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Approximation of the solution of stochastic differential equations driven by multifractional Brownian motion

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Abstract. The aim of this paper is to approximate the solution of a stochastic differential equations

$$dX(t) = F(X(t))dt + G(X(t))dB(t),\ X(0) = X_0,\ \ t \geq 0$$

on $\mathbb{R}^n$. We will use wavelet approximation of multifractional Brownian motion.

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1. Introduction

The fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$ is a zero mean Gaussian random process $\Big(B(t)\Big)_{t\geq 0}$ with continuous sample paths and with covariance function

$$E\Big(B(s)B(t)\Big) = \frac{1}{2}\Big(t^{2H} + s^{2H} - |s-t|^{2H}\Big).$$

For $H = \frac{1}{2}$ the fractional Brownian motion is the ordinary standard Brownian motion.

The fractional Brownian motion B has on any finite interval $[0,T]$ Hölder continuous paths with exponent $\gamma \in (0,H)$ (see [5]). Moreover, the quadratic variation on $[a,b] \subseteq [0,T]$ is

$$\lim_{|\Delta_n|\to 0} \sum_{i=1}^{n} \Big(B(t_i^n) - B(t_{i-1}^n)\Big)^2 \quad = \quad \begin{cases} \infty & \text{if } H < \frac{1}{2}, \\ b-a & \text{if } H = \frac{1}{2}, \\ 0 & \text{if } H > \frac{1}{2}, \end{cases} \tag{1.1}$$

where $\Delta_n = (a = t_0^n < \cdots < t_n^n = b)$ is a partition of $[a,b]$ with

$$|\Delta_n| = \max_{1\leq i\leq n} (t_i^n - t_{i-1}^n).$$

If $H \neq \frac{1}{2}$, then the convergence in (1.1) holds with probability 1 uniformly in the set of all partitions of $[a, b]$, while for $H = \frac{1}{2}$ the convergence in (1.1) holds in mean square uniformly in the set of all partitions of $[a, b]$. Note that, if $H \neq \frac{1}{2}$, then B is not a semimartingale, so the classical stochastic integration does not work. But the Hölder continuity of B will ensure the existence of integrals

$$\int_0^T G(u)dB(u),$$

defined in terms of fractional integration as investigated in [15] and [16] for the stochasticc process $(G(t))_{t\in[0,T]}$ with Hölder continuous paths of order $\alpha > 1 - H$. Moreover, the fractional Brownian motion is H-self similar, so for any $c > 0$ the process $\Big(c^H B(t/c)\Big)_{t\geq 0}$ is again a fractional Brownian motion, has stationary increments. Stochastic differential equations driven by fBm have received considerable attention during the last two decades. Fractional Brownian motion as driving noise is used in electrical engineering ([6]) or biophysics ([11]). Moreover, fBm has established itself also in financial modelling ([4],[8]).

The multifractional Brownian motion (mfBm) is obtained by replacing the constant parameter H of the fractional Brownian motion by a smooth enough functional parameter $H(\cdot)$. We denote by H a function defined on the real line and with values in a fixed interval $[a, b] \subset (0, 1)$. We assume that it is uniformly Hölder continuous of order $\beta > b$ on each compact subset of $\mathbb{R}$.

In this article we study the approximation of the Itô stochastic differential equation

$$dX(t) = F(X(t))dt + G(X(t))dB(t),\ X(0) = X_0,\ \ t \geq 0 \tag{1.2}$$

on $\mathbb{R}^n$. Here $F : \mathbb{R}^n \to \mathbb{R}^n,\ G : \mathbb{R}^n \to \mathbb{R}^n,\ B = \Big(B(t)\Big)_{t\geq 0},\ H \in (0, 1)$ is a 1-dimensional multifractional Brownian motion adapted to a filtration $F = (F_t)_{t\geq 0}$ on a probability space (Ω, K, P), and x_0 is a F_0 measurable random variable independent of B.

