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Bernstein quasi-interpolants on triangles

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Abstract. The aim of this paper is to provide some results on Bernstein quasi-interpolants of different types applied to functions defined on a triangle. Classical multivariate Bernstein operators and their extensions have been studied for about 25 years by various authors. Based on their representation as differential operators, we extend our previous results on the univariate case to the multivariate one and we define new families of Bernstein quasi-interpolants. Then we compare their approximation properties on various types of functions. Our approach seems to be distinct from another interesting extension given in [5, 6].

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1. Introduction and notations

The aim of this paper is to provide some results on Bernstein quasi-interpolants of different types applied to functions defined on a triangle. Classical multivariate Bernstein operators and their extensions have been studied for about 25 years by various authors (see references). These extensions are of Kantorovitch and Durrmeyer types. We only consider the latter together with the genuine case studied e.g. in [24, 27, 39, 47].

On the unit triangle $T := \{(x, y) \,|\, x, y \geq 0,\, 0 \leq x + y \leq 1\}$, the classical Bernstein quasi-interpolants are defined by

$$\mathcal{B}_n f(x,y) := \sum_{0 \leq i+j \leq n} f\left(\frac{i}{n}, \frac{j}{n}\right) \frac{n!}{i!j!k!} x^i y^j z^k, \quad z := 1-x-y, \quad k := n-i-j.$$

Using the notation $\alpha := (i, j) \in \Delta_n := \{(i, j) | 0 \leq i + j \leq n\}$, we often write them as

$$\mathcal{B}_n f := \sum_{\alpha \in \Delta_n} f\left(\frac{\alpha}{n}\right) B_\alpha^n, \quad B_\alpha^n(x, y, z) := \frac{n!}{i!j!k!} x^i y^j z^k$$

where $\{B_\alpha^n, \alpha \in \Delta_n\}$ is the Bernstein basis of $\mathbb{P}_n$. The Durrmeyer extension has been first developed by Derriennic [13][14] in the case of the Legendre weight and later by various authors in the general case of Jacobi weights [7][8]. With the sacalar product

$$\langle f,g\rangle := \int_T w(x,y)f(x,y)g(x,y)dxdy, \quad w(x,y)=x^p y^q z^r, \ \ p,q,r>-1$$

the multivariate Bernstein-Durrmeyer (abbr. BD) operator is defined by

$$\mathcal{M}_n f := \sum_{\gamma\in\Delta_n} \langle \tilde{B}^n_\gamma, f\rangle B^n_\gamma, \quad \text{where} \quad \tilde{B}^n_\gamma := B^n_\gamma/\langle 1, B^n_\gamma\rangle$$

The genuine Bernstein-Durrmeyer (abbr. GBD) case corresponds to the limit weight $w(x,y)=1/xyz$ and has been studied e.g. in [47]. Its definition involves line integrals along the sides of the triangle T.
Using the representation of the above operators as differential operators in the space $\mathbb{P}$ of bivariate polynomials, we extend our previous results on univariate operators [40, 42, 44, 45, 46] to the bivariate ones and we define new families of Bernstein quasi-interpolants (partial results are given in [41, 44]). Then we compare their approximation properties on various types of functions. Our approach seems to be distinct from another interesting extension given by Berdysheva, Jetter and Stöckler in [3]-[6].
Here is a brief outline of the paper. In sections 2 and 3, we compute the differential forms of the operator $\mathcal{B}_n$ and its inverse $\mathcal{A}_n$ on the space $\mathbb{P}_n$ of polynomials of total degree at most n and we define the associated quasi-interpolants $\mathcal{B}_n^{(r)}$, $0\le r\le n$ (abbr. QIs). Then, in sections 4 and 5 (resp. 6 and 7), we follow the same program for Bernstein-Durrmeyer operators $\mathcal{M}_n$ with Legendre weight $w=1$ (resp. the genuine Bernstein-Durrmeyer operators $\mathcal{G}_n$). In section 8, we give some partial results on the asymptotic expansions and convergence orders of these various quasi-interpolants. In section 9, we give some results on numerical experiments done on the approximations of two functions by Bernstein and genuine Bernstein-Durrmeyer operators. Finally, in Section 10, we set some open problems that would be useful to solve for the applications of those QIs to various problems in approximation theory and numerical analysis.

2. The classical Bernstein operator

2.1. $\mathcal{B}_n$ and its inverse $\mathcal{A}_n = \mathcal{B}_n^{-1}$ as operators on $\mathbb{P}_n$

The classical Bernstein operator

$$\mathcal{B}_n f := \sum_{\alpha\in\Delta_n} f\left(\frac{\alpha}{n}\right) B^n_\alpha$$

where $\{B^n_\alpha, \alpha\in\Delta_n\}$ is the Bernstein basis of $\mathbb{P}_n$, is an **isomorphism** of the space $\mathbb{P}_n$ of bivariate polynomials of total degree at most n. This can be proved in various ways. For example, let $\{\ell^n_\alpha, \alpha\in\Delta_n\}$ be the **Lagrange basis** of $\mathbb{P}_n$ (see e.g. Ciarlet [11], chapter 2) based on points $\{\frac{\alpha}{n}, \alpha\in\Delta_n\}$, then $\mathcal{B}_n\ell^n_\alpha = B^n_\alpha$ for $\alpha\in\Delta_n$. Similarly, let $\{\nu^n_\alpha, \alpha\in\Delta_n\}$ be the **Newton basis** of

$\mathbb{P}_n$ based on the same points $\{\frac{\alpha}{n}, \alpha \in \Delta_n\}$, defined for $|\alpha| = i + j = p \leq n$ and using the Pochammer symbol $(n)_p = n(n-1)\dots(n-p+1)$, by

$$\nu_\alpha^n = \prod_{k=0}^{i-1}(nx-k)\prod_{\ell=0}^{j-1}(ny-\ell)/(n)_p$$

then $\mathcal{B}_n \nu_\alpha^n = m_\alpha$ where $m_\alpha(x,y) = m_{i,j}(x,y) := x^i y^j$ are the monomials of $\mathbb{P}_n$. So the image of the Lagrange (resp. Newton) basis is the Bernstein (resp. monomial) basis.

Denoting $\mathcal{A}_n = \mathcal{B}_n^{-1}$ the inverse operator of $\mathcal{B}_n$ on $\mathbb{P}_n$, then we have $\mathcal{A}_n B_\alpha^n = \ell_\alpha^n$ and $\mathcal{A}_n m_\alpha = \nu_\alpha^n$ for all $\alpha \in \Delta_n$. These properties are used below for the computation of the coefficients of $\mathcal{A}_n$ expressed as a differential operator.

