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# On the rate of convergence of a new $q$-Szász-Mirakjan operator

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**Abstract.** In the present paper we introduce a new $q$-generalization of Szász-Mirakjan operators and we investigate their approximation properties. By using a weighted modulus of smoothness, we give local and global estimations for the error of approximation.

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## 1. Introduction

The aim of this paper is to study the approximation properties of a new Szász-Mirakjan type operator constructed by using $q$-Calculus. Firstly, we recall some basic definitions and notations used in quantum calculus, see, e.g., [6, pp. 7-13].

Let $q>0$. For any $n\in\mathbb{N}_0:=\{0\}\cup\mathbb{N}$ the $q$-integer $[n]_q$ is defined by

$$[n]_q:=1+q+\dots q^{n-1}\quad(n\in\mathbb{N}),\quad [0]_q:=0,$$

and the $q$-factorial $[n]_q!$ by

$$[n]_q!:=[1]_q[2]_q\dots[n]_q\quad(n\in\mathbb{N}),\quad [0]_q!:=1.$$

Also, the $q$-binomial coefficients are denoted by $\left[\begin{array}{c} n \\ k \end{array}\right]_q$ and are defined by

$$\left[\begin{array}{c} n \\ k \end{array}\right]_q=\frac{[n]_q!}{[k]_q!\,[n-k]_q!},\quad k=0,1,\dots,n.$$

The $q$-derivative of a function $f:\mathbb{R}\to\mathbb{R}$ is defined by

$$D_qf\,(x)=\frac{f(x)-f(qx)}{(1-q)x},\quad x\neq 0,\quad D_qf\,(0):=\lim_{x\to 0}D_qf\,(x)\,,$$

and the high $q$-derivatives $D_q^0f:=f$, $D_q^nf:=D_q\left(D_q^{n-1}f\right)$, $n\in\mathbb{N}$.

The product rule is

$$D_q\left(f(x)g(x)\right) = D_q\left(f(x)\right)g(x) + f(qx)D_q\left(g(x)\right). \tag{1.1}$$

We recall the $q$-Taylor theorem as it is given in [4, p. 103].

**Theorem 1.1.** *If the function $g(x)$ is capable of expansion as a convergent power series and $q$ is not a root of unity, then*

$$g(x) = \sum_{r=0}^{\infty} \frac{(x-a)_q^r}{[r]_q!} D_q^r g(a),$$

*where*

$$(x-a)_q^r = \prod_{s=0}^{r-1}(x - q^s a) = \sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} x^{r-k}(-a)^k.$$

## 2. Auxiliary results

Throughout the paper we consider $q \in (0,1)$.

We define a suitable $q$-difference operator as follows

$$\Delta_q^0 f_{k,s} = f_{k,s}, \tag{2.1}$$

$$\Delta_q^{r+1} f_{k,s} = q^r \Delta_q^r f_{k+1,s} - \Delta_q^r f_{k,s-1}, \quad r \in \mathbb{N}_0, \tag{2.2}$$

where $f_{k,s} = f\left(\frac{[k]_q}{q^s[n]_q}\right)$, $k \in \mathbb{N}_0$, $s \in \mathbb{Z}$.

The following lemma gives an expression for the $r$-th $q$-differences $\Delta_q^r f_{k,s}$ as a sum of multiplies of values of $f$.

**Lemma 2.1.** *The $q$-difference operator $\Delta_q^r$ defined by (2.1)-(2.2)satisfies*

$$\Delta_q^r f_{k,s} = \sum_{j=0}^{r}(-1)^{r-j} q^{j(j-1)/2} \begin{bmatrix} r \\ j \end{bmatrix}_q f_{k+j,j+s-r} \quad \textit{for} \quad r,k \in \mathbb{N}_0, \quad s \in \mathbb{Z}. \tag{2.3}$$

Taking into account the relations (2.1)-(2.2) and the formula

$$\begin{bmatrix} r+1 \\ j+1 \end{bmatrix}_q = q^{r-j} \begin{bmatrix} r \\ j \end{bmatrix}_q + \begin{bmatrix} r \\ j+1 \end{bmatrix}_q,$$

the identity (2.3) can be easily obtained by induction over $r \in \mathbb{N}_0$.

