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On elliptic partial differential equations with random coefficients

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Abstract. We consider stationary diffusion equations with random coefficients which cannot be bounded strictly away from zero and infinity by constants. We prove the existence of a unique solution to the corresponding weak formulation with different solution and test function spaces. Furthermore, the convergence of the Stochastic Galerkin solution is established under certain conditions.

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1. Introduction

In recent years there has been a growing interest in quantifying uncertainty in complex systems which are modeled via algebraic, ordinary or partial differential equations with random input data. For example, the stationary diffusion equation with a random coefficient is an instructive model problem. Thus, we consider the boundary value problem consisting of the random partial differential equation

$$-\nabla \cdot (\kappa \nabla u) = f$$

and some suitable boundary conditions. Thereby, the coefficient κ and also the forcing f are random functions. In previous works (see for example Babuška et al. [1, 3, 4] or Schwab et al. [5, 6, 14]) it is often assumed that there exist constants $\underline{\kappa}, \overline{\kappa} > 0$, such that

$$0 < \underline{\kappa} \leq \kappa(x, \omega) \leq \overline{\kappa} \qquad \text{a.e. and a.s.}$$

Then the theorem of Lax-Milgram can be used to prove the existence of a unique weak solution. In a first step towards a generalization of the problem setting Galvis and Sarkis [9] as well as Gittelson [11] investigate this random

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partial differential equation where the coefficient is modeled as a lognormal random field. That is, $\kappa(x) = \exp(G(x))$ with a Gaussian random field $G(x)$. In this case, however, there do not exist constants $\underline{\kappa}, \overline{\kappa} > 0$ as above and thus the Lax-Milgram theorem is not applicable. For this reason, the authors employ alternative techniques to prove the existence and uniqueness of the weak solution and to obtain a priori error estimates of the Stochastic Galerkin approximation to this solution. In the following we generalize these results to arbitrary random input fields which can be bounded by random variables $\kappa_{min}, \kappa_{max} > 0$ a.s., that is,

$$0 < \kappa_{min}(\omega) \leq \kappa(x, \omega) \leq \kappa_{max}(\omega) < \infty \qquad \text{a.e. and a.s.}$$

2. Setting and problem formulation

Let $D \subset \mathbb{R}^d,\ d \in \mathbb{N}$, be a bounded Lipschitz domain and $(\Omega, \mathfrak{A}, \mathbf{P})$ a probability space. We consider the following boundary value problem

$$\begin{aligned} -\nabla \cdot (\kappa(x,\omega)\nabla u(x,\omega)) &= f(x,\omega) \qquad & x \in D,\ \omega \in \Omega \\ u(x,\omega) &= 0 \qquad & x \in \partial D,\ \omega \in \Omega \end{aligned} \tag{2.1}$$

with random coefficient κ and random forcing f. We assume that the coefficient function $\kappa : D \times \Omega \to \mathbb{R}$ is a strongly measurable random variable with values in $\mathrm{L}^\infty(D)$ and that there exist real-valued random variables κ_{min} and κ_{max} such that

$$0 < \kappa_{min}(\omega) \leq \kappa(x, \omega) \leq \kappa_{max}(\omega) < \infty \qquad \text{a.e. and a.s.} \tag{2.2}$$

We define the pathwise bilinear form $b(\cdot, \cdot; \omega) : H^1(D) \times H^1(D) \to \mathbb{R}$ by

$$b(u, v; \omega) = \int_D \kappa(x, \omega) \nabla u(x) \cdot \nabla v(x)\, dx$$

for $\omega \in \Omega$ and we denote by $\langle g, v \rangle_{H^{-1}, \mathring{H}^1}$ the duality pairing between $g \in H^{-1}(D)$ and $v \in \mathring{H}^1(D)$. Now, assuming that f is a random variable with values in $H^{-1}(D)$, we consider a pathwise weak formulation of the boundary value problem:

Problem 2.1 (Pathwise Weak Formulation). Find a random variable $\tilde{u}$ with values in $\mathring{H}^1(D)$, such that

$$b(\tilde{u}(\omega), v; \omega) = \langle f(\omega), v \rangle_{H^{-1}, \mathring{H}^1} \quad \text{for all } v \in \mathring{H}^1(D) \tag{2.3}$$

holds almost surely.

Remark 2.2. In Problem 2.1 we look for a random variable $\tilde{u}$ with values in $\mathring{H}^1(D)$, such that

$$\mathbf{P}\left(b(\tilde{u}(\omega), v; \omega) = \langle f(\omega), v \rangle_{H^{-1}, \mathring{H}^1} \quad \text{for all } v \in \mathring{H}^1(D) \right) = 1. \tag{2.3a}$$

Due to the separability of $\mathring{H}^1(D)$ this problem is equivalent to the weaker problem formulation: Find a random variable $\tilde{u}$ with values in $\mathring{H}^1(D)$, such that for all $v \in \mathring{H}^1(D)$ there holds

$$\mathbf{P}\left(b(\tilde{u}(\omega), v; \omega) = \langle f(\omega), v\rangle_{H^{-1},\mathring{H}^1}\right) = 1. \tag{2.4}$$

