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# Perov's fixed point theorem for multivalued mappings in generalized Kasahara spaces

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**Abstract.** In this paper we give some corresponding results to Perov's fixed point theorem which was given in a complete generalized metric space. Our results will be given in a more general space, the so called generalized Kasahara space. We will also use the case of multivalued operators and give some fixed point results for multivalued Kannan, Reich and Caristi operators.

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## 1. Introduction and preliminaries

The classical Banach contraction principle was extended for contractive maps on spaces endowed with vector-valued metrics by Perov in 1964 (see [5]).We recall some notions regarding Perov's result.
Let $X$ be a nonempty set and $m \in \mathbb{N}$, $m \geq 1$. A mapping $d : X \times X \to \mathbb{R}^m$ is called a vector-valued metric on $X$ if the following statements are satisfied for all $x, y, z \in X$:

$d_1$) $d(x, y) \geq 0_m$, where $0_m := (0, 0, \ldots, 0) \in \mathbb{R}^m$;
$d_2$) $d(x, y) = 0_m \Rightarrow x = y$;
$d_3$) $d(x, y) = d(y, x)$;
$d_4$) $d(x, y) \leq d(x, z) + d(z, y)$.

We mention that if $\alpha, \beta \in \mathbb{R}^m$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$ and $c \in \mathbb{R}$ , then by $\alpha \leq \beta$ (respectively $\alpha < \beta$), we mean that $\alpha_i \leq \beta_i$ (respectively $\alpha_i < \beta_i$), for all $i = \overline{1, m}$ and by $\alpha \leq c$ we mean that $\alpha_i \leq c$, for all $i = \overline{1, m}$.

A set $X$ equipped with a vector-valued metric $d$ is called a generalized metric space. We will denote such a space with $(X, d)$. For generalized metric spaces, the notions of convergent sequence, Cauchy sequence, completeness, open subset and closed subset are similar to those for usual metric spaces.

Throughout this paper we denote by $\mathcal{M}_{m,m}(\mathbb{R}_+)$ the set of all $m \times m$ matrices with positive elements, by $\Theta$ the zero $m \times m$ matrix and by $I_m$ the identity $m \times m$ matrix. If $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$, then the symbol $A^\tau$ stands for the transpose matrix of $A$. Notice also that, for the sake of simplicity, we will make an identification between row and column vectors in $\mathbb{R}^m$.

A matrix $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ is said to be convergent to zero if and only if $A^n \to \Theta$ as $n \to \infty$ (see [11]). Regarding this class of matrices we have the following classical result in matrix analysis (see [1](Lemma 3.3.1, page 55), [6], [7](page 37), [11](page 12). More considerations can be found in [10].

**Theorem 1.1.** *Let $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$. The following statements are equivalent:*

i) $A^n \to \Theta$, *as* $n \to \infty$;

ii) *the eigenvalues of $A$ lies in the open unit disc, i.e., $|\lambda| < 1$, for all $\lambda \in \mathbb{C}$ with $det(A - \lambda I_m) = 0$;*

iii) *the matrix $I_m - A$ is non-singular and*

$$(I_m - A)^{-1} = I_m + A + A^2 + \ldots + A^n + \ldots;$$

iv) *the matrix $(I_m - A)$ is non-singular and $(I_m - A)^{-1}$ has nonnegative elements;*

v) *the matrices $Aq$ and $q^\tau A$ converges to zero for each $q \in \mathbb{R}^m$.*

The main result for self contractions on generalized metric spaces is Perov's fixed point theorem (see [5]):

**Theorem 1.2 (A.I. Perov).** *Let $(X, d)$ be a complete generalized metric space and the mapping $f : X \to X$ with the property that there exists a matrix $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ such that $d(f(x), f(y)) \leq Ad(x, y)$, for all $x, y \in X$. If $A$ is a matrix convergent to zero, then*

$p_1$) *there exists a unique $x^* \in X$ such that $x^* = f(x^*)$, i.e., the mapping $f$ has a unique fixed point;*

$p_2$) *the sequence of successive approximations $(x_n)_{n\in\mathbb{N}} \subset X$, $x_n = f^n(x_0)$ is convergent and it has the limit $x^*$, for all $x_0 \in X$;*