2.2. $\mathcal{B}_n$ as a differential operator

As in the univariate case (see e.g. [33], chapter 1, and [45]), the operator $\mathcal{B}_n$ has the following representation in $\mathbb{P}_n$:

$$\mathcal{B}_n = Id + \sum_{r=2}^{n}\sum_{k+\ell=r}\beta_{k,\ell}D^{k,\ell}$$

Note that the polynomial coefficients $\beta_{k,\ell}$ should be denoted $\beta_{k,\ell}^{(n)}$ since they depend on n. However, we omit the upper index for the sake of clarity.

Theorem. *The polynomial coefficients $\beta_{k,\ell}$ satisfy the recurrence relation, for $k,\ell \geq 1$*

$$n\left((k+1)\beta_{k+1,\ell} + (\ell+1)\beta_{k,\ell+1}\right)$$
$$= (1-x-y)\left(x(\partial_{10}\beta_{k,\ell} + \beta_{k-1,\ell}) + y(\partial_{01}\beta_{k,\ell} + \beta_{k,\ell-1})\right).$$

with $\beta_{0,0} = 1, \beta_{1,0} = \beta_{0,1} = 0$, and for $k,\ell \geq 1$

$$n(k+1)\beta_{k+1,0} = x(1-x)(\partial_{10}\beta_{k,0} + \beta_{k-1,0})$$

$$n(\ell+1)\beta_{0,\ell+1} = y(1-y)(\partial_{01}\beta_{0,\ell} + \beta_{0,\ell-1})$$

Proof. Using Taylor's formula

$$f(s,t) = f(x,y) + \sum_{r\geq 1}\frac{1}{r!}\left(\sum_{k+\ell=r}\binom{n}{k}(s-x)^k(t-y)^\ell D^{k,\ell}f(x,y)\right)$$

and applying the Bernstein operator

$$\mathcal{B}_n f(x,y) = f(x,y) + \sum_{n\geq 1}\frac{1}{n!}\left(\sum_{k+\ell=n}\binom{n}{k}B_n[(.-x)^k(.-y)^\ell](x,y)D^{k,\ell}f(x,y)\right)$$

we first obtain

$$\beta_{k,\ell}(x,y) := \frac{1}{n!}\binom{n}{k}B_n[(.-x)^k(.-y)^\ell](x,y).$$

or, setting $\phi_{k,\ell} = (.-x)^k(.-y)^\ell$ and $m := n-k-\ell$:

$$\beta_{k,\ell} = \frac{1}{k!\ell!(n-k-\ell)!}\sum_{i+j\leq n}\phi_{k,\ell}\left(\frac{i}{n},\frac{j}{n}\right)B_{i,j}^n$$

Let us compute the expression

$$z(xD^{1,0}+yD^{0,1})\beta_{k,\ell}=\frac{xzD^{1,0}+yzD^{0,1}}{k!\ell!m!}\mathcal{B}_n\phi_{k,\ell}$$

First we get

$$D^{1,0}\mathcal{B}_n\phi_{k,\ell}=-k\sum_{i+j\le n}\phi_{k-1,\ell}\left(\frac{i}{n},\frac{j}{n}\right)B^n_{i,j}+\sum_{i+j\le n}\phi_{k,\ell}\left(\frac{i}{n},\frac{j}{n}\right)D^{1,0}B^n_{i,j},$$

with

$$D^{1,0}B^n_{i,j}=n\left(B^{n-1}_{i-1,j}-B^{n-1}_{i,j}\right)$$

Moreover, we have

$$nxzB^n_{i-1,j}=izB^n_{i,j},\quad\text{and}\quad nxzB^n_{i,j}=(n-i-j)B^n_{i,j}$$

therefore

$$xzD^{1,0}\mathcal{B}_n\phi_{k,\ell}=-kxz\mathcal{B}_n\phi_{k-1,\ell}+z\sum i\phi_{k,\ell}\left(\frac{i}{n},\frac{j}{n}\right)B^n_{i,j}$$
$$-x\sum(n-i-j)\phi_{k,\ell}\left(\frac{i}{n},\frac{j}{n}\right)B^n_{i,j}.$$

Now, using the identities:

$$i=n\left(\frac{i}{n}-x\right)+nx,\quad\text{and}\quad i=n\left(z-n\left(\frac{i}{n}-x\right)-n\left(\frac{j}{n}-x\right)\right)$$

we obtain

$$xzD^{1,0}\mathcal{B}_n\phi_{k,\ell}=-kz\mathcal{B}_n\phi_{k-1,\ell}+n(1-y)\mathcal{B}_n\phi_{k+1,\ell}+nx\mathcal{B}_n\phi_{k,\ell+1}$$

In the same way, we also have

$$yzD^{0,1}\mathcal{B}_n\phi_{k,\ell}=-kz\mathcal{B}_n\phi_{k,\ell-1}+n(1-x)\mathcal{B}_n\phi_{k,\ell+1}+ny\mathcal{B}_n\phi_{k+1,\ell}$$

and finally

$$z(xD^{1,0}+yD^{0,1})\mathcal{B}_n\phi_{k,\ell}=-kz(x\mathcal{B}_n\phi_{k-1,\ell}+y\mathcal{B}_n\phi_{k,\ell-1})$$
$$+n(\mathcal{B}_n\phi_{k+1,\ell}+\mathcal{B}_n\phi_{k,\ell+1}),$$

which gives the following recurrence relation on the polynomial coefficients:

$$n(k+1)\beta_{k+1,\ell}+n(\ell+1)\beta_{k,\ell+1}=z\left(x(D^{1,0}\beta_{k,\ell}+\beta_{k-1,\ell})+y(D^{0,1}\beta_{k,\ell}+\beta_{k,\ell-1})\right).$$

□

Examples. Using the notations $X:=x(1-x),Y:=y(1-y)$, the first beta polynomials (depending on n) are given by

$$2n\beta_{2,0}=X,\qquad n\beta_{1,1}=-xy$$
$$6n^2\beta_{3,0}=X(1-2x),\qquad 2n^2\beta_{2,1}=-3xy(1-2x),$$
$$24n^3\beta_{4,0}=X(1+3(n-2)X),\qquad 6n^3\beta_{3,1}=-4xy(1+3(n-2)X),$$
$$4n^3\beta_{2,2}=xy(n-1-(n-2)(x+y)+3(n-2)xy)$$
$$5!n^4\beta_{5,0}=(1-2x)X(1+2(5n-6)X),\quad 24n^4\beta_{4,1}=-xy(1+2(5n-6)X)$$
$$12n^5\beta_{3,2}=10xy((n-1)(1-6x)-(n-2)y-(5n-6)x(x+3y-4xy))$$