Since every realization of the coefficient κ is bounded by assumption (2.2) and f is a random variable with values in $H^{-1}(D)$, by the theorem of Lax-Milgram (see e.g. [7] Theorem 2.7.7) there exists a mapping $\tilde{u} : \Omega \to \mathring{H}^1(D),\ \omega \mapsto \tilde{u}(\omega)$ satisfying

$$b(\tilde{u}(\omega), v; \omega) = \langle f(\omega), v\rangle_{H^{-1},\mathring{H}^1} \quad \text{for all } v \in \mathring{H}^1(D)$$

for almost all $\omega \in \Omega$. Furthermore, the estimate

$$\|\tilde{u}(\omega)\|_{H^1(D)} \leq C\frac{\|f(\omega)\|_{H^{-1}(D)}}{\kappa_{min}(\omega)} \qquad \text{a.s.} \tag{2.5}$$

holds, where $C > 0$ is a suitable constant which does not depend on $\omega \in \Omega$. This mapping $\tilde{u}$ is a.s. uniquely defined and measurable as is proved in the next Lemma.

Lemma 2.3. *Assume $\kappa : D \times \Omega \to \mathbb{R}$ is a strongly measurable random variable in $L^\infty(D)$ satisfying*

$$0 < \kappa_{min}(\omega) \leq \kappa(x, \omega) \leq \kappa_{max}(\omega) < \infty \quad \textit{a.e. and a.s.}$$

for real-valued random variables $\kappa_{min}, \kappa_{max}$, and f is a random variable with values in $H^{-1}(D)$. Then the mapping $\tilde{u} : \Omega \to \mathring{H}^1(D)$ is a random variable in $\mathring{H}^1(D)$ which is measurable with respect to the σ-algebra $\sigma(f, \kappa)$, generated by f and κ, and solves Problem 2.1.

Proof. From the assumptions on κ and f it follows that there exist sequences $(\kappa_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ of $\sigma(f, \kappa)$-measurable, simple random variables with values in $\mathrm{L}^\infty(D)$ and $H^{-1}(D)$, respectively, satisfying

$$\|\kappa - \kappa_n\|_{\mathrm{L}^\infty(D)} \to 0, \ \text{a.s.} \quad \text{and} \quad \|f - f_n\|_{H^{-1}(D)} \to 0, \ \text{a.s.} \quad \text{for } n \to \infty.$$

Then the result follows immediately from the properties of the pathwise bilinear form b and the convergence of the simple random variables. □

In analogy to variational formulations of boundary value problems with purely deterministic input data we want to study also the corresponding variational formulation for random input data which is sometimes referred to as "stochastic variational formulation". Such a formulation is obtained by defining a suitable bilinear form on a Hilbert space of random variables in $\mathring{H}^1(D)$, e.g. $a(u(\cdot), v(\cdot)) = \mathbf{E}_\mathbf{P} b(u(\cdot), v(\cdot); \cdot)$, and correspondingly by defining a linear form. However, since the coefficient κ is not bounded by constants but random variables we cannot directly use the Lax-Milgram theorem to prove existence and uniqueness of the weak solution. To address this problem we will define suitable solution and test function spaces to formulate the problem and to ensure the existence of a unique weak solution. The key observation is obtained as follows: Squaring inequality (2.5) and taking the expectation $\mathbf{E_P}$ with respect to the probability measure $\mathbf{P}$ yields

$$\mathbf{E_P}\left(\|\tilde{u}\|^2_{H^1(D)}\right) \leq C^2 \mathbf{E_P}\left(\frac{\|f\|^2_{H^{-1}(D)}}{\kappa^2_{min}}\right). \tag{2.6}$$

Hence, the pathwise solution $\tilde{u}$ is a second-order random variable in $\mathring{H}^1(D)$ if the second-order moment of the H^{-1}-norm of f, weighted with the reciprocal of the real-valued random variable κ^2_{min}, is finite. Thus, we need weighted function spaces in order to formulate the stochastic variational problem. Given a general real-valued random variable $\varrho > 0$ a.s. we introduce the spaces

$$U^m_\varrho := \mathrm{L}^2(\Omega, \mathfrak{A}, \varrho d\mathbf{P}; H^m(D)), \quad m \in \mathbb{Z}, \quad \text{and}$$
$$\mathring{U}^m_\varrho := \mathrm{L}^2(\Omega, \mathfrak{A}, \varrho d\mathbf{P}; \mathring{H}^m(D)), \quad m \in \mathbb{N}_0,$$

where the ϱ-weighted L^2-spaces are defined by

$$\mathrm{L}^2(\Omega, \mathfrak{A}, \varrho d\mathbf{P}; V) := \left\{\xi : \Omega \to V \text{ measurable} : \mathbf{E_P}\left(\|\xi\|^2_V \varrho\right) < \infty\right\}$$

with $V = H^m(D)$ or $\mathring{H}^m(D)$, respectively. Endowing the spaces U^m_ϱ and $\mathring{U}^m_\varrho$ with the inner product

$$(u, v)_{U^m_\varrho} = \mathbf{E_P}\left((u, v)_{H^m(D)}\varrho\right), \quad u, v \in U^m_\varrho$$