$p_3$) $d(x_n, x^*) \leq A^n(I_m - A)^{-1}d(x_0, x_1)$, *for all* $n \in \mathbb{N}$;

$p_4$) *if $g : X \to X$ satisfies the condition $d(f(x), g(x)) \leq \eta$, for all $x \in X$ and $\eta \in \mathbb{R}^m$, then by considering the sequence $(y_n)_{n\in\mathbb{N}} \subset X$, $y_n = g^n(x_0)$ one has $d(y_n, x^*) \leq (I_m - A)^{-1}\eta + A^n(I_m - A)^{-1}d(x_0, x_1)$, for all $n \in \mathbb{N}$.*

In this paper we give some corresponding results to Perov fixed point theorem. We will use the multivalued operators and we will adapt Perov's result to the context of generalized Kasahara spaces. In order to do this, we recall the following notions and results:

**Definition 1.3 (see [8]).** *Let $X$ be a nonempty set, $\to$ be an $L$-space structure on $X$, $(G, +, \leq, \xrightarrow{G})$ be an $L$-space ordered semigroup with unity, $0$ be the least element in $(G, \leq)$ and $d_G : X \times X \to G$ be an operator. The triple $(X, \to, d_G)$ is called a generalized Kasahara space if and only if the following compatibility* *condition between* $\to$ *and* $d_G$ *holds:*

$$\text{for all } (x_n)_{n\in\mathbb{N}} \subset X \text{ with } \sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) < +\infty$$
$$\Rightarrow (x_n)_{n\in\mathbb{N}} \text{ is convergent in } (X, \to).$$

**Example 1.4.** Let $\rho : X \times X \to \mathbb{R}^m_+$ be a generalized complete metric on a set $X$. Let $x_0 \in X$ and $\lambda \in \mathbb{R}^m_+$ with $\lambda \neq 0$. Let $d_\lambda : X \times X \to \mathbb{R}^m_+$ be defined by

$$d_\lambda(x, y) = \begin{cases} \rho(x, y) & , \text{ if } x \neq x_0 \text{ and } y \neq x_0, \\ \lambda & , \text{ if } x = x_0 \text{ or } y = x_0. \end{cases}$$

Then $(X, \xrightarrow{\rho}, d_\lambda)$ is a generalized Kasahara space.

In [3], S. Kasahara gives a useful tool which is used in proving the uniqueness of a fixed point.

**Lemma 1.5.** *Let* $(X, \to, d_G)$ *be a generalized Kasahara space. Then*

$$\text{for all } x, y \in X \text{ with } d_G(x, y) = d_G(y, x) = 0 \Rightarrow x = y.$$

For more considerations on generalized Kasahara spaces, see [8] and the references therein.

Through this paper, we consider $G = \mathbb{R}^m$. The functional $d_G$ will be denoted by $d$, which is not necessary a metric on $X$. In other words, we will consider the generalized Kasahara space $(X, \to, d)$ where $d : X \times X \to \mathbb{R}^m_+$ is a functional.

Finally, in the above setting, for a multivalued operator $F : X \multimap X$, we shall use the following notations:

$m_1$) $P(X) := \{Y \subset X \mid Y \neq \emptyset\}$, so $F : X \to P(X)$;

$m_2$) $Fix(F) := \{x^* \in X \mid x^* \in F(x^*)\}$, the set of all fixed points for $F$. For simplicity, we will use the notation $Fx$ instead of $F(x)$, where $x \in X$;

$m_3$) $Graph(F) = \{(x, y) \in X \times X \mid y \in Fx\}$, the graph of $F$.
We say that $F$ has closed graph, if and only if $Graph(F)$ is closed in $X \times X$ with respect to $\to$, i.e., if $(x_n)_{n\in\mathbb{N}} \subset X$ and $y_n \in Fx_n$, for all $n \in \mathbb{N}$ with $x_n \to x^* \in X$, as $n \to \infty$ and if $y_n \to y^*$, as $n \to \infty$ then $y^* \in Fx^*$.

