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Estimates with optimal constants using Peetre's K-functionals of order 2

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Abstract. We present estimates of the degree of approximation by positive linear operators which preserve linear function, with the K-functionals K_2^s and $K_{2,\varphi}^s$, $1 \le s \le \infty$.

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1. Introduction

Estimates with the second order modulus ω_2 given by

$$\omega_2(f,t) = \sup_{|x-y|\le 2t} \left| f(x) - f\left(\frac{x+y}{2}\right) + f(y) \right|, \; f \in \mathbf{C}[a,b], \; t > 0$$

were established by H. Esser in 1976, G. Freud in 1978, H. Gonska in 1984 and R. Păltănea in 1995.

In [8] is given the following axiomatic definition for the modulus of continuity:

Definition 1.1. *Let X be a linear space of functions $f : I \longrightarrow \mathbb{R}$ ($I \subset \mathbb{R}$ an interval) who include the space of algebric polynomials of degree at most r denoted by Π_r, $r \in \mathbb{N}$. A function $\Omega_r : X \times (0,\infty) \longrightarrow [0,\infty) \cup \{\infty\}$ is called a modulus of continuity of order r on X if and only if the following axioms are satisfied*

1. $\Omega_r(f,t_1) \le \Omega_r(f,t_2)$ *if* $0 < t_1 < t_2$
2. $\Omega_r(f+p,t) = \Omega_r(f,t)$ *if* $p \in \Pi_{r-1}$
3. $\Omega_r(0,t) = 0$.

Moreover, if there exists a constant $M > 0$ such that $\Omega_r(e_r,t) \le Mt^r$ for all $t > 0$, then the modulus ω_r is called normalized.

There are established estimates with different second order moduli based on the following general result:

Theorem 1.2. [1] [8, p. 20]*Let* $[c,d] \subset [a,b]$, $L : \mathbf{C}[a,b] \longrightarrow \mathbf{C}[c,d]$ *a positive linear operator such that* $Le_0 = e_0$ *and* $Le_1 = e_1$, Ω_2 *a second order modulus on* $\mathbf{C}[a,b]$, $f \in \mathbf{C}[a,b]$, $t > 0$ *and* $x \in (c,d)$. *Suppose that there exists a function* $\psi : [0,\infty) \longrightarrow [0,\infty)$ *such that* $\psi\left(\frac{|e_1 - xe_0|}{t}\right) \in \mathbf{C}[a,b]$ *and*

$$|\Delta(f; x_1, x, x_2)| \leq \delta_t(\psi; x_1, x, x_2)\,\Omega_2(f,t),\ a \leq x_1 < x < x_2 \leq b.$$

Then

$$|L(f,x) - f(x)| \leq L\left(\psi\left(\frac{|e_1 - xe_0|}{t}\right), x\right)\Omega_2(f,t).$$

The notations used are:

- e_k for the function $e_k(x) = x^k$, $k \in \mathbb{N} \cup \{0\}$;
- $\Delta(f; x_1, x, x_2) = \dfrac{x_2 - x}{x_2 - x_1} f(x_1) + \dfrac{x - x_1}{x_2 - x_1} f(x_2) - f(x)$ for $f : [a,b] \longrightarrow \mathbb{R}$, x_1, x, $x_2 \in [a,b]$, $x_1 \neq x_2$;
- $\delta_t(\psi; x_1, x, x_2) = \dfrac{x_2 - x}{x_2 - x_1}\psi\left(\dfrac{x - x_1}{t}\right) + \dfrac{x - x_1}{x_2 - x_1}\psi\left(\dfrac{x_2 - x}{t}\right)$ for $\psi : [0,\infty) \longrightarrow \mathbb{R}$, x_1, x, $x_2 \in [0,\infty)$, $x_1 < x < x_2$, $t > 0$.