2.3. $\mathcal{A}_n := \mathcal{B}_n^{-1}$ as a differential operator

2.3.1. First method: long recursion. The operator $\mathcal{A}_n$ has also the following representation in $\mathbb{P}_n$:

$$\mathcal{A}_n = Id + \sum_{p=2}^{n} \sum_{i+j=p} \alpha_{i,j} D^{i,j}$$

A first method, giving a long recursion, consists in deducing the polynomial coefficients from the identities $\mathcal{A}_n m_{k,\ell} = \nu_{k,\ell}^n$ for $0 \leq i+j \leq n$.

$$\nu_{k,\ell}^n = x^k y^\ell + \sum_{p=2}^{k+\ell} \sum_{i+j=p} \frac{k!}{(k-i)!} \frac{\ell!}{(\ell-j)!} x^{k-i} y^{\ell-j} \alpha_{i,j}$$

giving the (long) recursion

$$\alpha_{k,\ell} = \frac{\nu_{k,\ell}^n - m_{k,\ell}}{k!\ell!} - \sum_{(0,0)<(i,j)<(k,\ell)} \frac{x^{k-i}}{(k-i)!} \frac{y^{\ell-j}}{(\ell-j)!} \alpha_{i,j}.$$

2.3.2. Second method : expansion in the Newton basis. From the Taylor expansion of $f \in \mathbb{P}_n$:

$$f(.,.) = f(x,y) + \sum_{p=1}^{n} \sum_{k+\ell=p} \frac{(.-x)^k (.-y)^\ell}{k!\ell!} D^{k,l} f(x,y),$$

we deduce

$$\mathcal{A}_n f = f + \sum_{p=1}^{n} \left(\sum_{k+\ell=p} \mathcal{A}_n \left[\frac{(.-x)^k (.-y)^\ell}{k!\ell!} \right] D^{k,l} f(x,y) \right)$$

giving

$$\alpha_{k,\ell}(x,y) = \mathcal{A}_n \left[\frac{(.-x)^k (.-y)^\ell}{k!\ell!} \right]$$

and since $\mathcal{A}_n m_{ij} = \nu_{i,j}$, we obtain the compact form :

$$\alpha_{k,\ell}(x,y) = \frac{(-1)^p}{k!\ell!} \sum_{i=0}^{k} \sum_{j=0}^{\ell} \binom{k}{i} \binom{\ell}{j} (-1)^{i+j} x^{k-i} y^{\ell-j} \nu_{i,j}(x,y).$$

2.3.3. Third method : direct short recursion. At least for polynomials $\alpha_{k,0}$ and $\alpha_{0,\ell}$, we have the short recursions [45]

$$(k+1)(n-k)\alpha_{k+1,0} = -k(1-2x)\alpha_{k,0} - X\alpha_{k-1,0}.$$

$$(\ell+1)(n-\ell)\alpha_{0,\ell+1} = -k(1-2y)\alpha_{0,\ell} - Y\alpha_{0,\ell-1}.$$

Following the model of beta-polynomials:

$$(k+1)n\beta_{k+1,\ell} + n(\ell+1)\beta_{k,\ell+1} = z\left(x(D^{1,0}\beta_{k,\ell} + \beta_{k-1,\ell}) + y(D^{0,1}\beta_{k,\ell} + \beta_{k,\ell-1})\right).$$

it would be possible to get a recursion for the computation of these polynomials. However, it is still an open question.

2.3.4. A table of polynomials alpha. With the notations $X = x(1-x), Y = y(1-y), n_k := (n-1)\dots(n-k), [i,j] := \alpha_{i,j}$, here are the first polynomials alpha

$$2n_1[2,0] = X, \quad n_1[1,1] = xy, \quad 2n_1[2,0] = Y$$

$$3n_2[3,0] = (1-2x)X \quad n_2[2,1] = -xy(1_2 x),$$

$$n_2[1,2] = -xy(1-2y), \quad 3n_2[0,3] = (1-2y)Y$$

$$8n_3[4,0] = -X(2-(n+6)X), \quad 2n_3[3,1] = xy(2-(n+6)X),$$

$$4n_3[2,2] = xy(n-(n+6)(x+y-3xy))$$

$$30n_4[5,0] = (1-2x)X(6-(5n+12)X),$$

$$6n_4[4,1] = -xy(1-2x)(6-(5n+12)X)$$

$$6n_4[3,2] = -xy(n-6nx-(n+12)y+(5n+12)x(x+3y-4xy))$$

3. Bernstein quasi-interpolants

3.1. Quasi-interpolants of order r

Given $0 \le r \le n$, define the truncated inverse of order r

$$\mathcal{A}_n^{(r)} = Id + \sum_{p=2}^{r} \sum_{i+j=p} \alpha_{i,j} D^{i,j}$$

Then the Bernstein-quasi-interpolant (abbr. BQI) of order r is defined by

$$\mathcal{B}_n^{(r)} = \mathcal{A}_n^{(r)} \mathcal{B}_n$$

In other words, for all polynomial $p \in \mathbb{P}_n$, we have

$$\mathcal{B}_n^{(r)} p = \mathcal{B}_n p + \sum_{p=2}^{r} \sum_{i+j=p} \alpha_{i,j} D^{i,j} \mathcal{B}_n p$$

Theorem. *The operator $\mathcal{B}_n^{(r)}$ is exact on $\mathbb{P}_r$, for all $0 \le r \le n$.*
Proof. As $p = \mathcal{A}_n \mathcal{B}_n p = \mathcal{B}_n^{(n)} p$, we can write

$$p - \mathcal{B}_n^{(r)} p = \sum_{p=r+1}^{n} \sum_{i+j=p} \alpha_{i,j} D^{i,j} \mathcal{B}_n p$$

As $p \in \mathbb{P}_r$, we have $\mathcal{B}_n p \in \mathbb{P}_r$, thus $D^{i,j}\mathcal{B}_n p = 0$ for all (i,j) satisfying $i + j = p \ge r + 1$, thus $p - \mathcal{B}_n^{(r)} p = 0$. □

Therefore, we have constructed **a chain of intermediate operators** between the classical Bernstein operator and the identity operator which can be written in the form of the Lagrange interpolation operator $\mathcal{L}_n$ since $\mathcal{A}_n B_\alpha^n = \ell_\alpha^n$:

$$p = \mathcal{A}_n \mathcal{B}_n p = \sum_{\alpha \in \Delta_n} f\left(\frac{\alpha}{n}\right) \mathcal{A}_n B_\alpha^n = \sum_{\alpha \in \Delta_n} f\left(\frac{\alpha}{n}\right) \ell_\alpha^n = \mathcal{L}_n p$$