## 2. Main results

**Theorem 2.1.** *Let* $(X, \to, d)$ *be a generalized Kasahara space and* $F : X \to P(X)$ *be a multivalued operator. We assume that:*

i) *there exists* $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ *and for all* $x, y \in X$ *and* $u \in Fx$, *there exists* $v \in Fy$ *such that*
$$d(u, v) \le Ad(x, y);$$

ii) $Graph(F)$ *is closed in* $X \times X$ *with respect to* $\to$.

*If $A$ converges to zero, then $Fix(F) \neq \emptyset$. If, in addition, $(I_m - A)$ is non-singular, $(I_m - A)^{-1} \in \mathcal{M}_{m\times m}(\mathbb{R}_+)$ and*

$$\max\{d(u,v) \mid u \in Fx,\ v \in Fy\} \leq Ad(x,y),\ \textit{for all } x,y \in X$$

*then $F$ has a unique fixed point in $X$.*

*Proof.* Let $x_0 \in X$ and $x_1 \in Fx_0$. If $x_1 = x_0$ then $x_0 \in Fix(F)$. We assume that $x_1 \neq x_0$. Then by $i)$ there exists $x_2 \in Fx_1$ such that

$$d(x_1,x_2) \leq Ad(x_0,x_1).$$

Since $x_2 \in Fx_1$, if $x_2 = x_1$ then $x_1 \in Fix(F)$. If we consider $x_2 \neq x_1$ then there exists $x_3 \in Fx_2$ such that

$$d(x_2,x_3) \leq Ad(x_1,x_2) \leq A^2 d(x_0,x_1).$$

By induction, we construct the sequence of successive approximations for $F$ starting from $(x_0,x_1) \in Graph(F)$. This sequence has the following properties:

1°) $x_{n+1} \in Fx_n$, for all $n \in \mathbb{N}$;
2°) $d(x_n,x_{n+1}) \leq A^n d(x_0,x_1)$, for all $n \in \mathbb{N}$.

Next, we have the following estimation:

$$\sum_{n\in\mathbb{N}} d(x_n,x_{n+1}) \leq \sum_{n\in\mathbb{N}} A^n d(x_0,x_1) = (I_m - A)^{-1} d(x_0,x_1) < +\infty.$$

Since $(X, \to, d)$ is a generalized Kasahara space, the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent in $X$ with respect to $\to$. Hence there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. On the other hand, $F$ has closed graph, so $x^* \in Fix(F)$.

We prove now the uniqueness of the fixed point $x^*$.

Let $x^*, y^* \in Fix(F)$ such that $x^* \neq y^*$. Since $x^* \in Fx^*$ and $y^* \in Fy^*$, we get that

$$d(x^*,y^*) \leq \max_{\substack{u\in Fx^*\\ v\in Fy^*}} d(u,v) \leq Ad(x^*,y^*) \Leftrightarrow (I_m - A)d(x^*,y^*) \leq 0_m.$$

Since $I_m - A$ is a non-singular matrix and $(I_m - A)^{-1}$ has non-negative elements, it follows that $d(x^*,y^*) = 0_m$. By the same way of proof, we get that $d(y^*,x^*) = 0_m$. By Lemma 1.5, we obtain $x^* = y^*$. □

**Remark 2.2.** Let $X$ be a nonempty set and $\rho : X \times X \to \mathbb{R}^m_+$ be a complete generalized metric on $X$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $X$ and let $x \in X$. Then

$$x_n \xrightarrow{\rho} x \ \Leftrightarrow\ \rho(x_n,x) \to 0_m,\ \text{as } n \to \infty.$$

We have the following Maia type result:

**Corollary 2.3.** *Let $X$ be a nonempty set and $\rho : X \times X \to \mathbb{R}^m_+$ be a complete generalized metric on $X$. Let $d : X \times X \to \mathbb{R}^m_+$ be a functional and $F : X \to P(X)$ be a multivalued operator. We assume that*

i) *there exists* $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ *and for all* $x, y \in X$ *and* $u \in Fx$, *there exists* $v \in Fy$ *such that*

$$d(u,v) \le Ad(x,y);$$

ii) *Graph*$(F)$ *is closed in* $X \times X$ *with respect to* $\xrightarrow{\rho}$;

iii) *there exists* $c > 0$ *such that* $\rho(x,y) \le c \cdot d(x,y)$.