and the induced norm

$$\|u\|_{U^m_\varrho} = \sqrt{\mathbf{E_P}\left(\|u\|^2_{H^m(D)}\varrho\right)}, \quad u \in U^m_\varrho$$

these spaces are also Hilbert spaces and there exist isomorphisms to the corresponding tensor product spaces (see e.g. [13])

$$U^m_\varrho \cong H^m(D) \otimes \mathrm{L}^2(\Omega, \mathfrak{A}, \varrho d\mathbf{P}) \quad \text{and} \quad \mathring{U}^m_\varrho \cong \mathring{H}^m(D) \otimes \mathrm{L}^2(\Omega, \mathfrak{A}, \varrho d\mathbf{P}),$$

if $\mathrm{L}^2(\Omega, \mathfrak{A}, \varrho d\mathbf{P})$ is separable. Furthermore, we note that the seminorm

$$|u|_{U^1_\varrho} = \sqrt{\mathbf{E_P}\left(|u|^2_{H^1(D)}\varrho\right)} = \sqrt{\int_{D\times\Omega} |\nabla u(x,\omega)|^2 \varrho(\omega)\, dx\, d\mathbf{P}(\omega)}$$

is equivalent to the norm $\|\cdot\|_{U^1_\varrho}$ in $\mathring{U}^1_\varrho$ and that the dual space of $\mathring{U}^m_\varrho$ can be identified with the space $U^{-m}_{\varrho^{-1}}$. For convenience we denote by U^m or $\mathring{U}^m$ the spaces $H^m(D) \otimes \mathrm{L}^2(\Omega, \mathfrak{A}, \mathbf{P})$ or $\mathring{H}^m(D) \otimes \mathrm{L}^2(\Omega, \mathfrak{A}, \mathbf{P})$, respectively. On occasion we will replace $\mathbf{P}$ by another probability measure $\mathbf{Q}$ and write $U^m_\mathbf{Q} := H^m(D) \otimes \mathrm{L}^2(\Omega, \mathfrak{A}, \mathbf{Q})$ and $\mathring{U}^m_\mathbf{Q} := \mathring{H}^m(D) \otimes \mathrm{L}^2(\Omega, \mathfrak{A}, \mathbf{Q})$.

Then for a given $f \in U^{-1}_{\frac{1}{\kappa^2_{min}}}$ the stochastic weak formulation reads as follows:

Problem 2.4 (Stochastic Weak Formulation). Find $\hat{u} \in \mathring{U}^1$, such that

$$a(\hat{u}, v) = \langle f, v\rangle \quad \text{for all } v \in \mathring{U}^1_{\kappa^2_{min}}, \tag{2.7}$$

where the bilinear form a is given by

$$a(u,v) = \mathbf{E}_{\mathbf{P}}\left(\int\limits_D \kappa(x)\nabla u(x)\cdot\nabla v(x)\,dx\right) = \int\limits_\Omega b(u(\omega),v(\omega);\omega)\,d\mathbf{P}(\omega) \quad (2.8)$$

and the duality pairing between $f \in U^{-1}_{\frac{1}{\kappa^2_{min}}}$ and $v \in \mathring{U}^1_{\kappa^2_{min}}$ is given by

$$\langle f,v\rangle = \mathbf{E}_{\mathbf{P}}\left(\langle f,v\rangle_{H^{-1},\mathring{H}^1}\right) = \int\limits_\Omega \langle f(\omega),v(\omega)\rangle_{H^{-1},\mathring{H}^1}\,d\mathbf{P}(\omega).$$

It is important to note that the solution and test function spaces are now different spaces. Furthermore, the domain of the bilinear form a is a proper subset of $\mathring{U}^1 \times \mathring{U}^1_{\kappa^2_{min}}$, i.e., the bilinear form a is not defined or finite for all pairs $(u,v) \in \mathring{U}^1 \times \mathring{U}^1_{\kappa^2_{min}}$. Thus, an implicit requirement of the weak formulation is to find a solution $\hat{u}$ such that the related bilinear form $a(\hat{u},\cdot)$ is defined and finite for all test functions.

3. Existence and uniqueness of weak solution

In this section, we will present two alternative proofs of existence and uniqueness of a solution to the weak formulation (2.7). Both approaches have benefits and drawbacks but when combined appropriately they are a powerful tool to study weak solutions and their properties. First we state a theorem which is a generalization of the Lax-Milgram theorem where the bilinear form is not defined on a cartesian product.

Theorem 3.1. *Let Hilbert spaces X_1, X_2, Y_1, Y_2 with dense and continuous embeddings $X_2 \subset X_1$ and $Y_2 \subset Y_1$ and a bilinear form $a : X_1 \times Y_1 \supsetneq \mathcal{D}_a \to \mathbb{R}$ be given such that*

(i) the restricted bilinear forms $a\big|_{X_1\times Y_2} : X_1 \times Y_2 \to \mathbb{R}$ and $a\big|_{X_2\times Y_1} : X_2 \times Y_1 \to \mathbb{R}$ are continuous,

(ii) there holds the inf-sup condition with a constant $c > 0$

$$\inf_{u\in X_1\setminus\{0\}} \sup_{v\in Y_1\setminus\{0\}} \frac{|a(u,v)|}{\|u\|_{X_1}\|v\|_{Y_1}} \geq c > 0, \qquad \textit{and}$$

(iii) for any $v \in Y_1 \setminus \{0\}$ there exists $u \in X_2$ such that $a(u,v) > 0$.