3.2. Some open questions on BQIs

Among the open questions relative to the BQIs, the following seem particularly interesting:

1) Prove, as in the univariate case [50], that for $r \in \mathbb{N}$ fixed, the BQIs of order r are uniformly bounded, i.e. there exists a constant C_r such that

$$\|\mathcal{B}_n^{(r)}\|_\infty \leq C_r \quad \text{for all } \ n \geq r$$

2) Numerical experiments show that some functions f (e.g. of Runge type) are better approximated by intermediate polynomials $\mathcal{B}_n^{(r)} f$ rather than by their Lagrange interpolant. This is not quite surprising in view of the fact that $\|\mathcal{L}_n\|_\infty$ goes to infinity rather fastly when $n \to \infty$ (see e.g. [9]). Therefore the approximating polynomials generated in this way can be useful in practice, in approximation as well as in CAGD.

3) It would be interesting to have a direct formula giving the polynomial coefficients $\alpha_{i,j}$, or at least a short recursive formula.

4. Bernstein-Durrmeyer operators

For the sake of simplicity, we take $w = 1$ (Legendre) and we only consider Bernstein Durrmeyer quasi-interpolants (abbr. BDQIs) in that case. Of course, the same technique can be extended to general BDQIs with an arbitrary Jacobi weight. It would be also interesting to study the generalizations recently proposed in [3, 4]. Setting

$$\langle f, g \rangle := \int_T f(x,y) g(x,y) dx dy$$

since area$(T) = 1/2$, we have

$$\int_T B_\gamma^n = \frac{1}{(n+1)(n+2)}$$

whence the definition of the BD operator:

$$\mathcal{M}_n f := (n+1)(n+2) \sum_{\gamma \in \Delta_n} \langle B_\gamma^n, f \rangle B_\gamma^n$$

4.1. $\mathcal{M}_n$ and $\mathcal{K}_n = \mathcal{M}_n^{-1}$ as operators on $\mathbb{P}_n$

Consider a family of orthogonal polynomials $\{P_{k,\ell}, \ \ 0 \leq |\gamma| = k + \ell \leq n\}$ on T (see e.g. [12, 21, 22, 48]) whose expansion in the BB basis is the following:

$$P_\gamma = \sum_{\delta \in \Delta_n} p(\delta, \gamma) B_\delta^n$$

It is known (see e.g. [13]) that for $\gamma \in \Delta_s$, with $0 \leq s \leq n$, one has

$$\mathcal{M}_n P_\gamma = \rho_\gamma(n) P_\gamma,$$

where the eigenvalue is given by

$$\rho_\gamma(n) = \frac{[n]_s}{(n+3)_s} = \frac{\Gamma(n+1)}{\Gamma(n-s+1)} \frac{\Gamma(n+3)}{\Gamma(n+s+3)}$$

We use here the Pochhammer symbol defined by

$$(n)_s := n(n+1)\dots(n+s-1) = \frac{(n+s-1)!}{(n-1)!} = \frac{\Gamma(n+s)}{\Gamma(n)}$$

and we set

$$[n]_s := n(n-1)\dots(n-s+1) = \frac{n!}{(n-s)!} = \frac{\Gamma(n+1)}{\Gamma(n-s+1)}$$

Thus $\mathcal{M}_n$ is an automorphism of $\mathbb{P}_n$. Denoting $\mathcal{K}_n = \mathcal{M}_n^{-1}$, we have

$$\mathcal{K}_n P_\gamma = \rho_\gamma^{-1}(n) P_\gamma, \quad \gamma \in \Delta_n$$

4.2. $\mathcal{M}_n$ as a differential operator on $\mathbb{P}_n$

Like the classical Bernstein operator, the BD operator $\mathcal{M}_n$ can be expressed as a differential operator on $\mathbb{P}_n$:

$$\mathcal{M}_n = \sum_{r=0}^{n} \sum_{\delta\in\Delta_r} \mu_\delta^{(n)} D^\delta, \quad \mu_\delta^{(n)} \in \mathbb{P}_r$$

Therefore, for $|\gamma| = m \le n$:

$$\mathcal{M}_n P_\gamma = \sum_{r=0}^{m} \sum_{\delta\in\Delta_r} \mu_\delta^{(n)} D^\delta P_\gamma = \rho_\gamma(n) P_\gamma$$

As in Section 2.2, a direct expression of the polynomials $\mu_\delta^{(n)}$ for $\delta = (k,\ell) \in \Delta_r$, can be deduced from Taylor's formula:

$$\mu_\delta^{(n)} = \frac{1}{r!}\binom{r}{k} \mathcal{M}_n[(.-x)^k(.-y)^\ell]$$

4.3. $\mathcal{K}_n := \mathcal{M}_n^{-1}$ as a differential operator

One can also write $\mathcal{K}_n$ as a differential operator on $\mathbb{P}_n$:

$$\mathcal{K}_n = \sum_{r=0}^{n} \sum_{\delta\in\Delta_r} \kappa_\delta^{(n)} D^\delta, \quad \kappa_\delta^{(n)} \in \mathbb{P}_r$$

Therefore, for $|\gamma| = m \le n$, we have the long recursion:

$$\mathcal{K}_n P_\gamma = \sum_{r=0}^{m} \sum_{\delta\in\Delta_r} \kappa_\delta^{(n)} D^\delta P_\gamma = \rho_\gamma^{-1}(n) P_\gamma$$

For the computation of the polynomial coefficients κ, we did not use this method. Rather, we compute the polynomials $p_\gamma := \mathcal{M}_n m_\gamma$ from which we deduce $\mathcal{K}_n p_\gamma = m_\gamma$ as follows.