*Then the following statements hold:*

1) *if* $A$ *converges to zero, then* $Fix(F) \neq \emptyset$. *If, in addition,* $(I_m - A)$ *is non-singular,* $(I_m - A)^{-1} \in \mathcal{M}_{m\times m}(\mathbb{R}_+)$ *and*

$$\max\{d(u,v) \mid u \in Fx,\ v \in Fy\} \le Ad(x,y),\ \textit{for all } x, y \in X$$

*then* $F$ *has a unique fixed point in* $X$.

2) $\rho(x_n, x^*) \le c \cdot A^n (I_m - A)^{-1} d(x_0, x_1)$, *for all* $n \in \mathbb{N}$, *where* $x^* \in Fix(F)$ *and* $(x_n)_{n\in\mathbb{N}}$ *is the sequence of successive approximations for* $F$ *starting from* $(x_0, x_1) \in Graph(F)$.

*Proof.* By $i)$ and by following the proof of Theorem 2.1, we can construct a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for $F$ starting from $(x_0, x_1) \in Graph(F)$ such that $x_{n+1} \in Fx_n$ and $d(x_n, x_{n+1}) \le A^n d(x_0, x_1)$, for all $n \in \mathbb{N}$. By $iii)$ there exists $c > 0$ such that

$$\rho(x_n, x_{n+1}) \le c \cdot d(x_n, x_{n+1}) \le c \cdot A^n d(x_0, x_1),\ \text{for all } n \in \mathbb{N}.$$

Now let $p \in \mathbb{N}$, $p > 0$. Since $\rho$ is a metric, we have that

$$\begin{aligned}\rho(x_n, x_{n+p}) &\le \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \ldots + \rho(x_{n+p-1}, x_{n+p})\\ &\le c \cdot A^n d(x_0, x_1) + c \cdot A^{n+1} d(x_0, x_1) + \ldots + c \cdot A^{n+p-1} d(x_0, x_1).\end{aligned}$$

Thus, for all $n, p \in \mathbb{N}$ with $p > 0$, the following estimation holds

$$\rho(x_n, x_{n+p}) \le c \cdot A^n (I_m + A + \ldots + A^{p-1}) d(x_0, x_1). \qquad (2.1)$$

By letting $n \to \infty$, we get that $\rho(x_n, x_{n+p}) \to 0_m$, so $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete generalized metric space $(X, \rho)$. Therefore $(x_n)_{n\in\mathbb{N}}$ is convergent in $(X, \rho)$, so there exists $x^* \in X$ such that $x_n \xrightarrow{\rho} x$.

By $ii)$ it follows that $x^* \in Fix(F)$. The uniqueness of the fixed point $x^*$ follows from Theorem 2.1.

By letting $p \to \infty$ in (2.1), we get the estimation mentioned in the conclusion 2) of the corollary. □

**Corollary 2.4.** *Let* $(X, \to, d)$ *be a generalized Kasahara space where* $d$ *satisfies* $d(x,x) = 0_m$, *for all* $x \in X$. *Let* $F : X \to P(X)$ *be a multivalued operator. We assume that:*

i) *there exists* $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$, $B \in \mathcal{M}_{m,m}(\mathbb{R})$ *and for all* $x, y \in X$ *and* $u \in Fx$, *there exists* $v \in Fv$ *such that*

$$d(u,v) \le Ad(x,y) + Bd(y,u);$$

ii) *Graph*$(F)$ *is closed in* $X \times X$ *with respect to* $\to$.

*If* $A$ *converges to zero, then* $F$ *has at least one fixed point in* $X$.

*Proof.* Let $x_0 \in X$ and $x_1 \in Fx_0$. If $x_1 = x_0$ then $x_0 \in Fix(F)$. We assume that $x_1 \neq x_0$. Then by $i)$ there exists $x_2 \in Fx_1$ such that

$$d(x_1, x_2) \leq Ad(x_0, x_1) + Bd(x_1, x_1) = Ad(x_0, x_1).$$

By following the proof of Theorem 2.1, the conclusion follows. □

As an application of the previous results we present an existence theorem for a semi-linear inclusion systems.

