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On the approximation of the constant of Napier

Andrei Vernescu

Abstract. Starting from some older ideas of [12] and [6], we show new facts concerning the approximation of the constant of Napier.

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1. Introduction

Consider the two equivalent classical definitions of the real exponential function

$$\mathrm{e}^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots \frac{x^n}{n!} + \dots \tag{1.1}$$

respectively

$$\mathrm{e}^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n, \tag{1.2}$$

both convergences being uniform on compact subsets of $\mathbb{R}$.
Their speed of convergence is different. Concerning the Taylor-Maclaurin approximation (1.1) of the exponential, see D. S. Mitrinović [3], pp. 268-269. For the approximation given by (1.2), also in this classical book are given the following inequalities

$$0 \le \mathrm{e}^x - \left(1 + \frac{x}{n}\right)^n \le \frac{x^2\mathrm{e}^x}{n}, \quad \text{for } |x| < n \text{ and } n \in \mathbb{N}^*;$$
$$0 \le \mathrm{e}^{-x} - \left(1 - \frac{x}{n}\right)^n \le \frac{x^2\left(1+x\right)\mathrm{e}^{-x}}{2n}, \quad \text{for } 0 \le x < n,\ n \in \mathbb{N},\ n \ge 2;$$
$$0 \le \mathrm{e}^{-x} - \left(1 - \frac{x}{n}\right)^n \le \frac{x^2}{2n}, \quad \text{for } 0 \le x \le n \text{ and } n \in \mathbb{N}^*$$

(see [4], [5], [13], [14], [15]).

In [7] we gave some stronger inequalities, namely

i) If $x > 0$, $t > 0$ and $t > \frac{1-x}{2}$ then

$$\frac{x^2 e^x}{2t + x + \max\{x, x^2\}} < e^x - \left(1 + \frac{x}{t}\right)^t < \frac{x^2 e^x}{2t + x}. \tag{1.3}$$

ii) If $x > 0$, $t > 0$ and $t > \frac{x-1}{2}$ then

$$\frac{x^2 e^{-x}}{2t - x + x^2} < e^{-x} - \left(1 - \frac{x}{t}\right)^t < \frac{x^2 e^{-x}}{2t - 2x + \min\{x, x^2\}} \tag{1.4}$$

and we detailed the proof of (1.3) (for the proof of (1.4) see[12], pp. 258-260).

Also, note *en passant*, that the previous inequalities give by the simple particularization $x = 1$, the characterizations of the "speed" of convergence of four standard sequences related to the numbers e and $\frac{1}{e}$, namely [1)]

$$\frac{e}{2n+2} < e - \left(1 + \frac{1}{n}\right)^n < \frac{e}{2n+1} \quad \text{([8], pag. 38, [11])}$$
$$\frac{e}{2n+1} < \left(1 + \frac{1}{n}\right)^{n+1} - e < \frac{e}{2n} \quad \text{([10])}$$
$$\frac{1}{2ne} < \frac{1}{e} - \left(1 - \frac{1}{n}\right)^n < \frac{1}{(2n-1)e} \quad \text{([6], [7])}$$
$$\frac{1}{(2n-1)e} < \left(1 - \frac{1}{n}\right)^{n-1} - \frac{1}{e} < \frac{1}{(2n-2)e} \quad \text{([6], [7]).}$$

2. The main result

Now we will establish the best approximation of e by the family of sequences of general term $\left(1 + \frac{1}{n}\right)^{n+p}$, where p is a real parameter; this may suggest the best approximation of e^x, $x > 0$, by some algebraic functions.