4.4. The polynomials p_γ

In order to find the polynomial p_γ whose image by $\mathcal{M}_n$ is the monomial $m_\gamma := x^i y^j$, i.e. such that $\mathcal{K}_n m_\gamma = p_\gamma$, we write

$$p_\gamma := \sum_{\delta\in\Delta_n} c(\gamma,\delta) B_\delta^{(n)}$$

Setting

$$B_\gamma^n := B_{i,j}^n := \frac{n!}{i!j!k!} x^i y^j z^k, \quad k := n-i-j, \quad \text{for } \gamma := (i,j) \in \Delta_n,$$

$$B_\delta^n := B_{p,q}^n := \frac{n!}{p!q!r!} x^p y^q z^r, \quad r := n-p-q, \quad \text{for } \delta := (p,q) \in \Delta_n$$

and introducing the Gram matrix

$$G[\gamma,\delta] := \langle B_\gamma^n, B_\delta^n \rangle = \frac{1}{(n+1)^2} \frac{\binom{i+p}{i}\binom{j+q}{j}\binom{k+r}{k}}{\binom{2n+2}{n+1}}$$

we obtain

$$\mathcal{M}_n p_\gamma = \sum_{\delta\in\Delta_n} c(\gamma,\delta)\mathcal{M}_n B_\delta^{(n)} = \frac{1}{2}(n+1)(n+2) \sum_{\delta\in\Delta_n} c(\gamma,\delta) \left(\sum_{\theta\in\Delta_n} G[\delta,\theta] B_\theta^n \right)$$

$$\mathcal{M}_n p_\gamma = \frac{1}{2}(n+1)(n+2) \sum_{\theta\in\Delta_n} \left(\sum_{\delta\in\Delta_n} G[\theta,\delta] c(\gamma,\delta) \right) B_\theta^n$$

Now, we need the representation of the monomial m_γ in the BB basis:

$$m_{i,j} = \sum_{\theta\in\Delta_n} \frac{\binom{i}{r}\binom{j}{s}}{\binom{n}{r,s}} B_\theta^n, \quad \theta := (r,s)$$

By identification, we compute $c(\gamma,\delta)$ as the solution of the system of linear equations

$$\frac{1}{2}(n+1)(n+2) \sum_{\delta\in\Delta_n} G[\theta,\delta] c(\gamma,\delta) = \frac{\binom{i}{r}\binom{j}{s}}{\binom{n}{r,s}}, \quad \theta \in \Delta_n$$

4.5. A table of the first polynomials kappa

The list of the first kappa polynomials shows that they are more complex than alpha polynomials of section 2.3.4 :

$$n\kappa_{1,0}^{(n)} = 3x - 1, \quad n\kappa_{0,1}^{(n)} = 3y - 1$$

$$(n)_2\, \kappa_{2,0}^{(n)} = (n+9)x^2 - (n+7)x + 1$$

$$(n)_2\, \kappa_{1,1}^{(n)} = 2(n+9)xy - 4(x+y) + 1$$

$$(n)_2\, \kappa_{0,2}^{(n)} = (n+9)y^2 - (n+7)y + 1$$

$$(n)_3\, \kappa_{3,0}^{(n)} = 5(n+5)x^3 - (7n+31)x^2 + (2n+11)x - 1$$

$$(n)_3\, \kappa_{2,1}^{(n)} = 15(n+5)x^2 y - (n+13)x^2 - 4(2n+11)xy + (n+8)x + 5y - 1$$

$$(n)_3\, \kappa_{1,2}^{(n)} = 15(n+5)xy^2 - (n+13)y^2 - 4(2n+11)xy + 5x + (n+8)y - 1$$

$$(n)_3\, \kappa_{0,3}^{(n)} = 5(n+5)y^3 - (7n+31)y^2 + (2n+11)y - 1$$

$$(n)_4\, \kappa_{4,0}^{(n)} = \frac{1}{2}((n+4)(n+33)x^4 - 2(n^2+34n+113)x^3$$

$$+(n+4)(n+33))x^2 - 6(n+5)x + 2)$$

$$(n)_4\, \kappa_{3,1}^{(n)} = 2(n+4)(n+33)x^2y(x-1) - 2(3n+19)x^3$$

$$+(8n+39)x(x+2y) - 2(n+6)x - 6y + 1$$

$$(n)_4\, \kappa_{2,2}^{(n)} = 3(n+4)(n+33)x^2y^2 - (n^2+46n+189)xy(x+y) + (n+18)(x^2+y^2)$$

$$+(n+5)(n+18)xy - (n+9)(x+y) + 1$$

$$\kappa_{1,3}^{(n)}(x,y) = \kappa_{3,1}^{(n)}(y,x), \qquad \kappa_{0,4}^{(n)}(x,y) = \kappa_{4,0}^{(n)}(y,x).$$

5. Bernstein-Durrmeyer quasi-interpolants

5.1. Bernstein-Durrmeyer quasi-interpolants of order r

Given $0 \le r \le n$, define the truncated inverse of order r

$$\mathcal{K}_n^{(r)} = Id + \sum_{p=2}^{r} \sum_{i+j=p} \kappa_{i,j} D^{i,j}$$

Then the Bernstein-Durrmeyer quasi-interpolant (abbr. BDQI) of order r is defined by

$$\mathcal{M}_n^{(r)} = \mathcal{K}_n^{(r)} \mathcal{M}_n$$

In other words, for all polynomial $p \in \mathbb{P}_n$, we have

$$\mathcal{M}_n^{(r)} p = \mathcal{M}_n p + \sum_{p=2}^{r} \sum_{i+j=p} \kappa_{i,j} D^{i,j} \mathcal{M}_n p$$

Theorem. *The operator $\mathcal{M}_n^{(r)}$ is exact on $\mathbb{P}_r$, for all $0 \le r \le n$.*
The proof is the same as for BQIs.
Therefore, we have constructed a chain of intermediate operators between the Bernstein-Durrmeyer operator and the identity operator. The latter can be written in the form of the **orthogonal projector** $\mathcal{P}_n$ on the space $\mathbb{P}_n$. Indeed, since $\mathcal{M}_n$ is a self-adjoint isomorphism in that space, we have, for all $p \in \mathbb{P}_n$:

$$0 = \langle f - \mathcal{P}_n f, \mathcal{M}_n p\rangle = \langle \mathcal{M}_n(f - \mathcal{P}_n f), p\rangle$$

As $\mathcal{M}_n(f - \mathcal{P}_n f) \in \mathbb{P}_n$, this implies first that $\mathcal{M}_n f = \mathcal{M}_n \mathcal{P}_n f$, i.e. $\mathcal{M}_n \mathcal{K}_n \mathcal{M}_n f = \mathcal{M}_n \mathcal{P}_n f$ and second that $\mathcal{K}_n \mathcal{M}_n f = \mathcal{P}_n f$, in other words $\mathcal{K}_n \mathcal{M}_n = \mathcal{K}_n \mathcal{M}_n$, q.e.d. □