**Theorem 2.5.** *Let $\varphi, \psi : [0,1]^2 \to ]0, \frac{1}{2}]$ be two functions and $F_1, F_2 : [0,1]^2 \to P([0,1])$ be two multivalued operators defined as follows:*

$$F_1(x_1, x_2) = \left[\varphi(x_1, x_2), \frac{1}{2} + \varphi(x_1, x_2)\right] \text{ and}$$
$$F_2(x_1, x_2) = \left[\psi(x_1, x_2), \frac{1}{2} + \psi(x_1, x_2)\right].$$

*We assume that for each $(x_1, x_2), (y_1, y_2) \in [0,1]^2$ and each $u_1 \in F_1(x_1, x_2)$, $u_2 \in F_2(x_1, x_2)$, there exist $v_1 \in F_1(y_1, y_2)$ and $v_2 \in F_2(y_1, y_2)$ such that*

$$|u_1 - v_1| \leq a|x_1 - y_1| + b|x_2 - y_2|,$$
$$|u_2 - v_2| \leq c|x_1 - y_1| + d|x_2 - y_2|,$$

*for all $a, b, c, d \in \mathbb{R}_+$ with $|a + d \pm \sqrt{(a-d)^2 + 4bc}| < 2$.*

*Then the system*

$$\begin{cases} x_1 \in F_1(x_1, x_2) \\ x_2 \in F_2(x_1, x_2), \end{cases} \tag{2.2}$$

*has at least one solution in $[0,1]^2$.*

*Proof.* Let $F := (F_1, F_2) : [0,1]^2 \to P([0,1]^2)$. Then the system (2.2) can be represented as a fixed point problem of the form

$$x \in Fx, \text{ where } x = (x_1, x_2) \in [0,1]^2.$$

We consider the generalized Kasahara space $([0,1]^2, \xrightarrow{\rho_e}, d)$ where:

i) $\rho_e : [0,1]^2 \times [0,1]^2 \to \mathbb{R}^2_+$ is defined by

$$\rho_e(x, y) = (|x_1 - y_1|, |x_2 - y_2|),$$

for all $x = (x_1, x_2), y = (y_1, y_2) \in [0,1]^2$;

ii) $d : [0,1]^2 \times [0,1]^2 \to \mathbb{R}^2_+$ is defined by

$$d(x, y) = \begin{cases} \rho_e(x, y) & , \ x \neq \theta \text{ and } y \neq \theta \\ (1, 1) & , \ x = \theta \text{ or } y = \theta \end{cases},$$

for all $x = (x_1, x_2), y = (y_1, y_2) \in [0,1]^2$, where $\theta = (0, 0)$.

For each $x = (x_1, x_2), y = (y_1, y_2) \in [0, 1]^2$ and $u = (u_1, u_2) \in Fx$, there exists $v = (v_1, v_2) \in Fy$ such that

$$d(u, v) \leq Ad(x, y),$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix convergent to zero, having its eigenvalues in the open unit disc.

Since $Graph(F)$ is closed in $[0, 1]^2$ w.r.t. $\xrightarrow{\rho_e}$, Theorem 2.1 holds. $\square$

**Remark 2.6.** Some examples of matrix convergent to zero are:

a) any matrix $A = \begin{pmatrix} a & a \\ b & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $a + b < 1$;

b) any matrix $A = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $a + b < 1$;

c) any matrix $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $\max\{a, c\} < 1$;

In what follows, we present some results regarding the fixed points for multivalued Kannan and Reich operators. For our proofs, we will need the following result:

**Lemma 2.7.** *Let $A = (a_{ij})_{i,j=\overline{1,m}} \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ be a triangular matrix with*

$$\max\left\{a_{ii} \mid i = \overline{1, m}\right\} < \frac{1}{2}.$$

*Then the matrix $\Lambda = (I_m - A)^{-1}A$ is convergent to zero.*

*Proof.* Suppose that $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ 0 & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & a_{mm} \end{pmatrix} \in \mathcal{M}_{m,m}(\mathbb{R}_+)$. Then the eigenvalues of $\Lambda$ are $\lambda_i = \frac{a_{ii}}{1-a_{ii}}$, for all $i = \overline{1, m}$. Since all of the eigenvalues of $\Lambda$ are in the open unit disc, the conclusion follows from Theorem 1.1. $\square$