In what follows, the monomial of $m$ degree is denoted by $e_m$, $m \in \mathbb{N}_0$.

Let us denote by $[x_0, x_1, \ldots, x_n; f]$ the divided difference of the function $f$ with respect to the points $x_0, x_1, \ldots, x_n$.

**Lemma 2.2.** *For all $k, r \in \mathbb{N}_0$, $s \in \mathbb{Z}$, we have*

$$[x_{k,s-1}, \ldots, x_{k+r,s+r-1}; f] = \frac{q^{r(r+2s-1)/2}[n]_q^r}{[r]_q!} \Delta_q^r f_{k,r+s-1}, \tag{2.4}$$

*where* $x_{k,s-1} = \frac{[k]_q}{q^{s-1}[n]_q}$.

*Proof.* We use the mathematical induction with respect to $r$. For $r = 0$ the equality (2.4) follows immediately from (2.1). Let us assume that (2.4) holds true for some $r \geq 0$ and all $k \in \mathbb{N}_0$, $s \in \mathbb{Z}$.

We have

$$\begin{aligned}&[x_{k,s-1}, \ldots, x_{k+r+1,s+r}; f]\\ &= \frac{[x_{k+1,s}, \ldots, x_{k+r+1,s+r}; f] - [x_{k,s-1}, \ldots, x_{k+r,s+r-1}; f]}{x_{k+r+1,s+r} - x_{k,s-1}}.\end{aligned}$$

Since $x_{k+r+1,s+r} - x_{k,s-1} = \frac{[r+1]_q}{q^{r+s}[n]_q}$, by using (2.2) we get

$$\begin{aligned}&[x_{k,s-1}, \ldots, x_{k+r+1,s+r}; f]\\ &= \frac{q^{(r+1)(r+2s)/2}[n]_q^{r+1}}{[r+1]_q!}\left(q^r \Delta_q^r f_{k+1,r+s} - \Delta_q^r f_{k,r+s-1}\right)\\ &= \frac{q^{(r+1)(r+2s)/2}[n]_q^{r+1}}{[r+1]_q!}\Delta_q^{r+1} f_{k,r+s}.\end{aligned}$$

□

## 3. Construction of the operators

In 1987 A. Lupaş [9] introduced the first $q$-analogue of Bernstein operator and investigated its approximating and shape-preserving properties. Another $q$-generalization of the classical Bernstein polynomials is due to G. Phillips [13]. More properties of these two $q$-extensions were obtained over time in several papers such as [3], [10], [11], [1]. We mention that the comprehensive survey [12] due to S. Ostrovska gives a good perspective of the most important achievements during a decade relative to these operators.

Two of the known expansions in $q$-calculus of the exponential function are given as follows (see, e.g., [6, p. 31])

$$E_q(x) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{x^k}{[k]_q!}, \quad x \in \mathbb{R}, \quad |q| < 1,$$

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!}, \quad |x| < \frac{1}{1-q}, \quad |q| < 1.$$

It is obvious that $\lim\limits_{q \to 1^-} E_q(x) = \lim\limits_{q \to 1^-} e_q(x) = e^x$.

For $q \in (0, 1)$, in [2] A. Aral introduced the first $q$-analogue of the classical Szász-Mirakjan operators given by

$$S_n^q(f; x) = E_q\left(-[n]_q \frac{x}{b_n}\right) \sum_{k=0}^{\infty} f\left(\frac{[k]_q b_n}{[n]_q}\right) \frac{\left([n]_q x\right)^k}{[k]_q!(b_n)^k},$$

where $0 \leq x < \frac{b_n}{1-q^n}$, $(b_n)_n$ is a sequence of positive numbers such that $\lim\limits_n b_n = \infty$.