Then for any $f \in Y_1^*$ there exists a unique $u \in X_1$ satisfying

$$a(u,v) = \langle f,v\rangle \qquad \textit{for all } v \in Y_1.$$

Proof. The operator $T_a : X_1 \to Y_2^*$, $u \mapsto a(u,\cdot)$, is linear and continuous. The restricted operator $\hat{T}_a : X_1 \supsetneq \mathcal{D}(\hat{T}_a) \to Y_1^* \subset Y_2^*$ associated with T_a is densely defined, since $X_2 \subset \mathcal{D}(\hat{T}_a) \subset X_1$ is densely embedded, and injective and closed, because of the inf-sup condition (ii). Therefore it follows with Banach's closed range theorem (see e.g. [17] p. 205) that

$$\mathcal{R}(\hat{T}_a) = \mathcal{N}(\hat{T}_a^*)^{\perp}$$

where $\hat{T}_a^*$ is the adjoint operator of $\hat{T}_a$. Condition (iii) yields $\mathcal{N}(\hat{T}_a^*) = \{0\}$, thus $\mathcal{R}(\hat{T}_a) = Y_1^*$, which completes the proof. □

Corollary 3.2. *For any $f \in U^{-1}_{\frac{1}{\kappa^2_{min}}}$ there exists a unique $\hat{u} \in \mathring{U}^1$ satisfying the stochastic weak formulation* (2.7) *and the estimate*

$$\|\hat{u}\|_{U^1} \leq C\|f\|_{U^{-1}_{\frac{1}{\kappa^2_{min}}}} .$$

Proof. The Hilbert spaces $X_1 = \mathring{U}^1, X_2 = \mathring{U}^1_{\frac{\kappa^2_{max}}{\kappa^2_{min}}}, Y_1 = \mathring{U}^1_{\kappa^2_{min}}$ and $Y_2 = \mathring{U}^1_{\kappa^2_{max}}$ and the bilinear form a defined in (2.8) satisfy all conditions in Theorem 3.1. The continuous and dense embeddings and the continuity of the bilinear forms $a|_{\mathring{U}^1 \times \mathring{U}^1_{\kappa^2_{max}}} : \mathring{U}^1 \times \mathring{U}^1_{\kappa^2_{max}} \to \mathbb{R}$ and $a|_{\mathring{U}^1_{\frac{\kappa^2_{max}}{\kappa^2_{min}}} \times \mathring{U}^1_{\kappa^2_{min}}} : \mathring{U}^1_{\frac{\kappa^2_{max}}{\kappa^2_{min}}} \times \mathring{U}^1_{\kappa^2_{min}} \to \mathbb{R}$ follow immediately from the definition of the spaces. To verify the inf-sup condition (ii), we define for $u \in \mathring{U}^1$ the random variable v_R with values in $\mathring{H}^1(D)$ by

$$v_R := \begin{cases} \frac{u}{\kappa_{min}}, & \frac{\kappa_{max}}{\kappa_{min}} \leq R, \\ 0, & \text{otherwise,} \end{cases}$$

and denote by B_R the set

$$B_R := \left\{ \omega \in \Omega \, : \, \frac{\kappa_{max}(\omega)}{\kappa_{min}(\omega)} \leq R \right\}.$$

Thus we obtain $v_R \in \mathring{U}^1_{\kappa^2_{min}}$, since

$$|v_R|^2_{U^1_{\kappa^2_{min}}} = \int_{B_R} |u(\omega)|^2_{H^1(D)} \, d\mathbf{P}(\omega) \leq |u|^2_{U^1} < \infty,$$

and by assumption (2.2) on the coefficient κ there holds

$$|a(u, v_R)| = \int_{D \times B_R} \frac{\kappa(x,\omega)}{\kappa_{min}(\omega)} |\nabla u(x,\omega)|^2 \, dx \, d\mathbf{P}(\omega) \geq \int_{B_R} |u(\omega)|^2_{H^1(D)} \, d\mathbf{P}(\omega).$$

Since $\mathbf{P}\left(\Omega \setminus \bigcup_{R>0} B_R\right) = 0$, there exists for every $\delta > 0$ a $R > 0$ such that

$$\int_{B_R} |u(\omega)|^2_{H^1(D)} \, d\mathbf{P}(\omega) \geq (1 - \delta)|u|^2_{U^1}$$

and thus

$$\sup_{v\in \mathring{U}^1_{\kappa^2_{min}}\setminus\{0\}} \frac{|a(u,v)|}{|v|_{U^1_{\kappa^2_{min}}}} \geq \frac{|a(u,v_R)|}{|v_R|_{U^1_{\kappa^2_{min}}}} \geq \frac{(1-\delta)|u|^2_{U^1}}{|u|_{U^1}} = (1-\delta)|u|_{U^1}.$$

Because $\delta > 0$ can be chosen arbitrarily the inf-sup condition holds with constant $c = 1$. Condition (iii) is satisfied, since for any $v \in \mathring{U}^1_{\kappa^2_{min}} \setminus \{0\}$ we can define

$$u_R := \begin{cases} v\kappa_{min}, & \frac{\kappa_{max}}{\kappa_{min}} \leq R, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and the set } B_R \text{ as above,}$$

which satisfies $u_R \in \mathring{U}^1$ and $a(u_R, v) > 0$ for R large enough. Hence, by Theorem 3.1 the statement follows. □

Obviously, Corollary 3.2 is also true for problems with other boundary conditions as long as the seminorm is a norm in the corresponding function spaces.