5.2. Some open questions on BDQIs

Among the open questions relative to the BDQIs, the following seem particularly interesting:
1) Prove that for $r \in \mathbb{N}$ fixed, the BDQIs of order r are uniformly bounded for L^p norms i.e. there exists constants $C(r,p)$ such that

$$\|\mathcal{B}_n^{(r)}\|_p \leq C(r,p) \quad \text{for all } \ n \geq r$$

2) As for BQIs, numerical experiments show that some functions f (e.g. of Runge type) are better approximated by intermediate polynomials $\mathcal{M}_n^{(r)} f$ rather than by their L^2-orthogonal projection $\mathcal{P}_n f$ on $\mathbb{P}_n$. (This is not quite surprising in view of the fact that $\|\mathcal{L}_n\|_\infty$ goes to infinity fastly when $n \to \infty$). Therefore the approximating polynomials generated in this way can be useful in practice, both in approximation and in CAGD.
3) It would be interesting to have a direct formula giving the polynomial coefficients $\kappa_{i,j}$, or at least a recursive formula allowing their fast computation.
4) From the computational point of view, it would be also interesting to have a fast algorithm for the effective computation of scalar products $\langle B_\gamma^n, f\rangle$. Even though the Bernstein polynomials are Jacobi weights (up to a constant), using the corresponding Gauss-Jacobi cubature formulas seem rather complicated since weights and data points are distinct.

6. Genuine Bernstein-Durrmeyer operators

Let f_s denote the restriction of f to the edge opposite to the vertex $A_s = (e_s)$ (barycentric coordinates :s $e_1 = (1,0,0), e_2 = (0,1,0)$ and $e_3 = (0,0,1)$), let b_{k-1}^{n-2} be the univariate Bernstein polynomials on that edge, and let Δ_n^* be the set of indices $\gamma \in \Delta_n$ with no null component. Then the genuine Bernstein-Durrmeyer (abbr. GBD) operators are defined by

$$\mathcal{G}_n f := \sum_{r=1}^{3} f(e_r) B_{ne_r}^n + (n-1) \sum_{s=1}^{3} \sum_{k=1}^{n-1} \langle f_s, b_{k-1}^{n-2} \rangle B_k^n$$

$$+(n-1)(n-2) \sum_{\gamma \in \Delta_n^*} \langle f, B_\alpha^{n-3} \rangle B_\alpha^n$$

Note that in the second sum, $\langle f_s, b_{k-1}^{n-2}\rangle$ is a univariate sacalar product along the edge, and B_k^n is an abbreviation for B_α^n when $\alpha = (k, n-k, 0)$, $(k, 0, n-k)$ or $(0, k, n-k)$.
Like the classical Bernstein and the BD operators, the GBD operator $\mathcal{G}_n$ can be expressed as a differential operator on $\mathbb{P}_n$:

$$\mathcal{G}_n = \sum_{r=0}^{n} \sum_{\delta \in \Delta_r} \theta_\delta^{(n)} D^\delta, \quad \bar{\beta}_\delta^{(n)} \in \mathbb{P}_r$$

The inverse operator $\mathcal{H}_n := \mathcal{G}_n^{-1}$ can also be expressed as a differential operator on $\mathbb{P}_n$:

$$\mathcal{H}_n = \sum_{r=0}^{n} \sum_{\delta \in \Delta_r} \eta_\delta^{(n)} D^\delta, \quad \bar{\alpha}_\delta^{(n)} \in \mathbb{P}_r$$

7. Genuine Bernstein-Durrmeyer quasi-interpolants

7.1. Genuine Bernstein-Durrmeyer quasi-interpolants of order r

Given $0 \le r \le n$, define the truncated inverse of order r

$$\mathcal{H}_n^{(r)} = Id + \sum_{p=2}^{r} \sum_{i+j=p} \theta_{i,j} D^{i,j}$$

Then the Genuine Bernstein-Durrmeyer quasi-interpolant (abbr. GBDQI) of order r is defined by

$$\mathcal{G}_n^{(r)} = \mathcal{H}_n^{(r)} \mathcal{G}_n$$

$$\mathcal{G}_n^{(r)} := \sum_{|\gamma|=0}^{r} \eta_\gamma^{(n)} D^\gamma G_n, \quad 0 \le r \le n$$

Theorem. *The operator $\mathcal{G}_n^{(r)}$ is exact on $\mathbb{P}_r$, for all $0 \le r \le n$.*
The proof is the same as for BQIs and BDQIs.

7.2. A table of the first polynomials eta

With the notation $n_k := (n-1)\dots(n-k)$, here are the first polynomials

$$n_1 \eta_{20}^{(n)} = -X, \quad n_1 \bar{\eta}_{11}^{(n)} = 2xy$$

$$n_2 \eta_{30}^{(n)} = (1-2x)X, \quad n_2 \bar{\eta}_{21}^{(n)} = -3xy(1-2x)$$

$$2n_3 \eta_{40}^{(n)} = X((n+7)X-2), \quad n_3 \bar{\eta}_{31}^{(n)} = -2xy((n+7)X-2)$$

$$n_3 \eta_{22}^{(n)} = xy((n+7)(3xy-x-y)+n+1)$$

$$n_4 \eta_{5,0} := (1-2x)X(1-(n+3)X), \quad n_4 \eta_{4,1} := 5(2x-1)(1-(n+3)X)xy$$

$$n_4 \eta_{3,2} := (5(n+3)x(4xy-x-3y)+(n+1)(6x-1)+(n+11)y)xy$$

8. Asymptotic formulas for Bernstein type quasi-interpolants

We only sketch a study the convergence for polynomials though the results can be extended to smooth functions (this will be developed elsewhere). Given a polynomial $p \in \mathbb{P}$, we are interested in the following limits:

$$\lim n^{r+1}(\mathcal{Q}_n^{(2r)} p(x) - p(x)) \text{ and } \lim n^{r+1}(\mathcal{Q}_n^{(2r+1)} p(x) - p(x))$$

where $\mathcal{Q}_n^{(s)}, s = 2r, 2r+1$ is one of the three types of Bernstein QIs previously defined. For original operators (case $s = 0$), see also [1, 2, 33, 34, 48].