The operator $S_n^q$ reproduces linear functions and

$$S_n^q(e_2;x) = qx^2 + \frac{b_n}{[n]_q}x, \quad 0 \le x < \frac{b_n}{1-q^n}.$$

Motivated by this work, for $q \in (0,1)$ we give another $q$-analogue of the same class of operators as follows

$$S_{n,q}(f;x) = \sum_{k=0}^{\infty} \frac{\left([n]_q x\right)^k}{[k]_q!} q^{k(k-1)} E_q\left(-[n]_q q^k x\right) f\left(\frac{[k]_q}{[n]_q q^{k-1}}\right), \quad x \ge 0, \tag{3.1}$$

where $f \in \mathcal{F}(\mathbb{R}_+) := \{f : \mathbb{R}_+ \to \mathbb{R}, \text{ the series in } (3.1) \text{ is convergent}\}$.

Since $E_q(x)$ is convergent for every $x \in \mathbb{R}$, by using Theorem 1.1 and the property $D_q^r E_q(x) = q^{\frac{r(r-1)}{2}} E_q(q^r x)$ we obtain

$$\sum_{r=0}^{\infty} \frac{(-x)^r}{[r]_q!} q^{r(r-1)} E_q(q^r x) = E_q(0) = 1, \quad x \in \mathbb{R},$$

which yields that the operator $S_{n,q}$ is well defined.

For $q \to 1^-$, the above operators reduce to the classical Szász-Mirakjan operators. In this case, the approximation function $S_{n,q}f$ is defined on $\mathbb{R}_+$ for each $n \in \mathbb{N}$.

**Theorem 3.1.** *Let $q \in (0,1)$ and $S_{n,q}$, $n \in \mathbb{N}$, be defined by (3.1). For any $f \in \mathcal{F}(\mathbb{R}_+)$ we have*

$$S_{n,q}(f;x) = \sum_{r=0}^{\infty} \frac{([n]_q x)^r}{[r]_q!} q^{\frac{r(r-1)}{2}} \Delta_q^r f_{0,r-1}, \quad x \ge 0. \tag{3.2}$$

*Proof.* Let $f \in \mathcal{F}(\mathbb{R}_+)$.

By using (2.1), the operator $S_{n,q}$ can be expressed as follows

$$S_{n,q}(f;x) = \sum_{k=0}^{\infty} \frac{\left([n]_q x\right)^k}{[k]_q!} q^{k(k-1)} E_q\left(-[n]_q q^k x\right) \Delta_q^0 f_{k,k-1}.$$

Applying $q$-derivative operator to $S_{n,q}f$ and taking into account the product rule (1.1) and the property $D_q E_q(ax) = aE_q(aqx)$, (see e.g. [6, pp. 29-32]), we have

$$\begin{aligned} &D_q S_{n,q}(f;x) \\ &= [n]_q \sum_{k=0}^{\infty} \frac{\left([n]_q x\right)^k}{[k]_q!} q^{k(k+1)} E_q\left(-[n]_q q^{k+1} x\right) \left(\Delta_q^0 f_{k+1,k} - \Delta_q^0 f_{k,k-1}\right) \\ &= [n]_q \sum_{k=0}^{\infty} \frac{\left([n]_q x\right)^k}{[k]_q!} q^{k(k+1)} E_q\left(-[n]_q q^{k+1} x\right) \Delta_q^1 f_{k,k}. \end{aligned}$$

For $n \in \mathbb{N}$ and $x \in \mathbb{R}_+$, by induction with respect to $r \in \mathbb{N}$, we can prove

$$D_q^r S_{n,q}(f;x) = [n]_q^r q^{\frac{r(r-1)}{2}} \sum_{k=0}^{\infty} \frac{\left([n]_q x\right)^k}{[k]_q!} q^{k(2r+k-1)} E_q\left(-[n]_q q^{k+r} x\right) \Delta_q^r f_{k,k+r-1}.$$

Choosing $x = 0$, we deduce $D_q^r S_{n,q}(f;0) = [n]_q^r q^{\frac{r(r-1)}{2}} \Delta_q^r f_{0,r-1}$.
Choosing $a = 0$ in Theorem 1.1, we obtain

$$S_{n,q}(f;x) = \sum_{r=0}^{\infty} \frac{([n]_q x)^r}{[r]_q!} q^{\frac{r(r-1)}{2}} \Delta_q^r f_{0,r-1},$$

which completes the proof. □

**Corollary 3.2.** *Let $q \in (0,1)$ and $S_{n,q}$, $n \in \mathbb{N}$, be defined by (3.1). For any $f \in \mathcal{F}(\mathbb{R}_+)$ we have*

$$S_{n,q}(f;x) = \sum_{r=0}^{\infty} x^r \left[0, \frac{1}{[n]_q}, \frac{[2]_q}{q[n]_q}, \ldots, \frac{[r]_q}{q^{r-1}[n]_q}; f\right], \quad x \geq 0. \tag{3.3}$$