The K-functional $K_r^s(f,t) = K^s(f, t^r; \mathbf{C}[a,b], \mathbf{C^r}[a,b])$, $t > 0$, $1 \leq s \leq \infty$ defined for the Banach space $(\mathbf{C}[a,b], \|\cdot\|)$ and the semi-Banach subspace $(\mathbf{C^r}[a,b], |\cdot|_{C^r})$, $|f|_{C^r} = \left\|f^{(r)}\right\|$ by

$$K_r^s(f,t) = \inf_{g \in \mathbf{C^r}[a,b]} \left\|\left(\|f - g\|, t^r\left\|g^{(r)}\right\|\right)\right\|_s,\ 1 \leq s \leq \infty,$$

where $\|\cdot\|_s$, $1 \leq s < \infty$, is the Minkowski norm in $\mathbb{R}^2$ and $\|\cdot\|_\infty$ is the Chebychev norm in $\mathbb{R}^2$, respectively, is a modulus of continuity of order r normalized on $\mathbf{C}[a,b]$. An useful relation between the K-functionals is given by:

Lemma 1.3. [13] *Let* $1 \leq s < \infty$ *and* $r \in \mathbb{N}$. *Then for* $f \in \mathbf{C}[a,b]$ *and* $t > 0$

$$K_r^s(f,t) = \inf_{u>0}\left(1 + \frac{t^{rs}}{u^{rs}}\right)^{\frac{1}{s}} K_r^\infty(f,u) \text{ holds.} \tag{1.1}$$

In the weighted case, for $r \in \mathbb{N}$ and $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$ we denote by

$$\mathbf{C}_\varphi[0,1] = \left\{f \in \mathbf{C}(0,1) \mid (\exists) \lim_{x \to 0+} f(x)\varphi(x),\ \lim_{x \to 1-} f(x)\varphi(x) \in \mathbb{R}\right\}$$

and

$$\mathbf{W}^{\mathbf{r}}_{\mathbf{C}_{\varphi^{\mathbf{r}}}}[0,1] = \left\{f \in \mathbf{C^{r-1}}[0,1] \mid f^{(r)} \in \mathbf{C}_{\varphi^{\mathbf{r}}}[0,1]\right\}.$$

The K-functional $K_{r,\varphi}^s(f,t) = K^s\left(f, t^r; \mathbf{C}[0,1], \mathbf{W}^{\mathbf{r}}_{\mathbf{C}_{\varphi^{\mathbf{r}}}}[0,1]\right)$, $t > 0$, $1 \leq s \leq \infty$ defined for the Banach space $(\mathbf{C}[0,1], \|\cdot\|)$ and the semi-Banach subspace

[1] We refer here only the particular case when the operators preserves the linear functions.

$\left(\mathbf{W}^{\mathbf{r}}_{\mathbf{C}_{\varphi^{\mathbf{r}}}}[0,1], |\cdot|_{W^r_{C_{\varphi^r}}}\right)$, $|f|_{W^r_{C_{\varphi^r}}} = \left\|\varphi^r f^{(r)}\right\|$ by

$$K^s_{r,\varphi}(f,t) = \inf_{g\in \mathbf{W}^{\mathbf{r}}_{\mathbf{C}_{\varphi^{\mathbf{r}}}}[0,1]} \left\|\left(\|f-g\|, t^r\left\|\varphi^r g^{(r)}\right\|\right)\right\|_s, \ 1\le s\le\infty,$$

is a modulus of continuity of order r normalized on $\mathbf{C}[0,1]$ and we have

Lemma 1.4. [14] *Let* $1\le s<\infty$ *and* $r\in\mathbb{N}$. *Then for* $f\in\mathbf{C}[a,b]$ *and* $t>0$

$$K^s_{r,\varphi}(f,t) = \inf_{u>0}\left(1+\frac{t^{rs}}{u^{rs}}\right)^{\frac{1}{s}} K^\infty_{r,\varphi}(f,u) \text{ holds.} \tag{1.2}$$

In Section 2 are given estimates with the K-functional K^s_2 and in Section 3 are given estimates with the K-functional $K^s_{2,\varphi}$.