Suppose with F and G satisfy with probability 1:
$F \in C(\mathbb{R}^n \times [0, T], \mathbb{R}^n), G \in C^1(\mathbb{R}^n \times [0, T], \mathbb{R}^n)$ and
$F(\cdot, t), \dfrac{\partial G(\cdot, t)}{\partial x}, \dfrac{\partial G(\cdot, t)}{\partial t}$ are locally Lipschitz, $\forall t \in [0, T]$.

2. Wavelet approximation for $(B(t))_{t\in[0,1]}$

Let $\{2^{j/2}\Psi(2^j x - k) : (j, k) \in \mathbb{Z}^2\}$ be a Lamarie Meyer wavelet basis of $L^2(\mathbb{R})$ and denote by $\mathbf{\Psi}$ the function defined by

$$\mathbf{\Psi}(x, \theta) = \int_{\mathbb{R}} e^{ixy} \frac{\overline{\mathbf{\Psi}}(y)}{|y|^{\theta+\frac{1}{2}}} dy,$$

where $\overline{\mathbf{\Psi}(y)}$ is the Fourier transform. We use the following wavelet approximation of the multifractional Brownian motion $(B(t))_{t\in[0,1]}$ with Hurst index H investigated in [1].

$$B(t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} 2^{-jH(k/2^j)}(\mathbf{\Psi}(2^j t - k, H(k/2^j)) - \mathbf{\Psi}(-k, H(k/2^j)))\epsilon_{j,k}, \tag{2.1}$$

where $\epsilon_{j,k}$ are independent identically distributed $N(0,1)$ random variables. This process was introduced in [3] to model fBm with piecewise constant Hurst index and continuous path.

As in [2] and [12] we consider the following assumptions for Ψ: $\Psi \in C^1$ and there exists a constant $c > 0$ such that

$$|\sup_{\theta\in[a,b]} \mathbf{\Psi}(t,\theta)| \leq \frac{c}{(2+|t|)^2} \text{ and } |\sup_{\theta\in[a,b]} \mathbf{\Psi}'(t,\theta)| \leq \frac{c}{(2+|t|)^3} \text{ for all } t \in \mathbb{R}. \tag{2.2}$$

We consider the following *high frequency component* of the wavelet representation in (2.1)

$$V_1(t) = \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} 2^{-jH}(\mathbf{\Psi}(2^j t - k, H(k/2^j)) - \mathbf{\Psi}(-k, H(k/2^j)))\epsilon_{j,k}$$

and the *low frequency component*

$$V_2(t) = \sum_{j=-\infty}^{-1} \sum_{k=-\infty}^{\infty} 2^{-jH}(\mathbf{\Psi}(2^j t - k, H(k/2^j)) - \mathbf{\Psi}(-k, H(k/2^j)))\epsilon_{j,k}.$$

Obviously,

$$B(t) = V_1(t) + V_2(t) \text{ for each } t \in [0,1].$$

Let $N \in \mathbb{N}$. In the following we use two approximation components, corresponding to the components V_1, respectively V_2, namely

$$B_1^N(t) = \sum_{j=0}^{N} \sum_{|k|\leq \frac{2^{N+4}}{(N-j+1)^2}} 2^{-jH}(\mathbf{\Psi}(2^j t - k, H(k/2^j)) - \mathbf{\Psi}(-k, H(k/2^j)))\epsilon_{j,k}$$

and

$$B_2^N(t) = \sum_{j=-2^{[N/2]}}^{-1} \sum_{|k|\leq 2^{[N/2]}} 2^{-jH}(\mathbf{\Psi}(2^j t - k, H(k/2^j)) - \mathbf{\Psi}(-k, H(k/2^j)))\epsilon_{j,k}.$$

We denote

$$B^N(t) = B_1^N(t) + B_2^N(t) \text{ for each } t \in [0,1]. \tag{2.3}$$

Using Theorem 2 and Theorem 3 from [2] we have the following result:

Theorem 2.1. *The sequence $(B^N)_{N\in\mathbb{N}}$ converges to B almost surely in $\omega \in \Omega$ and uniformly in $t \in [0,1]$, i.e.*

$$\mathbb{P}\Big(\lim_{N\to\infty} \sup_{t\in[0,1]} |B^N(t) - B(t)| = 0\Big) = 1.$$

In the sequel we need the following result:

Theorem 2.2. *For all $N \in \mathbb{N}$ the approximating processes $(B^N(t))_{t\in[0,1]}$ are Lipschitz continuous with probability* 1.