An alternative method to prove existence and uniqueness of the solution to Problem 2.4 where the coefficient κ is a lognormal random field, is given in the work of Gittelson [11]. For this special case it can be shown that the unique pathwise solution $\tilde{u}$ is also the unique solution of the stochastic variational problem if it belongs to the solution space. Below we prove an analogous result for the more general assumptions (2.2) on the random coefficient.

Theorem 3.3. *For $f \in U^{-1}_{\frac{1}{\kappa^2_{min}}}$ the unique solution $\tilde{u}$ of Problem* 2.1 *belongs to $\mathring{U}^1$ and it solves also Problem* 2.4*. Furthermore, any solution $\hat{u} \in \mathring{U}^1$ of Problem* 2.4 *is $\sigma(f,\kappa)$-measurable and there holds*

$$\hat{u}(x,\omega) = \tilde{u}(x,\omega) \qquad a.e. \text{ and } a.s.$$

Proof. Recalling that $f \in U^{-1}_{\frac{1}{\kappa^2_{min}}}$ and utilizing the estimate (2.6) we obtain

$$\|\tilde{u}\|^2_{U^1} = \mathbf{E_P}\|\tilde{u}\|^2_{H^1(D)} \leq C^2 \mathbf{E_P}\frac{\|f\|^2_{H^{-1}(D)}}{\kappa^2_{min}} = C^2\|f\|^2_{U^{-1}_{\frac{1}{\kappa^2_{min}}}} < \infty.$$

Since $\tilde{u}$ satisfies equation (2.3), there holds for all $v \in \mathring{U}^1_{\kappa^2_{min}}$

$$b(\tilde{u}(\omega), v(\omega); \omega) = \langle f(\omega), v(\omega)\rangle_{H^{-1},\mathring{H}^1} \qquad \text{a.s.}$$

Taking the expectation yields $a(\tilde{u}, v) = \langle f, v\rangle$ for all $v \in \mathring{U}^1_{\kappa^2_{min}}$ and hence $\tilde{u}$ solves Problem 2.4. Now, we consider a random variable $\hat{u} \in \mathring{U}^1$ satisfying

$$a(\hat{u}, v) = \langle f, v\rangle \qquad \text{for all } v \in \mathring{U}^1_{\kappa^2_{min}}.$$

Then we define for $w \in \mathring{H}^1(D)$ and $A \in \mathfrak{A}$ the functions $v_{w,A}(x,\omega) := w(x)\frac{\mathbf{1}_A(\omega)}{\kappa_{min}(\omega)}$. It follows $v_{w,A} \in \mathring{U}^1_{\kappa^2_{min}}$ and we get

$$\mathbf{E_P}\frac{\mathbf{1}_A}{\kappa_{min}}b(\hat{u}, w; \cdot) = a(\hat{u}, v_{w,A}) = \langle f, v_{w,A}\rangle = \mathbf{E_P}\frac{\mathbf{1}_A}{\kappa_{min}}\langle f, w\rangle_{H^{-1},\mathring{H}^1}.$$

Since $A \in \mathfrak{A}$ can be chosen arbitrarily this implies for any $w \in \mathring{H}^1(D)$

$$b(\hat{u}(\omega), w; \omega) = \langle f(\omega), w\rangle_{H^{-1},\mathring{H}^1} \quad \text{a.s.}$$

Hence, the random variable $\hat{u}$ with values in $\mathring{H}^1(D)$ solves problem (2.4) and since its solution is almost surely unique and $\sigma(f,\kappa)$-measurable (cf. Lemma 2.3), there holds

$$\hat{u}(x,\omega) = \tilde{u}(x,\omega) \quad \text{a.e. and a.s.,}$$

i.e., the random variable $\hat{u}$ is measurable with respect to the σ-algebra $\sigma(f,\kappa)$. □

4. Stochastic Galerkin discretization

Let $\xi := (\xi_i)_{i\in I_\xi}$ with index set $I_\xi \subseteq \mathbb{N}$ be a sequence of real-valued so called "basic" random variables, such that there are measurable functions $\kappa_\xi,\ f_\xi : D \times \mathbb{R}^{|I_\xi|} \to \mathbb{R}$ satisfying

$$\kappa(x,\omega) = \kappa_\xi(x,\xi(\omega)) \qquad \text{and} \qquad f(x,\omega) = f_\xi(x,\xi(\omega)) \quad \text{a.e. and a.s.}$$

Thereby the index set I_ξ can be finite, i.e., $I_\xi = \{1,\ldots,M\}$, $M \in \mathbb{N}$, or the set of the natural numbers, i.e., $I_\xi = \mathbb{N}$. Sequences of basic random variables can be obtained with the help of Karhunen-Loève expansions (see e.g. [12]) or other series expansions (see e.g. [10]) of the input data.