2. General estimates with K^s_2, $1\le s\le\infty$

Theorem 2.1. *Let* $[c,d]\subseteq[a,b]$, $L:\mathbf{C}[a,b]\longrightarrow\mathbf{C}[c,d]$ *a positive linear operator such that* $Le_0=e_0$, $Le_1=e_1$ *and* $f\in\mathbf{C}[a,b]$. *Then for every* $x\in(c,d)$ *and* $t>0$, *we have*

$$|L(f,x)-f(x)| \le \left(2+\frac{L\left((e_1-xe_0)^2,x\right)}{2t^2}\right)K^\infty_2(f,t). \tag{2.1}$$

Conversely, if there exist A, $B\ge 0$ *such that*

$$|L(f,x)-f(x)| \le \left(A+B\frac{L\left((e_1-xe_0)^2,x\right)}{t^2}\right)K^\infty_2(f,t) \tag{2.2}$$

holds for all positive linear operator L, *any* $f\in\mathbf{C}[a,b]$, *any* $x\in(c,d)$ *and any* $t>0$, *then* $B\ge\frac{1}{2}$ *and* $A\ge 2$.

Proof. Let $g\in\mathbf{C}^{\mathbf{2}}[a,b]$, x_1, x, $x_2\in[a,b]$, $x_1<x<x_2$. We have

$$\begin{aligned}|\Delta(f;x_1,x,x_2)| &\le |\Delta(f-g;x_1,x,x_2)|+|\Delta(g;x_1,x,x_2)|\\ &\le 2\|f-g\|+|\Delta(g;x_1,x,x_2)|\end{aligned}$$

and

$$\begin{aligned}|\Delta(g;x_1,x,x_2)| &= \left|\frac{x_2-x}{x_2-x_1}(g(x_1)-g(x))+\frac{x-x_1}{x_2-x_1}(g(x_2)-g(x))\right|\\
&= \left|\frac{x_2-x}{x_2-x_1}\left(g'(x)(x_1-x)+\frac{g''(\xi_1)}{2}(x_1-x)^2\right)\right.\\
&\quad \left.+\frac{x-x_1}{x_2-x_1}\left(g'(x)(x_2-x)+\frac{g''(\xi_2)}{2}(x_2-x)^2\right)\right|\\
&= \frac{(x_2-x)(x-x_1)}{2(x_2-x_1)}\left|g''(\xi_1)(x-x_1)+g''(\xi_2)(x_2-x)\right|\\
&\le \frac{(x_2-x)(x-x_1)}{2}\|g''\|\end{aligned}$$

with ξ_i between x and x_i, $i = 1, 2$. Therefore

$$|\Delta(f; x_1, x, x_2)| \le 2\|f-g\| + \frac{(x_2-x)(x-x_1)}{2t^2} t^2 \|g''\| \le \left(2 + \frac{(x_2-x)(x-x_1)}{2t^2}\right) \max\left\{\|f-g\|, t^2\|g''\|\right\}.$$

Since g was arbitrary it follows that

$$|\Delta(f; x_1, x, x_2)| \le \left(2 + \frac{(x_2-x)(x-x_1)}{2t^2}\right) K_2^{\infty}(f,t). \tag{2.3}$$

If we take $\psi(u) = 2 + \frac{u^2}{2}$ then (2.3) means

$$|\Delta(f; x_1, x, x_2)| \le \delta_t(\psi; x_1, x, x_2) K_2^{\infty}(f,t),\ x_1 < x < x_2.$$

By Theorem 1.2 we have

$$|L(f,x) - f(x)| \le \left(2 + \frac{L\left((e_1 - xe_0)^2, x\right)}{2t^2}\right) K_2^{\infty}(f,t).$$

Now we prove the converse part. We consider the positive linear operator L defined by

$$L(h,x) = (1-x)h(0) + xh(1),\ h \in \mathbf{C}[0,1].$$

For $f = e_2$ we have $K_2^{\infty}(e_2, t) \le 2t^2$ and from (2.2) it follows

$$x(1-x) \le 2At^2 + 2Bx(1-x).$$

Passing to the limit $t \to 0$, we obtain $B \ge \frac{1}{2}$.