Proof. We write

$$\begin{aligned}
|B^N(s) - B^N(t)| &\le |B_1^N(s) - B_1^N(t)| + |B_2^N(s) - B_2^N(t)| \\
&\le \sum_{j=0}^{N} \sum_{|k|\le \frac{2^{N+4}}{(N-j+1)^2}} 2^{-jH} |\mathbf{\Psi}(2^j s - k, H(k/2^j)) - \mathbf{\Psi}(2^j t - k, H(k/2^j)))||\epsilon_{j,k}| \\
&+ \sum_{j=-2^{[N/2]}}^{-1} \sum_{|k|\le 2^{[N/2]}} 2^{-jH} |\mathbf{\Psi}(2^j s - k, H(k/2^j)) - \mathbf{\Psi}(2^j t - k, H(k/2^j))||\epsilon_{j,k}|.
\end{aligned}$$

Using the assumption (2.2) for $\mathbf{\Psi}$ and using that the set of indices of j and k is bounded, it follows that there exists a $c_N > 0$ (depending on ω) such that

$$|B^N(s) - B^N(t)| \le c_N |s - t| \text{ for all } s, t \in [0,1] \text{ and all } n \in \mathbb{N}.$$

□

3. Fractional integrals and derivatives

Let $a, b \in \mathbb{R}$, $a < b$ and $f, g : \mathbb{R} \to \mathbb{R}$. We use notions and results about fractional calculus, from [14] and [15]:

$$f(a+) := \lim_{\delta \searrow 0} f(a + \delta), \quad f(b-) := \lim_{\delta \searrow 0} f(b - \delta),$$

$$f_{a+}(x) = \mathbb{I}_{(a,b)}(f(x) - f(a+)), \quad g_{b-}(x) = \mathbb{I}_{(a,b)}(g(x) - g(b-)).$$

Note that for $\alpha > 0$ we have $(-1)^\alpha = e^{i\pi\alpha}$.

For $f \in L_1(a,b)$ and $\alpha > 0$ the **left-** and **right-sided fractional Rieman-Liouville integral of** f **of order** α on (a,b) is given for almost every x by

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) dy$$

and

$$I_{b-}^{\alpha} f(x) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y) dy.$$

For $p > 1$ let $I_{a+}^{\alpha}(L_p(a,b))$, be the class of functions f which have the representation $f = I_{a+}^{\alpha}\Phi$, where $\Phi \in L_p(a,b)$, and let $I_{b-}^{\alpha}(L_p(a,b))$ be the class of functions g which have the representation $g = I_{b-}^{\alpha}\varphi$, where $\varphi \in L_p(a,b)$. If $0 < \alpha < 1$, then the function Φ, respectively φ, in the representations above agree almost surely with the **left-sided** and respectively

right-sided fractional derivative of f **of order** α (in the Weyl representation)

$$\Phi(x) = D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}} dy \right) \mathbb{I}_{(a,b)}(x)$$

and

$$\varphi(x) = D_{b-}^{\alpha} g(x) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left(\frac{g(x)}{(b-x)^{\alpha}} + \alpha \int_x^b \frac{g(x)-g(y)}{(y-x)^{\alpha+1}} dy \right) \mathbb{I}_{(a,b)}(x).$$

The convergence at the singularity $y = x$ holds in the L_p-sense. Recall that $I_{a+}^{\alpha}(D_{a+}^{\alpha} f) = f$ for $f \in I_{a+}^{\alpha}(L_p(a,b))$, $\quad I_{b-}^{\alpha}(D_{b-}^{\alpha} g) = g$ for $g \in I_{b-}^{\alpha}(L_p(a,b))$ and

$$D_{a+}^{\alpha}(I_{a+}^{\alpha} f) = f, \quad D_{b-}^{\alpha}(I_{b-}^{\alpha} g) = g \text{ for } f, g \in L_1(a,b).$$