Then according to Theorem 3.3 the solution $\hat{u}$ of variational formulation (2.7) belongs to $\mathrm{L}^2(\Omega,\sigma(\xi),\mathbf{P};\mathring{H}^1(D))$ since κ and f are $\sigma(\xi)$-measurable. In the following we assume that the random variable $\xi = (\xi_i)_{i\in I_\xi}$ on the probability space $(\Omega,\mathfrak{A},\mathbf{P})$ has the distribution $F^{\mathbf{P}}_\xi$ and that any ξ_i, $i \in I_\xi$, possesses finite moments of arbitrary order, i.e., $\mathbf{E_P}|\xi_i|^n < \infty$, $n \in \mathbb{N}$, and a continuous distribution function $F^{\mathbf{P}}_{\xi_i}$.

In order to apply the Stochastic Galerkin Method we define the space

$$U_{N,K,p} := U_p \otimes U_{N,K} \subset \mathring{U}^1$$

which serves as solution space for the Stochastic Galerkin approximation. The space U_p is a finite-dimensional subspace of $\mathring{H}^1(D)$ obtained by a uniform p version of the Finite Element Method and $U_{N,K}$ is a finite-dimensional subspace of $\mathrm{L}^2(\Omega,\sigma(\xi_1,\ldots,\xi_K),\mathbf{P}) \subseteq \mathrm{L}^2(\Omega,\sigma(\xi),\mathbf{P})$ with $\{1,\ldots,K\} \subseteq I_\xi$. Since we want to use generalized polynomial chaos (see e.g. [15, 16]), i.e. polynomials in the underlying basic random variables ξ, we construct the finite dimensional space $U_{N,K}$ as follows,

$$U_{N,K} := \operatorname{span}\left\{\xi^\alpha := \prod_{i\in I_\xi}\xi_i^{\alpha_i},\ \alpha \in \Lambda_{N,K}\right\}.$$

We choose the index set

$$\Lambda_{N,K} \subset \Lambda := \{\alpha \in \mathbb{N}_0^{|I_\xi|} \, : \, \alpha \text{ has only finitely many non-zero entries}\}$$

such that the total degree of the multivariate polynomials is bounded,

$$\Lambda_{N,K} = \{\alpha \in \Lambda \, : \, \alpha_i = 0 \; \forall \; i > K, \; |\alpha| \leq N \; \}, \qquad |\alpha| := \sum_{i \in I_\xi} \alpha_i.$$

As discretized test function space we choose

$$V_{N,K,p} := \left\{ \frac{u}{\kappa_{min}} \, : \, u \in U_{N,K,p} \right\}.$$

Then for a given $f \in U^{-1}_{\frac{1}{\kappa^2_{min}}}$ the discrete version of the weak formulation (2.7) reads as follows:

Problem 4.1 (Discrete Weak Formulation). Find $\hat{u}_{N,K,p} \in U_{N,K,p}$, such that

$$a(\hat{u}_{N,K,p}, v) = \langle f, v \rangle \quad \text{for all } v \in V_{N,K,p}. \tag{4.1}$$

The existence of a unique Stochastic Galerkin solution $\hat{u}_{N,K,p} \in U_{N,K,p}$ to problem (4.1) can be proved under the assumptions in the following lemma.

Lemma 4.2. *If* $\frac{\kappa^2_{max}}{\kappa^2_{min}} \in L^r(\Omega, \mathfrak{A}, \mathbf{P})$ *for some* $r > 1$ *then for any* $f \in U^{-1}_{\frac{1}{\kappa^2_{min}}}$ *there exists a unique* $\hat{u}_{N,K,p} \in U_{N,K,p}$ *such that*

$$a(\hat{u}_{N,K,p}, v) = \langle f, v \rangle \qquad \text{for all } v \in V_{N,K,p}.$$

Proof. The result follows from Theorem 3.1 with the Hilbert spaces

$$X_1 = U_{N,K,p} \subset \mathring{U}^1, \qquad X_2 = U_{N,K,p} \subset \mathring{U}^1_{\frac{\kappa^2_{max}}{\kappa^2_{min}}},$$

$$Y_1 = V_{N,K,p} \subset \mathring{U}^1_{\kappa^2_{min}} \quad \text{and} \quad Y_2 = V_{N,K,p} \subset \mathring{U}^1_{\kappa^2_{max}}$$

due to $\frac{\kappa^2_{max}}{\kappa^2_{min}} \in \mathrm{L}^r(\Omega, \mathfrak{A}, \mathbf{P})$ for some $r > 1$ and a discrete version of the inf-sup condition for the bilinear form a. □

Now, we want to investigate the approximation error of this Stochastic Galerkin solution $\hat{u}_{N,K,p}$. Employing the discrete inf-sup condition we get a quasi-optimal result for the Galerkin solution, i.e., the error can be bounded by a best approximation error in another – a stronger – norm.