For $f(x) = \alpha(4x-1)$, $x \in \left[0, \frac{1}{2}\right]$, $f(x) = \alpha(3-4x)$, $x \in \left(\frac{1}{2}, 1\right]$, $\alpha > 0$, we have $K_2^{\infty}(f,t) \le \|f\| = \alpha$ and from (2.2) it follows for $x = \frac{1}{2}$ that

$$2\alpha \le \left(A + \frac{B}{4t^2}\right)\alpha.$$

Passing to the limit $t \to \infty$, we obtain $A \ge 2$. □

Corollary 2.2. *Under the conditions of theorem we have*

$$|L(f,x) - f(x)| \le \max\left\{2, \frac{L\left((e_1 - xe_0)^2, x\right)}{2t^2}\right\} K_2^1(f,t) \tag{2.4}$$

and

$$|L(f,x) - f(x)| \le \left(2^{s'} + \frac{L\left((e_1 - xe_0)^2, x\right)^{s'}}{2^{s'} t^{2s'}}\right)^{\frac{1}{s'}} K_2^s(f,t) \tag{2.5}$$

where $1 < s < \infty$ *and* $s' = \dfrac{s}{s-1}$.

Conversely

- *if there exist* A, $B \geq 0$ *such that*

$$|L(f,x) - f(x)| \leq \max\left\{A,\ B\frac{L\left((e_1 - xe_0)^2, x\right)}{t^2}\right\} K_2^1(f,t) \tag{2.6}$$

holds for all positive linear operator L, *any* $f \in \mathbf{C}[a,b]$, *any* $x \in (c,d)$ *and any* $t > 0$, *then* $B \geq \dfrac{1}{2}$ *and* $A \geq 2$.

- *if there exist* A, $B \geq 0$ *such that*

$$|L(f,x) - f(x)| \leq \left(A + B\frac{L\left((e_1 - xe_0)^2, x\right)^{s'}}{t^{2s'}}\right)^{\frac{1}{s'}} K_2^s(f,t) \tag{2.7}$$

holds for all positive linear operator L, *any* $f \in \mathbf{C}[a,b]$, *any* $x \in (c,d)$ *and any* $t > 0$, *then* $B \geq \dfrac{1}{2^{s'}}$ *and* $A \geq 2^{s'}$.

Proof. Using the estimate (2.1), we obtain

$$\begin{aligned} |L(f,x) - f(x)| &\leq \left(2 + \frac{L\left((e_1 - xe_0)^2, x\right)}{2u^2}\right) K_2^\infty(f,u) \\ &\leq \max\left\{2, \frac{L\left((e_1 - xe_0)^2, x\right)}{2t^2}\right\}\left(1 + \frac{t^2}{u^2}\right) K_2^\infty(f,u), \end{aligned}$$

where $u > 0$ is arbitrary. Hence, by Lemma 1.3, we find (2.4). For $1 < s < \infty$, by (2.1) and Hölder's inequality, we have

$$|L(f,x) - f(x)| \leq \left(2^{s'} + \frac{L\left((e_1 - xe_0)^2, x\right)^{s'}}{2^{s'} t^{2s'}}\right)^{\frac{1}{s'}} \left(1 + \frac{t^{2s}}{u^{2s}}\right)^{\frac{1}{s}} K_2^\infty(f,u),$$

where $u > 0$ is arbitrary. Hence, by Lemma 1.3, we find (2.5).