Consider the known limited expansion

$$(1 + x)^{\frac{1}{x}} = e\left(1 - \frac{1}{2}x + \frac{11}{24}x^2 - \frac{7}{16}x^3\right) + O(x^4), \tag{2.1}$$

and also the limited binomial one

$$(1 + x)^p = 1 + \frac{p}{1!}x + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + O(x^4). \tag{2.2}$$

[1)] Using the notations $e_n = \left(1 + \frac{1}{n}\right)^n$, $f_n = \left(1 + \frac{1}{n}\right)^{n+1}$, $g_n = \left(1 - \frac{1}{n}\right)^n$, $h_n = \left(1 - \frac{1}{n}\right)^{n-1}$ and applying the GM-AM inequality for the numbers $a_1 = a_2 = a_3 = \ldots = a_n = 1 + \frac{1}{n}$, $a_{n+1} = 1$, we obtain that the sequence $(e_n)_n$ is strictly increasing (see [9]). Applying the GM-AM inequality for the numbers $b_1 = b_2 = b_3 = \ldots = b_n = 1 - \frac{1}{n}$, $b_{n+1} = 1$, we obtain analogously that the sequence $(g_n)_n$ is strictly increasing. The identities $f_n = \frac{1}{g_{n+1}}$ and $h_n = \frac{1}{e_{n-1}}$ show us that the sequences $(f_n)_n$ and $(h_n)_n$ are strictly decreasing. Therefore $e_n < e < f_n$ and $g_n < \frac{1}{e} < h_n$.

Remark. The formula (2.1) is can be obtained in a classical way, using the well-known limited expansions $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O(x^5)$ and $\exp y = 1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + O(y^5)$. Then

$$\frac{1}{e}(1+x)^{\frac{1}{x}} = \frac{1}{e}\exp\left(\frac{1}{x}\ln(1+x)\right) =$$

$$= \exp\left(\frac{1}{x}\ln(1+x) - 1\right) = \exp\left(-\frac{x}{2} + \frac{x^2}{2} - \frac{x^3}{4} + O(x^4)\right) =$$

$$= \left(\sum_{k=0}^{3}\frac{1}{k!}\left(-\frac{x}{2} + \frac{x^2}{2} - \frac{x^3}{4} + O(x^4)\right)^k\right) + O(x^4)$$

and some standard calculations give (2.1).

Multiplying (2.1) and (2.2), part by part, performing the usual calculations and replacing x by $\frac{1}{n}$ $(n = 1, 2, 3, \ldots)$, we obtain

$$\begin{aligned}\left(1+\tfrac{1}{n}\right)^{n+p} = \mathrm{e} + \left(p - \frac{1}{2}\right)\frac{\mathrm{e}}{n} + \frac{12p^2 - 24p + 11}{24}\cdot\frac{\mathrm{e}}{n^2} + \\ + \frac{8p^4 - 36p^2 + 50p - 21}{48}\cdot\frac{\mathrm{e}}{n^3} + O\left(\frac{1}{n^4}\right).\end{aligned} \tag{2.3}$$

From (2.3), we see that

$$\lim_{n\to\infty} n\left(\left(1+\frac{1}{n}\right)^{n+p} - \mathrm{e}\right) = \begin{cases} 0, & \text{for } p = \dfrac{1}{2} \\ \left(p - \dfrac{1}{2}\right)\mathrm{e} & \text{for } p \neq \dfrac{1}{2} \end{cases}. \tag{2.4}$$

For $p = \frac{1}{2}$ it results that the term in $\frac{1}{n}$ of (2.3) vanishes and we have

$$\left(1+\frac{1}{n}\right)^{n+1/2} = \mathrm{e} + \frac{\mathrm{e}}{12n^2} - \frac{\mathrm{e}}{12n^3} + O\left(\frac{1}{n^4}\right)$$

and so

$$n^2\left(\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} - \mathrm{e}\right) = \frac{\mathrm{e}}{12} - \frac{\mathrm{e}}{12n} + O\left(\frac{1}{n^2}\right),$$

which conducts us to the equality

$$\lim_{n\to\infty} n^2\left(\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} - \mathrm{e}\right) = \frac{\mathrm{e}}{12}. \tag{2.5}$$

Another way to obtain (2.5) consists in a (repeated) use of the *L'Hospital*'s rule, but this gives no idea of the provenance of the result.