For completeness we denote

$$D_{a+}^{0} f(x) = f(x), D_{b-}^{0} g(x) = g(x), D_{a+}^{1} f(x) = f'(x), D_{b-}^{1} g(x) = g'(x).$$

Let $0 \le \alpha \le 1$. The **fractional integral** of f with respect to g is defined as

$$\begin{aligned} \int_a^b f(x) dg(x) &= (-1)^{\alpha} \int_a^b D_{a+}^{\alpha} f_{a+}(x) D_{b-}^{1-\alpha} g_{b-}(x) dx \\ &\quad + f(a+)(g(b-) - g(a+)) \end{aligned} \tag{3.1}$$

if $f_{a+} \in I_{a+}^{\alpha}(L_p(a,b)), g_{b-} \in I_{b-}^{1-\alpha}(L_q(a,b))$ for $\frac{1}{p} + \frac{1}{q} \le 1$.

In our investigations we will take $p = q = 2$. If $0 \le \alpha < \frac{1}{2}$, then the integral in (3.1) can be written as

$$\int_a^b f(x) dg(x) = (-1)^{\alpha} \int_a^b D_{a+}^{\alpha} f(x) D_{b-}^{1-\alpha} g_{b-}(x) dx \tag{3.2}$$

if $f \in I_{a+}^{\alpha}(L_2(a,b))$, $f(a+)$ exists, $g_{b-} \in I_{b-}^{1-\alpha}(L_2(a,b))$ (see [15]).

4. The stochastic integral

Without loss of generality we consider $0 < T \le 1$, because for arbitrary $T > 0$ we can rescale the time variable using the H-self similarity property of the multifractional Brownian motion meaning that $\Big(B(ct)\Big)_{t \ge 0}$ and $\Big(c^H B(t)\Big)_{t \ge 0}$ are equal in distribution for every $c > 0$.

We will define the $\int_0^T G(u) dB(u)$ Itô integral instead of $\int_0^t G(u) dB(u)$ and use

$$\int_0^t G(u) dB(u) = \int_0^T \mathbb{I}_{[0,t]}(u) G(u) dB(u) \text{ for } t \in [0,T]$$

(by Theorem 2.5, p. 345, in [15]).

We consider $\alpha > 1 - H$. It follows by (3.2) that

$$\int_0^T G(u)dB(u) = (-1)^\alpha \int_0^T D_{0+}^\alpha G(u) D_{T-}^{1-\alpha} B_{T-}(u)du \tag{4.1}$$

for $G \in I_{0+}^\alpha(L_2(0,T))$, where $G(0+)$ exists and $B_{T-} \in I_{T-}^{1-\alpha}(L_2(0,T))$.

The condition $G \in I_{0+}^\alpha(L_2(0,T))$ (with probability 1) means that $G \in L_2(0,T)$ and

$$\mathcal{I}_\varepsilon(x) = \int_0^{x-\varepsilon} \frac{G(x) - G(y)}{(x-y)^{\alpha+1}} dy \text{ for } x \in (0,T)$$

converges in $L^2(0,T)$ as $\varepsilon \searrow 0$.

The condition $B_{T-} \in I_{T-}^{1-\alpha}(L_2(0,T))$ means $B_{T-} \in L_2(0,T)$ and

$$\mathcal{J}_\varepsilon(x) = \int_{x+\varepsilon}^{T} \frac{B(x) - B(y)}{(y-x)^{2-\alpha}} dy \text{ for } x \in (0,T)$$

converges in $L_2(0,T)$ as $\varepsilon \searrow 0$ This condition for B is fulfilled for $\alpha > 1 - H$, since the multifractional Brownian motion B is almost surely Hölder continuous with exponent $\gamma \in (0, H)$ (see [5]).

We will use (3.2) for the integrals with respect the approximating processes $\Big(B_N(t)\Big)_{t\in[0,T]}$. Observe that $B_{N,T-} \in I_{T-}^{1-\alpha}(L_2(0,T))$, which follows from the Lipschitz continuity property in Theorem 2.2. We have

$$\int_0^T G(u)dB_N(u) = (-1)^\alpha \int_0^T D_{0+}^\alpha G(u) D_{T-}^{1-\alpha} B_{N,T-}(u)du \tag{4.2}$$

for $G \in I_{0+}^\alpha(L_2(0,T))$, where $G(0+)$ exists.