For the converse part we make the same choices like in Theorem 2.1. $\square$

Example 2.3. We consider the Bernstein-type operator $P_{n,m} : \mathbf{C}[0,1] \longrightarrow \mathbf{C}[0,1]$ (see [12], [3])

$$P_{n,m}(f,x) = \sum_{k=0}^{n} b_{n,k,m}(x) \cdot f\left(\frac{k}{n}\right)$$

where

$b_{n,k,m}(x) = \binom{n-m}{k} x^k (1-x)^{n-m-k+1}$ for $0 \leq k < m$,

$b_{n,k,m}(x) = \binom{n-m}{k} x^k (1-x)^{n-m-k+1} + \cdots + \binom{n-m}{k-m} x^{k-m+1} (1-x)^{n-k}$

for $m \leq k \leq n-m$ and

$b_{n,k,m}(x) = \binom{n-m}{k-m} x^{k-m+1}(1-x)^{n-k}$ for $n-m < k \leq n$,
with $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $m < \dfrac{n}{2}$. We have

$$\begin{aligned} P_{n,m}(e_0, x) &= 1 \\ P_{n,m}(e_1, x) &= x \\ P_{n,m}\left((e_1 - xe_0)^2, x\right) &= \left(1 + \frac{m(m-1)}{n}\right)\frac{x(1-x)}{n}. \end{aligned}$$

Theorem 2.1 implies for $f \in \mathbf{C}[0,1]$, $x \in (0,1)$ and $t = \sqrt{\dfrac{x(1-x)}{n}}$

$$|P_{n,m}(f,x) - f(x)| \leq \left[2 + \frac{1}{2}\left(1 + \frac{m(m-1)}{n}\right)\right] K_2^{\infty}\left(f, \sqrt{\frac{x(1-x)}{n}}\right).$$

From Corollary 2.2 we obtain

$$|P_{n,m}(f,x) - f(x)| \leq \max\left\{2, \frac{1}{2}\left(1 + \frac{m(m-1)}{n}\right)\right\} K_2^{1}\left(f, \sqrt{\frac{x(1-x)}{n}}\right)$$

and

$$|P_{n,m}(f,x) - f(x)| \leq \left[2^{s'} + \frac{1}{2^{s'}}\left(1 + \frac{m(m-1)}{n}\right)^{s'}\right]^{\frac{1}{s'}} K_2^{s}\left(f, \sqrt{\frac{x(1-x)}{n}}\right)$$

where $1 < s < \infty$ and $s' = \dfrac{s}{s-1}$. In particular, for $m = 0$ or $m = 1$ we obtain the estimates for the Bernstein operators.

3. General estimates with $K_{2,\varphi}^{s}$, $1 \leq s \leq \infty$

Theorem 3.1. *Let $L : \mathbf{C}[0,1] \longrightarrow \mathbf{C}[0,1]$ be a positive linear operator such that $Le_0 = e_0$, $Le_1 = e_1$ and $f \in \mathbf{C}[0,1]$. Then for all $x \in (0,1)$ and $t > 0$ we have*

$$|L(f,x) - f(x)| \leq \left(2 + \frac{L\left((e_1 - xe_0)^2, x\right)}{t^2\varphi^2(x)}\right) K_{2,\varphi}^{\infty}(f,t). \tag{3.1}$$

Conversely, if there exist A, $B \geq 0$ such that

$$|L(f,x) - f(x)| \leq \left(A + B\frac{L\left((e_1 - xe_0)^2, x\right)}{t^2\varphi^2(x)}\right) K_{2,\varphi}^{\infty}(f,t) \tag{3.2}$$

holds for all positive linear operator L, any $f \in \mathbf{C}[0,1]$, any $x \in (0,1)$ and any $t > 0$, then $B \geq 1$ and $A \geq 2$.

Proof. Let $g \in \mathbf{W}^{\mathbf{2}}_{\mathbf{C}_{\varphi^2}}[0,1]$, x_1, x, $x_2 \in [0,1]$, $x_1 < x < x_2$. We have

$$\begin{aligned} |\Delta(f; x_1, x, x_2)| &\leq |\Delta(f-g; x_1, x, x_2)| + |\Delta(g; x_1, x, x_2)| \\ &\leq 2\|f-g\| + |\Delta(g; x_1, x, x_2)| \end{aligned}$$

and

$$|\Delta(g;x_1,x,x_2)| = \left|\frac{x_2-x}{x_2-x_1}\left(g(x_1)-g(x)\right)+\frac{x-x_1}{x_2-x_1}\left(g(x_2)-g(x)\right)\right|$$