8.1. Bernstein QIs

For beta and alpha polynomials, we define the polynomials

$$\bar{\beta}_{k,\ell} = \lim n^r \beta_{k,\ell} \quad \text{for} \quad k+\ell = 2r-1 \text{ or } 2r$$

$$\bar{\alpha}_{k,\ell} = \lim n^r \alpha_{k,\ell} \quad \text{for} \quad k+\ell = 2r-1 \text{ or } 2r$$

From the recurrence formulas of section 2.2, we immediately deduce the following

Theorem. *The following recurrence relations hold:*

$$(k+1)\bar{\beta}_{k+1,\ell} + (\ell+1)\bar{\beta}_{k,\ell+1} = z\left(x\bar{\beta}_{k-1,\ell} + y\bar{\beta}_{k,\ell-1}\right) \quad \text{for} \quad k+\ell = 2r-1,$$

$$(k+1)\bar{\beta}_{k+1,\ell} + (\ell+1)\bar{\beta}_{k,\ell+1} = z\left(xD^{1,0}\bar{\beta}_{k,\ell} + yD^{0,1}\bar{\beta}_{k,\ell}\right) \quad \text{for} \quad k+\ell = 2r.$$

We have not yet obtained the general formulas for alpha-polynomials. However, for polynomials $\bar{\alpha}_{k,0}$ and $\bar{\alpha}_{0,\ell}$, we deduce from the recurrence formulas of section 2.3.4 :

$$(2r+1)\bar{\alpha}_{2r+1,0} = -2r(1-2x)\bar{\alpha}_{2r,0} - X\bar{\alpha}_{2r-1,0} \qquad (2r+2)\bar{\alpha}_{2r+2,0} = -X\bar{\alpha}_{2r,0},$$

$$(2r+1)\bar{\alpha}_{0,2r+1} = -2r(1-2y)\bar{\alpha}_{0,2r} - Y\bar{\alpha}_{0,2r-1} \qquad (2r+2)\bar{\alpha}_{0,2r+2} = -Y\bar{\alpha}_{0,2r},$$

Here is a table of the first polynomials:

(k,ℓ)	$\beta_{k,\ell}$	$\alpha_{k,\ell}$
$(2,0)$	$X/2$	$-X/2$
$(1,1)$	$-xy$	xy
$(3,0)$	$(1-2x)X/6$	$(1-2x)X/3$
$(2,1)$	$-xy(1-2x)/2$	$-xy(1-2x)$
$(4,0)$	$X^2/8$	$X^2/8$
$(3,1)$	$-xyX/2$	$-xyX/2$
$(2,2)$	$xy(z+3xy)/4$	$xy(z+3xy)/4$

The asymptotic formulas are obtained as follows. For any polynomial f:

$$f - \mathcal{B}_n^{(q)} f = \sum_{p \geq q+1} \sum_{i+j=p} \alpha_{i,j} D^{i,j} f$$

For $q = 2r-1$, we get

$$n^r(f - \mathcal{B}_n^{(2r)} f) = \sum_{p \geq 2r} \sum_{i+j=p} n^r \alpha_{i,j} D^{i,j} \mathcal{B}_n f$$

As $\lim n^r \alpha_{i,j} = \bar{\alpha}_{i,j}$ for $i+j = 2r$, $\lim n^r \alpha_{i,j} = 0$ for $i+j = p > 2r$ and $\lim D^{i,j} \mathcal{B}_n f = D^{i,j} f$, we obtain:

$$\lim n^r(f - \mathcal{B}_n^{(2r)} f) = \sum_{i+j=2r} \bar{\alpha}_{i,j} D^{i,j} f$$

Similarly, for $q = 2r$, we get

$$n^{r+1}(f - \mathcal{B}_n^{(2r+1)} f) = \sum_{p \geq 2r+1} \sum_{i+j=p} n^{r+1} \alpha_{i,j} D^{i,j} \mathcal{B}_n f$$

As $\lim n^{r+1}\alpha_{i,j} = \bar{\alpha}_{i,j}$ for $i+j = 2r+1, 2r+2$, $\lim n^{r+1}\alpha_{i,j} = 0$ for $i+j = p > 2r+2$ and $\lim D^{i,j}\mathcal{B}_n f = D^{i,j} f$, we obtain:

$$\lim n^{r+1}(f - \mathcal{B}_n^{(2r+1)} f) = \sum_{p=2r+1}^{2r+2} \sum_{i+j=p} \bar{\alpha}_{i,j} D^{i,j} f$$

Examples.

$$\lim n(f - \mathcal{B}_n^{(2)} f) = -\frac{1}{2}(X D^{2,0} f - xy D^{1,1} f + Y D^{0,2} f)$$

$$\lim n^2(f - \mathcal{B}_n^{(3)} f) = \sum_{|\gamma|=3} \bar{\alpha}_\gamma D^\gamma f + \sum_{|\gamma|=4} \bar{\alpha}_\gamma D^\gamma f$$

$$= \frac{1}{3}(1-2x)XD^{3,0}f - xy(1-2x)D^{2,1}f - xy(1-2xy)D^{1,2}f + (1-2y)YD^{0,3}f$$

$$+\frac{1}{8}X^2D^{4,0}f - \frac{1}{2}xyXD^{3,1}f + \frac{1}{4}xy(z+3xy)D^{2,2}f - \frac{1}{2}xyYD^{1,3}f + \frac{1}{8}Y^2D^{0,4}f$$

8.2. Bernstein-Durrmeyer QIs

For lambda and kappa polynomials, we define

$$\bar{\lambda}_{k,\ell} = \lim n^r \lambda_{k,\ell} \quad \text{for} \quad k+\ell = 2r-1 \text{ or } 2r$$

$$\bar{\kappa}_{k,\ell} = \lim n^r \kappa_{k,\ell} \quad \text{for} \quad k+\ell = 2r-1 \text{ or } 2r$$

Here is a table of the first polynomials $\bar{\kappa}_{k,\ell}$:

(k,ℓ)	$\bar{\kappa}_{k,\ell}$
$(1,0)$	$3x-1$
$(2,0)$	$-X$
$(1,1)$	$2xy$
$(3,0)$	$-X(5x-2)$
$(2,1)$	$x(15xy-x-8y+1)$
$(4,0)$	$X^2/2$
$(3,1)$	$-2xyX$
$(2,2)$	$xy(3xy-(x+y)+1)$

As for Bernstein QIs, we deduce, for any polynomial p :

$$\lim n^r(f - \mathcal{M}_n^{(2r)} f) = \sum_{i+j=2r} \bar{\kappa}_{i,j} D^{i,j} f, \quad q = 2r-1$$

Similarly, for $q = 2r$, we get

$$\lim n^{r+1}(f - \mathcal{M}_n^{(2r+1)} f) = \sum_{p=2r+1}^{2r+2} \sum_{i+j=p} \bar{\kappa}_{i,j} D^{i,j} f, \quad q = 2r$$

Examples.