A result for multivalued Kannan operators is presented bellow:

**Theorem 2.8.** *Let $(X, \rightarrow, d)$ be a generalized Kasahara space and $F : X \rightarrow P(X)$ be a multivalued operator. We assume that:*

i) *there exists $A = (a_{ij})_{i,j=\overline{1,m}} \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ a triangular matrix such that $\max\limits_{i=\overline{1,m}} a_{ii} < \frac{1}{2}$ and for all $x, y \in X$ and $u \in Fx$, there exists $v \in Fy$ such that*

$$d(u, v) \leq A[d(x, u) + d(y, v)];$$

ii) *$Graph(F)$ is closed in $X \times X$ with respect to $\rightarrow$.*

*Then $F$ has at least one fixed point in $X$.*

*Proof.* Let $x_0 \in X$ and $x_1 \in Fx_0$. If $x_1 = x_0$, then we already have a fixed point for $F$ ($x_0 \in Fix(F)$). Assuming that $x_1 \neq x_0$, then by $i$), there exists $x_2 \in Fx_1$ such that

$$d(x_1, x_2) \leq A[d(x_0, x_1) + d(x_1, x_2)] \Leftrightarrow d(x_1, x_2) \leq (I_m - A)^{-1} A d(x_0, x_1).$$

We denote $\Lambda = (I_m - A)^{-1} A$ and we have

$$d(x_1, x_2) \leq \Lambda d(x_0, x_1).$$

By taking into account Lemma 2.7 and by following the proof of Theorem 2.1, replacing $A$ with $\Lambda$, the conclusion follows. $\square$

Next we present a result regarding the fixed points for the multivalued operators of Reich type:

**Theorem 2.9.** *Let $(X, \to, d)$ be a generalized Kasahara space and $F : X \to P(X)$ be a multivalued operator. We assume that:*

i) *there exist $A = (a_{ij})_{i,j=\overline{1,m}}$, $B = (b_{ij})_{i,j=\overline{1,m}}$, $C = (c_{ij})_{i,j=\overline{1,m}} \in \mathcal{M}_{m,m}(\mathbb{R}_+)$, where*

   1) *$C$ is a triangular matrix with $\max\limits_{i=\overline{1,m}} c_{ii} < \frac{1}{2}$*

   2) *$A + B \leq C$, i.e., $a_{ij} + b_{ij} \leq c_{ij}$, for all $i, j = \overline{1,m}$*

   *and for all $x, y \in X$ and $u \in Fx$, there exists $v \in Fy$ such that*

$$d(u, v) \leq Ad(x, y) + Bd(x, u) + Cd(y, v);$$

ii) *$Graph(F)$ is closed in $X \times X$ with respect to $\to$.*

*Then $F$ has at least one fixed point in $X$.*

*Proof.* Let $x_0 \in X$ and $x_1 \in Fx_0$. If $x_1 = x_0$, then we already have a fixed point for $F$ ($x_0 \in Fix(F)$). Assuming that $x_1 \neq x_0$, then by $i$), there exists $x_2 \in Fx_1$ such that

$$d(x_1, x_2) \leq Ad(x_0, x_1) + Bd(x_0, x_1) + Cd(x_1, x_2)$$
$$\Leftrightarrow d(x_1, x_2) \leq (I_m - C)^{-1}(A + B)d(x_0, x_1) \leq (I_m - C)^{-1} C d(x_0, x_1).$$

We denote $\Lambda = (I_m - C)^{-1} C$. By taking into account Lemma 2.7 and by following the proof of Theorem 2.1, replacing $A$ with $\Lambda$, the conclusion follows. $\square$

Some other fixed point results can be established for the multivalued Caristi operators:

**Definition 2.10.** *Let $(X, \to, d)$ be a generalized Kasahara space and $F : X \to P(X)$ be a multivalued operator. Let $\varphi : X \to \mathbb{R}^m_+$ be a functional. We say that $F$ is a multivalued Caristi operator if for all $x \in X$, there exists $y \in Fx$ such that*

$$d(x, y) \leq \varphi(x) - \varphi(y).$$

For more considerations on multivalued Caristi operators see [4] and [2].

