U, \tag{1.8}$$

where the operators $D_\delta^n f$ and $R^n f$ are given by Definition 1.3 *and Definition* 1.5*, respectively.*

Remark 1.8. When $\lambda = 0$ in (1.8), we get the Al-Oboudi differential operator, and when $\lambda = 1$ we obtain the Ruscheweyh differential operator.

Also, for $n = 0$, we have

$$\mathcal{D}_{\lambda\delta}^0 f(z) = \frac{1}{1-\lambda+\lambda\delta}[(1-\lambda) D_\delta^0 f(z) + \lambda\delta R^0 f(z)] = f(z), \quad z \in U.$$

Remark 1.9. $\mathcal{D}_{\lambda\delta}^n$ is a linear operator and for $f \in \mathcal{A}$,

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j,$$

by using (1.3) and (1.7), we have

$$\mathcal{D}_{\lambda\delta}^n f(z) = z + \frac{1}{1-\lambda+\lambda\delta} \sum_{j=2}^{\infty} \left[(1-\lambda)\left(1+(j-1)\delta\right)^n + \lambda\delta C_{n+j-1}^{n}\right] a_j z^j, \tag{1.9}$$

$z \in U$.

2. Main results

Theorem 2.1. *If* $0 \le \alpha < 1$, $f \in \mathcal{A}_m$ *and*

$$\mathrm{Re}\left[\left(\mathcal{D}_{\lambda\delta}^{n+1} f(z)\right)' + \frac{\lambda\delta z(\delta n + \delta - 1)\left(R^n f(z)\right)''}{(1-\lambda+\lambda\delta)(n+1)}\right] > \alpha\,, \quad z \in U \tag{2.1}$$

then

$$\mathrm{Re}\ (\mathcal{D}_{\lambda\delta}^{n} f(z))' > \gamma\,, \quad z \in U,$$

where

$$\gamma = \gamma(\alpha) = 2\alpha - 1 \,+\, \frac{2(1-\alpha)}{\delta m}\beta\left(\frac{1}{\delta m}\right)$$

and

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t}dt\,.$$

Proof. Let $f \in \mathcal{A}_m$,

$$f(z) = z + \sum_{j=m+1}^{\infty} a_j z^j, \quad z \in U.$$

If

$$h(z) = \frac{1 + (2\alpha - 1)z}{1+z}\,, \quad z \in U,$$

then (2.1) is equivalent to

$$\left(\mathcal{D}_{\lambda\delta}^{n+1} f(z)\right)' + \frac{\lambda\delta z(\delta n + \delta - 1)\left(R^n f(z)\right)''}{(1-\lambda+\lambda\delta)(n+1)} \prec h(z), \quad z \in U. \tag{2.2}$$

Using the properties of $\mathcal{D}_{\lambda\delta}^n f$, $D_\delta^n f$ and $R^n f$, we obtain

$$\begin{aligned}
&\left(\mathcal{D}_{\lambda\delta}^{n+1} f(z)\right)' + \frac{\lambda\delta z(\delta n + \delta - 1)\left(R^n f(z)\right)''}{(1-\lambda+\lambda\delta)(n+1)} \\
= \;& \frac{[(1-\lambda)D_\delta^{n+1} f(z) + \lambda\delta R^{n+1} f(z)]'}{1-\lambda+\lambda\delta} + \frac{\lambda\delta z(\delta n + \delta - 1)\left(R^n f(z)\right)''}{(1-\lambda+\lambda\delta)(n+1)} \\
= \;& \frac{1-\lambda}{1-\lambda+\lambda\delta}\left[(D_\delta^n f(z))' + \delta z\left(D_\delta^n f(z)\right)''\right] \\
& + \frac{\lambda\delta}{1-\lambda+\lambda\delta}\left[\frac{z\left(R^n f(z)\right)' + nR^n f(z)}{n+1}\right]' + \frac{\lambda\delta z(\delta n + \delta - 1)\left(R^n f(z)\right)''}{(1-\lambda+\lambda\delta)(n+1)} \\
= \;& \frac{1-\lambda}{1-\lambda+\lambda\delta}\left(D_\delta^n f(z)\right)' + \frac{(1-\lambda)\delta}{1-\lambda+\lambda\delta} z\left(D_\delta^n f(z)\right)'' \\
& + \frac{\delta\lambda\left[\left(R^n f(z)\right)' + z\left(R^n f(z)\right)'' + n\left(R^n f(z)\right)'\right]}{(1-\lambda+\lambda\delta)(n+1)}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\lambda\delta z(\delta n+\delta-1)\left(R^n f(z)\right)''}{(1-\lambda+\lambda\delta)(n+1)} \\
=\;& \frac{1}{1-\lambda+\lambda\delta}\left[(1-\lambda)\left(D_\delta^n f(z)\right)'+\lambda\delta\left(R^n f(z)\right)'\right] \\
&+\delta z\frac{1}{1-\lambda+\lambda\delta}\left[(1-\lambda)\left(D_\delta^n f(z)\right)''+\lambda\delta\left(R^n f(z)\right)''\right] \\
=\;& \left(\mathcal{D}_{\lambda\delta}^n f(z)\right)'+\delta z\left(\mathcal{D}_{\lambda\delta}^n f(z)\right)'', \quad z\in U. \qquad (2.3)
\end{aligned}
$$

Then, from (2.2) and (2.3), we have

$$\left(\mathcal{D}_{\lambda\delta}^n f(z)\right)'+\delta z\left(\mathcal{D}_{\lambda\delta}^n f(z)\right)'' \prec h(z), \quad z\in U. \qquad (2.4)$$