$$= |\frac{x_2-x}{x_2-x_1}\left[g'(x)(x_1-x)+\int_x^{x_1} g''(u)(x_1-u)\,du\right]$$

$$+\frac{x-x_1}{x_2-x_1}\left[g'(x)(x_2-x)+\int_x^{x_2} g''(u)(x_2-u)\,du\right]|$$

$$= \left|\frac{x_2-x}{x_2-x_1}\int_x^{x_1} g''(u)(x_1-u)\,du - \frac{x_1-x}{x_2-x_1}\int_x^{x_2} g''(u)(x_2-u)\,du\right|$$

$$\le \frac{x_2-x}{x_2-x_1}\cdot\left\|\varphi^2 g''\right\|\cdot\int_{x_1}^{x}\frac{u-x_1}{\varphi^2(u)}du+\frac{x-x_1}{x_2-x_1}\cdot\left\|\varphi^2 g''\right\|\cdot\int_x^{x_2}\frac{x_2-u}{\varphi^2(u)}du.$$

Let us now make use of the fact that the function $u\mapsto\dfrac{t-u}{u(1-u)}$, $u\in(0,t)$, $t\in(0,1]$ is decreasing [8] and we obtain

$$|\Delta(g;x_1,x,x_2)| \le$$

$$\le \frac{x_2-x}{x_2-x_1}\cdot\left\|\varphi^2 g''\right\|\cdot\int_{1-x}^{1-x_1}\frac{1-u-x_1}{\varphi^2(1-u)}du+\frac{x-x_1}{x_2-x_1}\cdot\left\|\varphi^2 g''\right\|\cdot\int_x^{x_2}\frac{x_2-u}{\varphi^2(u)}du$$

$$\le \frac{x_2-x}{x_2-x_1}\cdot\left\|\varphi^2 g''\right\|\cdot\int_{1-x}^{1-x_1}\frac{x-x_1}{\varphi^2(1-x)}du+\frac{x-x_1}{x_2-x_1}\cdot\left\|\varphi^2 g''\right\|\cdot\int_x^{x_2}\frac{x_2-x}{\varphi^2(x)}du$$

$$= \frac{(x_2-x)(x-x_1)}{\varphi^2(x)}\left\|\varphi^2 g''\right\|$$

therefore

$$\begin{aligned}|\Delta(f;x_1,x,x_2)| &\le 2\|f-g\|+\frac{(x_2-x)(x-x_1)}{t^2\varphi^2(x)}t^2\left\|\varphi^2 g''\right\|\\ &\le \left(2+\frac{(x_2-x)(x-x_1)}{t^2\varphi^2(x)}\right)\max\left\{\|f-g\|,\,t^2\left\|\varphi^2 g''\right\|\right\}.\end{aligned}$$

Since g was arbitrary it follows that

$$|\Delta(f;x_1,x,x_2)| \le \left(2+\frac{(x_2-x)(x-x_1)}{t^2\varphi^2(x)}\right)K_{2,\varphi}^{\infty}(f,t). \tag{3.3}$$

If we take $\psi(u)=2+\dfrac{u^2}{\varphi^2(x)}$ then (3.3) means

$$|\Delta(f;x_1,x,x_2)| \le \delta_t(\psi;x_1,x,x_2)\,K_{2,\varphi}^{\infty}(f,t),\ x_1<x<x_2.$$

By Theorem 1.2 we have

$$|L(f,x)-f(x)| \leq \left(2+\frac{L\left((e_1-xe_0)^2,x\right)}{t^2\varphi^2(x)}\right) K_{2,\varphi}^{\infty}(f,t).$$

Now we prove the converse part. To show that $A \geq 2$, we consider the positive linear operator L defined by

$$L(h,x)=(1-x)h(0)+xh(1),\ h \in \mathbf{C}[0,1].$$

For $f(x)=\alpha(4x-1)$, $x \in \left[0,\frac{1}{2}\right]$, $f(x)=\alpha(3-4x)$, $x \in \left(\frac{1}{2},1\right]$, $\alpha>0$, we have $K_{2,\varphi}^{\infty}(f,t) \leq \|f\|=\alpha$ and from (3.2) it follows for $x=\frac{1}{2}$ that

$$2\alpha \leq \left(A+\frac{B}{t^2}\right)\alpha.$$

Passing to the limit $t \to \infty$, we obtain $A \geq 2$.