$$\lim n(f - \mathcal{M}_n^{(2)} f) = -XD^{2,0}f + 2xyD^{1,1}f - YD^{0,2}f$$

$$\lim n^2(f - \mathcal{M}_n^{(3)} f) = \sum_{|\gamma|=3} \bar{\alpha}_\gamma D^\gamma f + \sum_{|\gamma|=4} \bar{\alpha}_\gamma D^\gamma f$$

$$= -X(5x-2)D^{3,0}f - x(15xy-x-8y+1)D^{2,1}f - yx(15xy-8x-y+1)D^{1,2}f$$

$$-(5y-2)YD^{0,3}f+\frac{1}{2}X^2D^{4,0}f-2xyXD^{3,1}f$$

$$+xy(3xy-(x+y)+1)D^{2,2}f-2xyYD^{1,3}f+\frac{1}{2}Y^2D^{0,4}f$$

8.3. Genuine Bernstein-Durrmeyer QIs

For and polynomials, we define

$$\bar{\theta}_{k,\ell}=\lim n^r\theta_{k,\ell}\quad\text{for}\quad k+\ell=2r-1\text{ or }2r$$

$$\bar{\eta}_{k,\ell}=\lim n^r\eta_{k,\ell}\quad\text{for}\quad k+\ell=2r-1\text{ or }2r$$

Here is a table of the first polynomials:

(k,ℓ)	$\bar{\eta}_{k,\ell}$
$(2,0)$	$-X$
$(1,1)$	$2xy$
$(3,0)$	$(1-2x)X$
$(2,1)$	$-3xy(1-2x)$
$(4,0)$	$X^2/2$
$(3,1)$	$-2xyX$
$(2,2)$	$xy(3xy-(x+y)+1)$

9. Numerical experiments on Bernstein quasi-interpolants

We present some numerical tests on the following functions

$$f_1(x,y)=\frac{1}{1+16((x-1/3)^2+(y-1/3)^2)}$$

$$f_2(x,y)=\exp(-x^2-y^2)$$

using classical and genuine Bernstein quasi-interpolants of various degrees and orders.

We denote the uniform errors respectively by $eb_n^{(r)}f:=\|f-\mathcal{B}_n^{(r)}f\|$ for Bernstein QIs and by $eg_n^{(r)}:=\|f-\mathcal{G}_n^{(r)}f\|$ for genuine Bernstein-Durrmeyer QIs.

(n,r)	$eb_n^{(r)}f_1$	$eb_n^{(r)}f_2$	(n,r)	$eg_n^{(r)}f_1$	$eg_n^{(r)}f_2$
$(8,0)$	0.38	3.6(-2)	$(5,1)$	0.6	$8.8(-2)$
$(8,3)$	8.4(-2)	2.3(-3)	$(5,3)$	0.3	$8.8(-3)$
$(8,5)$	2.4(-2)	1.2(-4)	$(5,4)$	0.25	$1.2(-3)$
$(8,8)$	0.12	2.0(-6)	(5,5)	0.14	4.8(-4)
$(15,0)$	0.26	2.0(-2)	(10,0)	0.46	5.2(-2)
$(15,4)$	4.6(-2)	4.4(-5)	(10,2)	0.25	5.2(-3)
$(15,8)$	1.2(-2)	6.0(-8)	(10,4)	0.15	4.0(-4)
$(15,9)$	5.6(-3)	3.0(-8)	(10,6)	8.4(-2)	4.8(-5)
$(15,10)$	9.2(-3)	3.4(-9)	(10,7)	0.12	2.6(-4)
$(15,15)$	1.5(-2)	5.0(-11)	(10,10)		

We see that the behaviours of QIs are quite different for f_1 and f_2.
1) f_1 is a rational function of Runge type : the Lagrange interpolants for $n = 8$ and $n = 15$ both give bad results. However, the errors $eb_n^{(r)} f_1$ seem to have a minimum value for some intermediate QIs, for example for $(n, r) = (8, 5)$ and $(n, r) = (15, 9)$. A similar fact occurs for the errors $eg_n^{(r)} f_1$ where the minimum value is obtained for $(n, r) = (10, 6)$. However the errors are higher than those obtained by Bernstein QIs for $n = 8$.
2) f_1 is a good analytic function with a nice behaviour: the Lagrange interpolant gives the best results. The errors slowly decrease from $r = 0$ to $r = n$. If one does not want a very high precision, the first QIs can be taken as approximants of the given function. For the genuine Durrmeyer operator, the errors for $n = 10$ are higher than those obtained by Bernstein QIs for $n = 8$, except maybe the minimum value for $(n, r) = (10, 6)$.
We also compared the above results with those obtained using the BD operator with Legendre weight (the errors are denoted $ed_n^{(r)} f$). For the two tested functions, the results were worse. We only give them for the exponential functionf_2.

(n, r)	$eb_n^{(r)} f_2$	$eg_n^{(r)} f_2$	$ed_n^{(r)} f_2$
$(5, 0)$	$5.6(-2)$	$8.8(-2)$	0.18
$(5, 3)$	$4.6(-3)$	$8.8(-3)$	4.2(-3)
$(5, 4)$	$6.4(-4)$	$1.2(-3)$	2.3(-3)
(5,5)	6.4(-4)	8.8(-4)	1.6(-3)

As a conclusion of these tests (and of other tests done on various functions), the classical Bernstein QIs seem a priori to be the more efficient. Of course, the values of f on uniform lattices of points of the triangle must be available. If the function is only known by its moments or other mean integral values, then one could consider the approximation by BDQIs with convenient Jacobi weights or by GDQIs.

10. Some applications

In this final section, we briefly present some possible applications of the above quasi-interpolants to various problems in approximation, CAGD and numerical analysis.

- in approximation, the Hausdorff moment problem in T consists in finding a function f having given moments $\mu_\gamma(f) := \int_T f(x, y) x^k y^\ell dxdy$ for some indices $\gamma = (k, \ell) \in \mathbb{N}^2$. Such a function can be approximated by the Bernstein-Durrmeyer quasi-interpolants of Section 5. Indeed, scalar products $\langle f, B_\alpha^n \rangle$ are directly computable from moments, so $\mathcal{M}_n f$ is easily obtained together with its partial derivatives.
- in CAGD, when one is interested in approximating a function defined on a uniform lattice of points in the triangle T, Bernstein quasi-interpolants of Section 3 can sometimes offer an alternative to strict interpolation at

those points since their norms seem to be uniformly bounded in n for a given order r.

- in numerical analysis, it would be perhaps interesting to derive cubature formulas from integration of Bernstein quasi-interpolants. In the same way, approximate formulas for partial derivatives can be obtained by computing derivatives of Bernstein or Bernstein-Durrmeyer type quasi-interpolants.

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