**Theorem 2.11.** *Let $(X, \to, d)$ be a generalized Kasahara space and $F : X \to P(X)$ be a multivalued Caristi operator, having closed graph with respect to $\to$. Then $F$ has at least one fixed point in $X$.*

*Proof.* Let $x_0 \in X$. Then there exists $x_1 \in Fx_0$. If $x_1 = x_0$ then $x_0 \in Fix(F)$ and the proof is complete. If $x_1 \neq x_0$ then

$$d(x_0, x_1) \leq \varphi(x_0) - \varphi(x_1).$$

Since $x_1 \in Fx_0$, there exists $x_2 \in Fx_1$. If $x_2 = x_1$ then $x_1 \in Fix(F)$ and the proof is complete. If $x_2 \neq x_1$ then

$$d(x_1, x_2) \leq \varphi(x_1) - \varphi(x_2).$$

By induction, there exists $x_{n+1} \in Fx_n$ such that

$$d(x_n, x_{n+1}) \leq \varphi(x_n) - \varphi(x_{n+1}), \text{ for all } n \in \mathbb{N}.$$

We have the following estimations

$$\sum_{n \in \mathbb{N}} d(x_n, x_{n+1}) \leq \varphi(x_0) - \varphi(x_{n+1}) \leq \varphi(x_0) < +\infty.$$

Since $(X, \to, d)$ is a Kasahara space, the sequence $(x_n)_{n \in \mathbb{N}}$ is convergent in $(X, \to)$. So there exists $x^* \in X$ such that $x_n \to x^*$, as $n \to \infty$.

Since $Graph(F)$ is closed, $x^* \in Fix(F)$. □

By taking into account the Remark 2.2, we have the following result:

**Corollary 2.12.** *Let $X$ be a nonempty set and $\rho : X \times X \to \mathbb{R}^m_+$ be a complete generalized metric on $X$. Let $d : X \times X \to \mathbb{R}^m_+$ be a functional. Let $\varphi : X \to \mathbb{R}^m_+$ be a functional.*

*Let $F : X \to P(X)$ be a multivalued operator such that*

i) *$Graph(F)$ is closed in $X \times X$ with respect to $\xrightarrow{\rho}$;*
ii) *for all $x \in X$, there exists $y \in Fx$ such that $d(x, y) \leq \varphi(x) - \varphi(y)$;*
iii) *there exists $c > 0$ such that $\rho(x, y) \leq c \cdot d(x, y)$.*

*Then $F$ has at least one fixed point in $X$.*

*Proof.* By *ii)* and the proof of the Theorem 2.11, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in $X$ such that

1) $x_{n+1} \in Fx_n$, for all $n \in \mathbb{N}$;
2) $d(x_n, x_{n+1}) \leq \varphi(x_n) - \varphi(x_{n+1})$, for all $n \in \mathbb{N}$.

By *iii)* there exists $c > 0$ such that

$$\rho(x_n, x_{n+1}) \leq c \cdot d(x_n, x_{n+1}) \leq c \cdot (\varphi(x_n) - \varphi(x_{n+1})), \text{ for all } n \in \mathbb{N}.$$

We will prove that the series $\sum_{n \in \mathbb{N}} \rho(x_n, x_{n+1})$ is convergent. For this purpose, we need to show that the sequence of its partial sums is convergent in $\mathbb{R}^m_+$.

Denote by $s_n = \sum_{k=0}^{n} \rho(x_k, x_{k+1})$. Then $s_{n+1} - s_n = \rho(x_{n+1}, x_{n+2}) \geq 0$, for each $n \in \mathbb{N}$. Moreover $s_n \leq \sum_{k=0}^{n} \left[c\varphi(x_k) - c\varphi(x_{k+1})\right] \leq c\varphi(x_0)$. Hence $(s_n)_{n\in\mathbb{N}}$ is upper bounded and increasing in $\mathbb{R}^m_+$. So the sequence $(s_n)_{n\in\mathbb{N}}$ is convergent in $\mathbb{R}^m_+$. It follows that the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and, from the completeness of the metric space $(X, \rho)$, convergent to a certain element $x^* \in X$. The conclusion follows from $i)$. □

For more considerations on multivalued Kannan, Reich and Caristi operators, see [9] and the references therein.

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