To show that $B \geq 1$, we choose

$$L(h,x)=\left(1-x^{\beta}\right)h(0)+x^{\beta}h\left(x^{1-\beta}\right),\ \beta \in (0,1),\ h \in \mathbf{C}[0,1]$$

and $f(x)=x^{1+\alpha}$, $\alpha>0$. We have $f \in \mathbf{W}^{\mathbf{2}}_{\mathbf{C}_{\varphi^{\mathbf{2}}}}[0,1]$ and then

$$K_{2,\varphi}^{\infty}(f,t) \leq t^2\left\|\varphi^2 f''\right\|=t^2\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha}}.$$

We replace it in (3.2) and passing to the limit $t \to 0$, we obtain

$$x^{1+\alpha}\left(x^{-\alpha\beta}-1\right) \leq B \cdot \frac{x\left(x^{-\beta}-1\right)}{1-x} \cdot \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha}}$$

i.e.

$$B \geq x^{\alpha}(1-x) \cdot \frac{\left(x^{-\alpha\beta}-1\right)}{x^{-\beta}-1} \cdot \frac{(\alpha+1)^{\alpha}}{\alpha^{\alpha+1}}.$$

Passing to the limit $\beta \to 0$, we obtain $B \geq x^{\alpha}(1-x)\frac{(\alpha+1)^{\alpha}}{\alpha^{\alpha}}$. Since x is arbitrary, this implies $B \geq \frac{1}{\alpha+1}$. Passing to the limit $\alpha \to 0$, we obtain $B \geq 1$. □

Corollary 3.2. *Under the conditions of theorem we have*

$$|L(f,x)-f(x)| \leq \max\left\{2,\ \frac{L\left((e_1-xe_0)^2,x\right)}{t^2\varphi^2(x)}\right\} K_{2,\varphi}^{1}(f,t) \tag{3.4}$$

and

$$|L(f,x)-f(x)| \leq \left(2^{s'}+\frac{L\left((e_1-xe_0)^2,x\right)^{s'}}{t^{2s'}\varphi^{2s'}(x)}\right)^{\frac{1}{s'}} K_{2,\varphi}^{s}(f,t) \tag{3.5}$$

where $1 < s < \infty$ *and* $s' = \dfrac{s}{s-1}$.
Conversely

- *if there exist* A, $B \geq 0$ *such that*

$$|L(f,x) - f(x)| \leq \max\left\{A,\ B\frac{L\left((e_1 - xe_0)^2, x\right)}{t^2\varphi^2(x)}\right\} K^1_{2,\varphi}(f,t) \qquad (3.6)$$

  holds for all positive linear operator L, *any* $f \in \mathbf{C}[0,1]$, *any* $x \in (0,1)$ *and any* $t > 0$, *then* $B \geq 1$ *and* $A \geq 2$.
- *if there exist* A, $B \geq 0$ *such that*

$$|L(f,x) - f(x)| \leq \left(A + B\frac{L\left((e_1 - xe_0)^2, x\right)^{s'}}{t^{2s'}\varphi^{2s'}(x)}\right)^{\frac{1}{s'}} K^s_{2,\varphi}(f,t) \qquad (3.7)$$

  holds for all positive linear operator L, *any* $f \in \mathbf{C}[0,1]$, *any* $x \in (0,1)$ *and any* $t > 0$, *then* $B \geq 1$ *and* $A \geq 2^{s'}$.

Proof. We use the estimate (3.1) and Lemma 1.4 (see also the proof of Corollary 2.2). For the converse part we make the same choices like in Theorem 3.1. □

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