Let $\Big(Z(t)\Big)_{t\in[0,T]}$ be a cádlág process. Its **generalized quadratic variation process** $\Big([Z](t)\Big)_{t\in[0,T]}$ is defined as

$$[Z](t) = \lim_{\varepsilon \searrow 0} \varepsilon \int_0^1 \int_0^t \frac{1}{u}(Z_{t-}(s+u) - Z_{t-}(s))^2 ds du + (Z(t) - Z(t-))^2,$$

if the limit exists uniformly in probability (see [16]).

In particular, if B is a multifractional Brownian motion with Hurst index $H \in (\frac{1}{2}, 1)$ and B^N is an approximation of B as given in (2.3), it is easy to verify that

$$[B](t) = 0 \quad \text{and} \quad [B^N](t) = 0 \quad \text{for each } \ t \in [0,T], \tag{4.3}$$

because B is locally Hölder continuous and B^N is Lipschitz continuous. The **Itô formula** for change of variable for fractional integrals is given in the next theorem.

Theorem 4.1 ([16], **Theorem 5.8, p. 170).** *Let* $\left(Z(t)\right)_{t\in[0,T]}$ *be a continuous process with generalized quadratic variation* $[Z]$. *Let* $Q : \mathbb{R}\times[0,T]\to\mathbb{R}$ *be a random function such that a.s. we have* $Q\in\mathcal{C}^1(\mathbb{R}\times[0,T])$ *and* $\frac{\partial^2 Q}{\partial x^2}\in\mathcal{C}(\mathbb{R}\times[0,T])$. *Then, for* $t_0, t\in[0,T]$ *we have*

$$\begin{aligned} Q(Z(t),t)-Q(Z(t_0),t_0) &= \int_{t_0}^{t}\frac{\partial Q}{\partial x}(Z(s),s)dZ(s)+\int_{t_0}^{t}\frac{\partial Q}{\partial t}(Z(s),s)ds \\ &\quad+\int_{t_0}^{t}\frac{\partial^2 Q}{\partial^2 x}(Z(s),s)d[Z]s. \end{aligned}$$

Let $1-H<\alpha<\frac{1}{2}$ and let $G\in I^{\alpha}_{0+}(L_2(0,T))$ such that $G(0+)$ exists. We define the processes

$$Z(t)=\int_0^t G(s)dB(s) \text{ and } Z_N(t)=\int_0^t G(s)dB_N(s),\quad t\in(0,T].$$

Then by Theorem 5.6, p. 167 in [16] it follows that

$$[Z](t)=0 \text{ and } [Z_N](t)=0.$$

Using Theorem 4.1, it follows that, if $Q:\mathbb{R}\times[0,T]\to\mathbb{R}$ is a random function such that a.s. we have $Q\in\mathcal{C}^1(\mathbb{R}\times[0,T])$ and $\frac{\partial^2 Q}{\partial x^2}\in\mathcal{C}(\mathbb{R}\times[0,T])$, then for $t_0, t\in[0,T]$ we have

$$\begin{aligned} Q(Z(t),t)-Q(Z(t_0),t_0) &= \int_{t_0}^{t}\frac{\partial Q}{\partial x}(Z(s),s)G(s)dB(s) \\ &\quad+\int_{t_0}^{t}\frac{\partial Q}{\partial t}(Z(s),s)ds \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} Q(Z_N(t),t)-Q(Z_N(t_0),t_0) &= \int_{t_0}^{t}\frac{\partial Q}{\partial x}(Z_N(s),s)G(s)dB_N(s) \\ &\quad+\int_{t_0}^{t}\frac{\partial Q}{\partial t}(Z_N(s),s)ds. \end{aligned} \tag{4.5}$$