*Proof.* The identity (3.3) is obtained from the above theorem and (2.4) by choosing $k = s = 0$. □

**Corollary 3.3.** *For all $n \in \mathbb{N}$, $x \in \mathbb{R}_+$ and $0 < q < 1$, we have*

$$S_{n,q}(e_0;x) = 1, \tag{3.4}$$
$$S_{n,q}(e_1;x) = x, \tag{3.5}$$
$$S_{n,q}(e_2;x) = x^2 + \frac{1}{[n]_q} x. \tag{3.6}$$

*Moreover, for $m \in \mathbb{N}_0$ and $0 < q < 1$, the operator $S_{n,q}$ defined by (3.1) can be expressed as*

$$S_{n,q}(e_m;x) = \sum_{r=0}^{m} x^r \left[0, \frac{1}{[n]_q}, \frac{[2]_q}{q[n]_q}, \ldots, \frac{[r]_q}{q^{r-1}[n]_q}; e_m\right], \quad x \geq 0. \tag{3.7}$$

*Proof.* Since for any distinct points $x_0, \ldots, x_r$, the divided difference

$$[x_0, \ldots, x_r; e_m] = \begin{cases} 0 & if \quad m < r, \\ 1 & if \quad m = r, \\ x_0 + \ldots + x_r & if \quad m = r+1, \end{cases}$$

(see e.g. [5, p.63]), the identities (3.4)-(3.7) are obvious. □

**Lemma 3.4.** *For $m \in \mathbb{N}_0$ and $q \in (0,1)$ we have*

$$S_{n,q}(e_m;x) \leq A_{m,q}(1 + x^m), \quad x \geq 0, \quad n \in \mathbb{N}, \tag{3.8}$$

*where $A_{m,q}$ is a positive constant depending only on $q$ and $m$.*

*Proof.* Let $m \in \mathbb{N}$. From (3.7) we get

$$S_{n,q}(e_m;x) \le (1+x^m)\sum_{r=1}^{m}\left[0,\frac{1}{[n]_q},\ldots,\frac{[r]_q}{q^{r-1}[n]_q};e_m\right].$$

Applying the well known Lagrange's Mean Value Theorem, we can write

$$S_{n,q}(e_m;x) \le (1+x^m)\sum_{r=1}^{m}\binom{m}{r}(\xi_r)^{m-r},$$

where $0<\xi_r<\frac{[r]_q}{q^{r-1}[n]_q}$, $0<r\le m$.

Consequently, we have

$$\begin{aligned} S_{n,q}(e_m;x) &\le (1+x^m)\sum_{r=1}^{m}\binom{m}{r}\frac{[r]_q^{m-r}}{q^{(r-1)(m-r)}[n]_q^{m-r}} \\ &\le (1+x^m)[m]_q^{m-1}\sum_{r=1}^{m}\binom{m}{r}\frac{1}{q^{(r-1)(m-r)}q^{m-r+r^2}} \\ &\le A_{m,q}(1+x^m), \end{aligned}$$

where

$$A_{m,q} := [m]_q^{m-1}\left(1+\frac{1}{q^m}\right)^m, \quad m\ge 1. \tag{3.9}$$

For $m=0$ we can take $A_{0,q}=\frac{1}{2}$. □

Examining relation (3.6) it is clear that the sequence of the operators $(S_{n,q})_n$ does not satisfies the conditions of Bohman-Korovkin theorem.

Further on, we consider a sequence $(q_n)_n$, $q_n \in (0,1)$, such that

$$\lim_n q_n = 1. \tag{3.10}$$

The condition (3.10) guarantees that $[n]_{q_n} \to \infty$ for $n\to\infty$.