Let

$$p(z)=\left(\mathcal{D}_{\lambda\delta}^n f(z)\right)', \quad z\in U. \qquad (2.5)$$

In view of (1.9), we get

$$
\begin{aligned}
p(z) &= 1+\frac{1}{1-\lambda+\lambda\delta}\sum_{j=m+1}^{\infty}\left[(1-\lambda)\left(1+(j-1)\delta\right)^n+\lambda\delta C_{n+j-1}^n\right] j a_j z^{j-1} \\
&= 1+b_m z^m+b_{m+1}z^{m+1}+\dots, \quad z\in U.
\end{aligned}
$$

and from (2.4), we obtain

$$p(z)+\delta z p'(z)\prec h(z), \quad z\in U. \qquad (2.6)$$

By applying now Lemma 1.1, we have

$$p(z)\prec q(z)\prec h(z), \quad z\in U,$$

where

$$
\begin{aligned}
q(z) &= \frac{1}{\delta m z^{1/\delta m}}\int_0^z h(t)t^{\frac{1}{\delta m}-1}dt \\
&= \frac{1}{\delta m z^{1/\delta m}}\int_0^z \left[2\alpha-1+2(1-\alpha)\frac{1}{1+t}\right] t^{\frac{1}{\delta m}-1}dt \\
&= \frac{2\alpha-1}{\delta m z^{1/\delta m}}\int_0^z t^{\frac{1}{\delta m}-1}dt \;+\; \frac{2(1-\alpha)}{\delta m z^{1/\delta m}}\int_0^z \frac{t^{\frac{1}{\delta m}-1}}{1+t}dt \\
&= 2\alpha-1\;+\;\frac{2(1-\alpha)}{\delta m z^{1/\delta m}}\int_0^z \frac{t^{\frac{1}{\delta m}-1}}{1+t}dt, \quad z\in U.
\end{aligned}
$$

The function q is convex, it is the best dominant and because $q(U)$ is symmetric with respect to the real axis, we get

$$\operatorname{Re}\,\left(\mathcal{D}_{\lambda\delta}^n f(z)\right)'=\operatorname{Re}p(z)>\operatorname{Re}q(1)=\gamma(\alpha)=2\alpha-1\;+\;\frac{2(1-\alpha)}{\delta m}\beta\left(\frac{1}{\delta m}\right).$$

□

Example 2.2. If $f\in\mathcal{A}$, $n=1$, $\lambda=1/2$, $\delta=1$ and $\alpha=1/2$, then $\gamma(\alpha)=\ln 2$ and the inequality

$$\operatorname{Re}\left[f'(z)+3zf''(z)+z^2f'''(z)\right]>\frac{1}{2}, \quad z\in U,$$

implies that

$$\mathrm{Re}\,[f'(z)+zf''(z)] > \ln 2, \quad z \in U.$$

Theorem 2.3. *Let $m \in \mathbb{N}$, $\delta > 0$ and let r be a convex function with $r(0) = 1$ and h a function such that*

$$h(z) = r(z) + m\delta z r'(z)\,, \quad z \in U.$$

If $f \in \mathcal{A}_m$, then the following subordination

$$\left(\mathcal{D}_{\lambda\delta}^{n+1} f(z)\right)' + \frac{\lambda\delta z(\delta n+\delta-1)\left(R^n f(z)\right)''}{(1-\lambda+\lambda\delta)(n+1)} \prec h(z) = r(z) + m\delta z r'(z)\,, \quad z \in U \tag{2.7}$$

implies that

$$(\mathcal{D}_{\lambda\delta}^{n} f(z))' \prec r(z)\,, \quad z \in U,$$

and the result is sharp.

Proof. By using (2.3) and (2.5), the subordination (2.7) is equivalent to

$$p(z) + \delta z p'(z) \prec h(z) = r(z) + m\delta z r'(z)\,, \quad z \in U.$$

Hence, from Lemma 1.2, we conclude that

$$p(z) \prec r(z)\,, \quad z \in U,$$

that is,

$$(\mathcal{D}_{\lambda\delta}^{n} f(z))' \prec r(z)\,, \quad z \in U,$$

and the result is sharp. □

Theorem 2.4. *Let $m \in \mathbb{N}$ and let r be a convex function with $r(0) = 1$ and h a function such that*

$$h(z) = r(z) + m z r'(z)\,, \quad z \in U.$$

If $f \in \mathcal{A}_m$, then the following subordination

$$(\mathcal{D}_{\lambda\delta}^{n} f(z))' \prec h(z) = r(z) + m z r'(z)\,, \quad z \in U \tag{2.8}$$

implies that

$$\frac{\mathcal{D}_{\lambda\delta}^{n} f(z)}{z} \prec r(z)\,, \quad z \in U,$$

and the result is sharp.

Proof. Let

$$p(z) = \frac{\mathcal{D}_{\lambda\delta}^{n} f(z)}{z}\,, \quad z \in U. \tag{2.9}$$

Differentiating (2.9), we have

$$(\mathcal{D}_{\lambda\delta}^{n} f(z))' = p(z) + z p'(z)\,, \quad z \in U,$$

and consequently, (2.8) becomes

$$p(z) + z p'(z) \prec h(z) = r(z) + m z r'(z)\,, \quad z \in U.$$

Hence, by applying Lemma 1.2, we conclude that

$$p(z) \prec r(z)\,, \quad z \in U,$$

that is,

$$\frac{\mathcal{D}^n_{\lambda\delta} f(z)}{z} \prec r(z)\,, \quad z \in U,$$

and the result is sharp. □

Remark 2.5. For $m = 1$ and $\delta = 1$, the above theorems were obtained by G. I. Oros and G. Oros in [4].

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