So, the best approximation of e by the sequences of general term $\left(1+\frac{1}{n}\right)^{n+p}$ is the one corresponding to $p = \frac{1}{2}$.

3. A two-sided estimate

The equality (2.5) suggests us to search a two sided estimate of the form

$$\frac{\mathrm{e}}{12(n+\alpha)^2} < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} - \mathrm{e} < \frac{\mathrm{e}}{12(n+\beta)^2} \tag{3.1}$$

where α and β are two real constants.

Professor Ioan Gavrea communicated me ([1]) a convenient left part of (3.1), namely for $\alpha = \frac{1}{2}$, we have

$$\frac{\mathrm{e}}{12\left(n+\frac{1}{2}\right)^2} < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} - \mathrm{e}. \tag{3.2}$$

We present here his proof. Let

$$a_n = \frac{\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}}{\mathrm{e}}$$

be and $b_n = \ln a_n$, that is

$$b_n = \left(n+\frac{1}{2}\right)[\ln(n+1) - \ln n] - 1.$$

We have successively

$$b_n = \left(n+\frac{1}{2}\right)\left[\ln\left(n+\frac{1}{2}+\frac{1}{2}\right) - \ln\left(n+\frac{1}{2}-\frac{1}{2}\right)\right] - 1$$

$$= \left(n+\frac{1}{2}\right)\left[\ln\left(n+\frac{1}{2}\right)\left(1+\frac{1}{2\left(n+\frac{1}{2}\right)}\right) - \ln\left(n+\frac{1}{2}\right)\left(1-\frac{1}{2\left(n+\frac{1}{2}\right)}\right)\right] - 1$$

$$= \left(n+\frac{1}{2}\right)\left[\ln\left(1+\frac{1}{2\left(n+\frac{1}{2}\right)}\right) - \ln\left(1-\frac{1}{2\left(n+\frac{1}{2}\right)}\right)\right] - 1$$

$$= u\left[\ln\left(1+\frac{1}{2u}\right) - \ln\left(1-\frac{1}{2u}\right)\right] - 1,$$

where we have denoted $n + \frac{1}{2} = u$

Using now the well known expansions

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \quad |x| < 1$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots \quad |x| < 1$$

(uniform convergent in every compact $K \subset (-1,1)$) and performing the usual calculations, we obtain

$$b_n = 2n\left(\frac{1}{2n} + \frac{1}{3}\frac{1}{(2n)^3} + \frac{1}{5}\frac{1}{(2n)^5} + \ldots\right) - 1 = \frac{1}{12n^2} + \frac{1}{8n^4} + \ldots > \frac{1}{12n^2}$$

(because of $n > 0$). Therefore (using that $\mathrm{e}^x > 1 + x$, for $x > 0$) we have

$$\frac{\left(1+\frac{1}{n}\right)^{n+1/2}}{\mathrm{e}} = a_n = \mathrm{e}^{b_n} > \mathrm{e}^{\frac{1}{12u^2}} > 1 + \frac{1}{12u^2}$$

and so

$$\left(1+\frac{1}{n}\right)^{n+1/2} > \mathrm{e}\left(1 + \frac{1}{12\left(n+\frac{1}{2}\right)^n}\right),$$

that gives (3.2).

The problem of finding of an adequate constant β in (3.1) remains open.

4. Concluding remarks

The previous results, concerning the approximation of the number e by the sequence $\left(1+\frac{1}{n}\right)^{n+p}$ conduct to the idea to search a similar approximation of the exponential. We mention that an approximation of the exponential using the rational functions was given by J. Karamata (see [2]).

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Andrei Vernescu
"Valahia" University of Târgovişte
Department of Sciences
Bd. Unirii 18
130082 Târgovişte
Romania
e-mail: avernescu@clicknet.ro