5. Stochastic differential equations driven by multifractional Brownian motion

Let $\left(B(t)\right)_{t\geq 0}$ be a multifractional Brownian motion with Hurst parameter H such that $H>\frac{1}{2}$. We investigate stochastic differential equations of the form

$$\begin{aligned} dX(t) &= F(X(t),t)dt + G(X(t),t)dB(t), \\ X(t_0) &= X_0, \end{aligned} \tag{5.1}$$

where $t_0 \in (0,T]$, X_0 is a random vector in $\mathbb{R}^n$ and the random functions F and G satisfy with probability 1 the following conditions:

(C1) $F \in C(\mathbb{R}^n \times [0,T], \mathbb{R}^n), G \in C^1(\mathbb{R}^n \times [0,T], \mathbb{R}^n)$;

(C2) for each $t \in [0,T]$ the functions $F(\cdot,t), \dfrac{\partial G(\cdot,t)}{\partial x^i}, \dfrac{\partial G(\cdot,t)}{\partial t}$ are locally Lipschitz for each $i \in \{1,\ldots,n\}$.

We consider the pathwise auxiliary partial differential equation on $\mathbb{R}^n \times \mathbb{R} \times [0,T]$

$$\begin{aligned} \frac{\partial K}{\partial z}(y,z,t) &= G(K(y,z,t),t), \\ K(Y_0,Z_0,t_0) &= X_0, \end{aligned} \tag{5.2}$$

where Y_0 is an arbitrary random vector in $\mathbb{R}^n$ and Z_0 an arbitrary random variable in $\mathbb{R}$. From the theory of differential equations it follows that with probability 1 there exists a local solution $K \in C^1(\mathbb{R}^n \times [0,T], \mathbb{R}^n)$ in a neighbourhood V of (Y_0,Z_0,t_0) with partial derivatives being Lipschitz in the variable y and

$$\det\left(\frac{K^i}{\partial y^j}(y,z,t)\right)_{1\le i,j\le n} \neq 0.$$

We have for $(x,y,t) \in V$

$$\frac{\partial^2 K}{\partial z^2}(y,z,t) = \sum_{j=1}^{n} \frac{\partial G}{\partial x^j}(K(y,z,t),t)G^j(K(y,z,t),t).$$

We also consider the pathwise differential equation (in matrix representation) on $[0,T]$

$$\begin{aligned} dY(t) &= \left(\frac{K}{\partial y}(Y(t),B(t),t)\right)^{-1}\Big[F(K(Y(t),B(t),t),t) - \frac{\partial K}{\partial t}(Y(t),B(t),t)\Big]dt \\ Y(t_0) &= Y_0, \end{aligned}$$

which has a unique local solution on a maximal interval $(t_0^1, t_0^2) \subseteq [0,T]$ with $t_0 \in (t_0^1, t_0^2)$ (see [13]).

Applying the Itô formula, see Theorem 4.1 and relation (4.4), to the random function $Q(z,t) = K(Y(t),z,t)$ (in fact, successively for $K^1,\ldots,K^n$)

and the fractional Brownian motion B we obtain

$$
\begin{aligned}
&K(Y(t),B(t),t)-K(Y(t_0),B(t_0),t_0)\\
&=\sum_{j=1}^{n}\int_{t_0}^{t}\frac{\partial K}{\partial y^j}(Y(s),B(s),s)dY^j(s)+\int_{t_0}^{t}\frac{\partial K}{\partial z}(Y(s),B(s),s)dB(s)\\
&\qquad+\int_{t_0}^{t}\frac{\partial K}{\partial t}(Y(s),B(s),s)ds\\
&=\sum_{j=1}^{n}\int_{t_0}^{t}\frac{\partial K}{\partial y^j}(Y(s),B(s),s)dY^j(s)\\
&\qquad+\int_{t_0}^{t}G(K(Y(s),B(s),s),s)dB(s)+\int_{t_0}^{t}\frac{\partial K}{\partial t}(Y(s),B(s),s)ds\\
&=\int_{t_0}^{t}F(K(Y(s),B(s),s),s)ds+\int_{t_0}^{t}G(K(Y(s),B(s),s),s)dB(s).
\end{aligned}
$$