**Theorem 3.5.** *Let $(q_n)_n$ be a sequence satisfying (3.10) and let the operators $S_{n,q_n}$, $n\in\mathbb{N}$, be defined by (3.1). For any compact $J\subset\mathbb{R}_+$ and for each $f\in C(\mathbb{R}_+)$ we have*

$$\lim_{n\to\infty} S_{n,q_n}(f;x)=f(x), \quad \text{uniformly in } x\in J.$$

*Proof.* Replacing $q$ by a sequence $(q_n)_n$ with the given conditions, the result follows from (3.4)-(3.6) and the well-known Bohman-Korovkin theorem (see [7], pp. 8-9). □

## 4. Error of approximation

Let $\alpha\in\mathbb{N}$. We denote by $B_\alpha(\mathbb{R}_+)$ the weighted space of real-valued functions $f$ defined on $\mathbb{R}_+$ with the property $|f(x)|\le M_f(1+x^\alpha)$ for all $x\in\mathbb{R}_+$, where $M_f$ is a constant depending on the function $f$. We also consider the weighted subspace $C_\alpha(\mathbb{R}_+)$ of $B_\alpha(\mathbb{R}_+)$ given by

$$C_\alpha(\mathbb{R}_+) := \{f\in B_\alpha(\mathbb{R}_+) : f \text{ continuous on } \mathbb{R}_+\}.$$

Endowed with the norm $\|\cdot\|_\alpha$, where $\|f\|_\alpha := \sup_{x\in\mathbb{R}_+} \frac{|f(x)|}{1+x^\alpha}$, both $B_\alpha(\mathbb{R}_+)$ and $C_\alpha(\mathbb{R}_+)$ are Banach spaces.

We can give estimates of the error $|S_{n,q}(f;\cdot) - f|$, $n \in \mathbb{N}$, for unbounded functions by using a weighted modulus of smoothness associated to the space $B_\alpha(\mathbb{R}_+)$.

We consider

$$\Omega_\alpha(f;\delta) := \sup_{\substack{x\geq 0\\ 0<h\leq\delta}} \frac{|f(x+h)-f(x)|}{1+(x+h)^\alpha}, \delta > 0,\ \alpha\in\mathbb{N}. \tag{4.1}$$

It is evident that for each $f \in B_\alpha(\mathbb{R}_+)$, $\Omega_\alpha(f;\cdot)$ is well defined and

$$\Omega_\alpha(f;\delta) \leq 2\|f\|_\alpha, \delta > 0, \quad f \in B_\alpha(\mathbb{R}_+),\ \alpha\in\mathbb{N}.$$

The weighted modulus of smoothness $\Omega_\alpha(f;\cdot)$ possesses the following properties ([8]).

$$\Omega_\alpha(f;\lambda\delta) \leq (\lambda+1)\Omega_\alpha(f;\delta), \quad \delta>0, \lambda>0, \tag{4.2}$$
$$\Omega_\alpha(f;n\delta) \leq n\Omega_\alpha(f;\delta), \quad \delta>0, n\in\mathbb{N},$$
$$\lim_{\delta\to 0^+} \Omega_\alpha(f;\delta) = 0.$$

**Theorem 4.1.** *Let $(q_n)_n$ be a sequence satisfying (3.10). Let $q_0 = \inf_{n\in\mathbb{N}} q_n$ and $\alpha\in\mathbb{N}$. For each $n\in\mathbb{N}$ and every $f \in B_\alpha(\mathbb{R}_+)$ one has*

$$|S_{n,q_n}(f;x) - f(x)| \leq C_{\alpha,q_0}(1+x^{\alpha+1})\Omega_\alpha\left(f;\sqrt{1/[n]_{q_n}}\right), \quad x\geq 0, \tag{4.3}$$

*where $C_{\alpha,q_0}$ is a positive constant independent of $f$ and $n$.*

*Proof.* Let $n\in\mathbb{N}$, $f\in B_\alpha(\mathbb{R}_+)$ and $x\geq 0$ be fixed. Setting $\mu_{x,\alpha}(t) := 1+(x+|t-x|)^\alpha$ and $\psi_x(t) := |t-x|$, $t\geq 0$, relations (4.1) and (4.2) imply

$$\begin{aligned}|f(t)-f(x)| &\leq (1+(x+|t-x|)^\alpha)\left(1+\frac{1}{\delta}|t-x|\right)\Omega_\alpha(f;\delta)\\ &= \mu_{x,\alpha}(t)\left(1+\frac{1}{\delta}\psi_x(t)\right)\Omega_\alpha(f;\delta), \quad t\geq 0.\end{aligned}$$