Lemma 4.3. *If* $\frac{\kappa^2_{max}}{\kappa^2_{min}} \in L^r(\Omega, \mathfrak{A}, \mathbf{P})$ *for some* $r > 1$ *and* $\hat{u} \in \mathring{U}^1_{\frac{\kappa^2_{max}}{\kappa^2_{min}}}$ *then the following estimate holds*

$$|\hat{u} - \hat{u}_{N,K,p}|_{U^1} \leq \widetilde{C} \inf_{z \in U_{N,K,p}} |\hat{u} - z|_{U^1_{\frac{\kappa^2_{max}}{\kappa^2_{min}}}}$$

with a constant $\widetilde{C} > 0$ *(independent of* N, K *and* p*) for the solutions* $\hat{u}$ *and* $\hat{u}_{N,K,p}$ *of the weak formulation* (2.7) *and the discrete weak formulation (*4.1*), respectively.*

Proof. Utilizing $a(\hat{u}-z,v) = a(\hat{u}_{N,K,p}-z,v)$ for all $v \in V_{N,K,p}$ and the discrete inf-sup condition we obtain

$$|\hat{u}-\hat{u}_{N,K,p}|_{U^1} \le |\hat{u}-z|_{U^1} + |\hat{u}_{N,K,p}-z|_{U^1} \le |\hat{u}-z|_{U^1} + |\hat{u}-z|_{U^1_{\frac{\kappa^2_{max}}{\kappa^2_{min}}}} \le 2|\hat{u}-z|_{U^1_{\frac{\kappa^2_{max}}{\kappa^2_{min}}}}$$

for all $z \in U_{N,K,p}$. □

Consequently we measure the error in the stronger $U^1_{\frac{\kappa^2_{max}}{\kappa^2_{min}}}$-norm and we assume the following.

Assumption 4.4. *Let* $q := \mathbf{E_P}\frac{\kappa^2_{max}}{\kappa^2_{min}} < \infty$ *and assume* $\frac{\kappa^2_{max}}{\kappa^2_{min}}$ *is* $\sigma(\xi)$*-measurable, i.e., there exists a measurable transformation* $t_{\frac{\kappa^2_{max}}{\kappa^2_{min}}} : \mathbb{R}^{|I_\xi|} \to \mathbb{R}^+$ *with* $\frac{\kappa^2_{max}}{\kappa^2_{min}} = t_{\frac{\kappa^2_{max}}{\kappa^2_{min}}}(\xi)$.

Then the measure $\mathbf{Q}$ with $d\mathbf{Q} = \frac{1}{q}\kappa^2_{max}\kappa^{-2}_{min}d\mathbf{P}$ is a probability measure. In the following we consider the function spaces $U^m_{\mathbf{Q}}$ and $\mathring{U}^m_{\mathbf{Q}}$ instead of $U^m_{\frac{\kappa^2_{max}}{\kappa^2_{min}}}$ and $\mathring{U}^m_{\frac{\kappa^2_{max}}{\kappa^2_{min}}}$, $m \in \mathbb{Z}$, which coincide with $U^m_{\frac{\kappa^2_{max}}{\kappa^2_{min}}}$ and $\mathring{U}^m_{\frac{\kappa^2_{max}}{\kappa^2_{min}}}$ but are much easier to handle due to the corresponding probability space $(\Omega, \mathfrak{A}, \mathbf{Q})$ at hand.

Corollary 4.5. *If* $\frac{\kappa^2_{max}}{\kappa^2_{min}} \in L^r(\Omega, \mathfrak{A}, \mathbf{P})$ *for some* $r > 1$ *and Assumption* 4.4 *is fulfilled there holds for* $\hat{u} \in \mathring{U}^1_{\mathbf{Q}}$ *with a suitable constant* $C > 0$ *(independent of* N, K *and* p*)*

$$|\hat{u}-\hat{u}_{N,K,p}|_{U^1} \le C \inf_{z \in U_{N,K,p}} |\hat{u}-z|_{U^1_{\mathbf{Q}}} \tag{4.2}$$

for the solutions $\hat{u}$ *and* $\hat{u}_{N,K,p}$ *of the corresponding weak formulation* (2.7) *and discrete weak formulation* (4.1).

Proof. This result follows immediately from Lemma 4.3 and Assumption 4.4. □

By choosing a suitable $z \in U_{N,K,p}$ and applying the triangle inequality to the right-hand side of (4.2) we can identify different sources of the approximation error. To see this, we introduce some notations: We denote by

$\Pi_{\mathring{U}^1_{\mathbf{Q},N,K,p}} : \mathring{U}^1_{\mathbf{Q}} \to U_{N,K,p}$

the orthogonal projection onto $U_{N,K,p}$, and by

$\Pi_{\mathring{U}^1_{\mathbf{Q},N,K}} : \mathring{U}^1_{\mathbf{Q}} \to \mathring{H}^1(D) \otimes U_{N,K}$

the orthogonal projection onto $\mathring{H}^1(D) \otimes U_{N,K}$, both with respect to the $U^1_{\mathbf{Q}}$-norm. Assuming $\hat{u} \in \mathring{U}^1_{\mathbf{Q}}$ the approximation error of the Stochastic Galerkin approximation to the exact solution can be estimated using (4.2) with $z = \Pi_{\mathring{U}^1_{\mathbf{Q},N,K,p}} \hat{u}$ as

$$|\hat{u} - \hat{u}_{N,K,p}|_{U^1} \leq C \left[|\hat{u} - \Pi_{\mathring{U}^1_{\mathbf{Q},N,K}} \hat{u}|_{U^1_{\mathbf{Q}}} + |\Pi_{\mathring{U}^1_{\mathbf{Q},N,K}} \hat{u} - \Pi_{\mathring{U}^1_{\mathbf{Q},N,K,p}} \hat{u}|_{U^1_{\mathbf{Q}}} \right]. \quad (4.3)$$

Hence this error has two components, namely an approximation error due to discretizing in the stochastic dimension and an approximation error due to discretizing in the spatial dimension.