Therefore,

$$X(t):=K(Y(t),B(t),t)$$

satisfies

$$X(t)=X_0+\int_{t_0}^{t}F(X(s),s)ds+\int_{t_0}^{t}G(X(s),s)dB(s).$$

Instead of the process $\left(B(t)\right)_{t\in[0,1]}$ we consider its approximations $\left(B^N(t)\right)_{t\in[0,1]}$ given in (2.3). For each $N\in\mathbb{N}$ we consider the pathwise differential equation (in matrix representation)

$$
\begin{aligned}
dY_N(t) &= \left(\frac{\partial K}{\partial y}(Y_N(t),B^N(t),t)\right)^{-1}\Big[F(K(Y_N(t),B^N(t),t),t)\\
&\qquad -\frac{\partial K}{\partial t}(Y_N(t),B^N(t),t)\Big]dt\\
Y_N(t_0) &= Y_0,
\end{aligned}
$$

which has a unique local solution Y_N on a maximal interval $(t^1,t^2)\subset(t_0^1,t_0^2)$ of existence which contains t_0. Applying the Itô formula, see Theorem 4.1 and (4.5), to the random function $Q(z,t)=K(Y_N(t),z,t)$ (in fact, successively for $K^1,\dots,K^n$) and the process B_N we obtain

$$K(Y_N(t),B^N(t),t)-K(Y_N(t_0),B^N(t_0),t_0)$$

$$=\sum_{j=1}^{n}\int_{t_0}^{t}\frac{\partial K}{\partial y^j}(Y_N(s),B_N(s),s)dY_N^j(s)+\int_{t_0}^{t}\frac{\partial K}{\partial z}(Y_N(s),B^N(s),s)dB^N(s)$$

$$+\int_{t_0}^{t}\frac{\partial K}{\partial t}(Y_N(s),B^N(s),s)ds$$

$$=\sum_{j=1}^{n}\int_{t_0}^{t}\frac{\partial K}{\partial y^j}(Y_N(s),B^N(s),s)dY_N^j(s)+\int_{t_0}^{t}G(K(Y_N(s),B^N(s),s),s)dB^N(s)$$

$$+\int_{t_0}^{t}\frac{\partial K}{\partial t}(Y_N(s),B^N(s),s)ds$$

$$=\int_{t_0}^{t}F(K(Y_N(s),B^N(s),s),s)ds+\int_{t_0}^{t}G(K(Y_N(s),B^N(s),s),s)dB^N(s).$$

Therefore,

$$X_N(t):=K(Y_N(t),B^N(t),t)$$

satisfies

$$X_N(t)=X_0+\int_{t_0}^{t}F(X_N(s),s)ds+\int_{t_0}^{t}G(X_N(s),s)dB^N(s),\quad t\in(t_1,t_2).$$

By Theorem 7.2 [13] it follows that we have the following pathwise property

$$\lim_{N\to\infty}\sup_{t\in(t_1,t_2)}\|Y_N(t)-Y(t)\|=0.$$

Then the continuity properties of K and (2.4) imply that for a.e. $\omega\in\Omega$ it holds

$$\lim_{N\to\infty}\sup_{t\in(t_1,t_2)}\|X_N(t)-X(t)\|=0.$$

By this we have proved the main result of our paper:

Theorem 5.1. *Let B be a multifractional Brownian motion approximated through the processes B^N given in* (2.1) *and* (2.3). *Let $F,G:\mathbb{R}^n\times[0,T]\to\mathbb{R}^n$ be random functions satisfying with probability* 1 *the conditions* (C1) *and* (C2). *Let $t_0\in(0,T]$ be fixed. Then, each of the stochastic equations*

$$X(t)=X_0+\int_{t_0}^{t}F(X(s),s)ds+\int_{t_0}^{t}G(X(s),s)dB(s),$$

$$X_N(t) = X_0 + \int_{t_0}^{t} F(X_N(s), s)ds + \int_{t_0}^{t} G(X_N(s), s)dB^N(s), \quad N \in \mathbb{N}$$

admits almost surely a unique local solution on a common interval (t_1, t_2) *(which is independent of* N *and contains* t_0*). Moreover, we have the following approximation result*

$$P\left(\lim_{N\to\infty} \sup_{t\in(t_1,t_2)} \|X_N(t) - X(t)\| = 0\right) = 1.$$

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