By using the Cauchy inequality for linear positive operators which preserve the constants, we obtain

$$\begin{aligned}|S_{n,q_n}(f;x)-f(x)| &\leq S_{n,q_n}(|f-f(x)|;x) \qquad (4.4)\\ &\leq \left(S_{n,q_n}(\mu_{x,\alpha};x) + \frac{1}{\delta}S_{n,q_n}(\mu_{x,\alpha}\psi_x;x)\right)\Omega_\alpha(f;\delta)\\ &\leq \sqrt{S_{n,q_n}(\mu^2_{x,\alpha};x)}\left(1+\frac{1}{\delta}\sqrt{S_{n,q_n}(\psi_x^2;x)}\right)\Omega_\alpha(f;\delta).\end{aligned}$$

Since

$$\begin{aligned}\mu^2_{x,\alpha}(t) &= (1+(x+|t-x|)^\alpha)^2 \leq 2\left(1+(2x+t)^{2\alpha}\right)\\ &\leq 2\left(1+2^{2\alpha}\left((2x)^{2\alpha}+t^{2\alpha}\right)\right),\end{aligned}$$

and taking into account (3.4) and (3.8) we get

$$S_{n,q_n}(\mu_{x,\alpha}^2;x) \le B_{\alpha,q_n}^2(1+x^{2\alpha}), \tag{4.5}$$

where $B_{\alpha,q_n}^2 = 2^{\alpha+1}\left(2^{2\alpha}+A_{2\alpha,q_n}\right)$.
According to (3.4)-(3.6) we have $S_{n,q_n}(\psi_x^2;x) = \frac{1}{[n]_{q_n}}x$.
By choosing $\delta := \sqrt{\frac{1}{[n]_{q_n}}}$ in (4.3), from (4.5) follows

$$|S_{n,q_n}(f;x) - f(x)| \le B_{\alpha,q_n}\sqrt{1+x^{2\alpha}}(1+\sqrt{x})\Omega_\alpha\left(f;\sqrt{\frac{1}{[n]_{q_n}}}\right).$$

Finally, since $1+\sqrt{x} \le \sqrt{2}\sqrt{1+x}$ and $(1+x^{2\alpha})(1+x) \le 4(1+x^{\alpha+1})$ for $x \ge 0$ and $\alpha \in \mathbb{N}$, we obtain

$$|S_{n,q_n}(f;x) - f(x)| \le C_{\alpha,q_0}(1+x^{\alpha+1})\Omega_\alpha\left(f;\sqrt{1/[n]_{q_n}}\right), \quad x \ge 0,$$

where $q_0 := \inf\limits_{n\in\mathbb{N}} q_n$ and $C_{\alpha,q_0} := 2\sqrt{2}B_{\alpha,q_0}$. □

On the basis of Theorem 4.1 we give the following global estimate.

**Corollary 4.2.** *Let $(q_n)_n$ be a sequence satisfying (3.10) and $\alpha \in \mathbb{N}$. For each $n \in \mathbb{N}$ and every $f \in B_\alpha(\mathbb{R}_+)$ one has*

$$\|S_{n,q_n}(f;\cdot) - f\|_{\alpha+1} \le C_{\alpha,q_0}\Omega_\alpha\left(f;\sqrt{1/[n]_{q_n}}\right),$$

*where $C_{\alpha,q_0}$ is a positive constant independent of $f$ and $n$.*

**Remark 4.3.** For any function $f \in B_\alpha(\mathbb{R}_+)$, $\alpha \in \mathbb{N}$, the rate of convergence of the operators $S_{n,q_n}(f;\cdot)$ to $f$ in weighted norm is $\sqrt{\frac{1}{[n]_{q_n}}}$ which is faster than $\sqrt{\frac{b_n}{[n]_{q_n}}}$ obtained in [2].

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