The spatial approximation error can be bounded using standard arguments from the theory of Finite Element Methods (FEMs). Here, we have employed a p version of the FEM (see e.g. [2]). Under the assumptions of Corollary 2.2 in [2] there holds the following.

Corollary 4.6. *If* $\frac{\kappa^2_{max}}{\kappa^2_{min}} \in L^r(\Omega, \mathfrak{A}, \mathbf{P})$ *for some* $r > 1$ *and Assumption* 4.4 *is satisfied then for* $\hat{u} \in U^k_{\frac{\kappa^2_{max}}{\kappa^3_{min}}} \cap \mathring{U}^1_{\mathbf{Q}}$ *with constant* $\widetilde{C} > 0$ *(independent of* N*,* K*,* p *and* $\hat{u}$*) there holds*

$$|\Pi_{\mathring{U}^1_{\mathbf{Q},N,K}} \hat{u} - \Pi_{\mathring{U}^1_{\mathbf{Q},N,K,p}} \hat{u}|_{U^1_{\mathbf{Q}}} \leq \widetilde{C} p^{-(k-1)} \|\hat{u}\|_{U^k_{\frac{\kappa^2_{max}}{\kappa^3_{min}}}}.$$

Proof. From Corollary 2.2 in [2] it follows

$$\sqrt{\kappa_{min}(\omega)} \left| \Pi_{\mathring{U}^1_{\mathbf{Q},N,K}} \hat{u}(\omega) - \Pi_{\mathring{U}^1_{\mathbf{Q},N,K,p}} \hat{u}(\omega) \right|_{H^1(D)} \leq \widetilde{C} p^{-(k-1)} \|\hat{u}(\omega)\|_{H^k(D)}$$

with a constant $\widetilde{C}$ independent of N, K, p, $\omega \in \Omega$ and $\hat{u}$. Squaring and taking the expectation $\mathbf{E}_{\mathbf{Q}}$ with respect to $\mathbf{Q}$ leads to

$$\mathbf{E}_{\mathbf{Q}} \left| \Pi_{\mathring{U}^1_{\mathbf{Q},N,K}} \hat{u} - \Pi_{\mathring{U}^1_{\mathbf{Q},N,K,p}} \hat{u} \right|^2_{H^1(D)} \leq \widetilde{C}^2 p^{-2(k-1)} \mathbf{E}_{\mathbf{Q}} \frac{\|\hat{u}\|^2_{H^k(D)}}{\kappa_{min}}.$$

□

We note that analogous results to Corollary 4.6 can be obtained for h or h-p versions of the FEM by using Theorem 2.1 in [2].

The first term on the right-hand side of inequality (4.3) can be estimated with the help of generalized polynomial chaos expansions. In view of Assumption 4.4 the random variable $\xi = (\xi_i)_{i \in I_\xi}$ as a random variable on the probability space $(\Omega, \mathfrak{A}, \mathbf{Q})$ has the distribution $F^{\mathbf{Q}}_\xi(dy) = \frac{1}{q} t_{\frac{\kappa^2_{max}}{\kappa^2_{min}}}(y) F^{\mathbf{P}}_\xi(dy)$. Assuming $\mathbf{E}_{\mathbf{Q}} |\xi_i|^n < \infty$ for all $i \in I_\xi$ and $n \in \mathbb{N}$ the multivariate orthonormal polynomials $\{q_\alpha(\xi),\ \alpha \in \Lambda\}$ in $\mathrm{L}^2(\Omega, \mathfrak{A}, \mathbf{Q})$ exist. Hence, in order to expand any random variable $u \in \mathrm{L}^2(\Omega, \sigma(\xi), \mathbf{Q}; \mathring{H}^1(D))$ in this generalized polynomial chaos the polynomials $\{q_\alpha(\xi),\ \alpha \in \Lambda\}$ have to be dense in $\mathrm{L}^2(\Omega, \sigma(\xi), \mathbf{Q})$. Some necessary conditions to establish this property are discussed in [8]. If the polynomials lie dense and $\hat{u} \in \mathrm{L}^2(\Omega, \sigma(\xi), \mathbf{Q}; \mathring{H}^1(D))$ then the solution possesses a generalized polynomial chaos expansion $\{q_\alpha(\xi),\ \alpha \in \Lambda\}$, i.e.,

$$\hat{u}(x,\omega) = \sum_{\alpha\in\Lambda} \hat{u}_\alpha(x) q_\alpha(\xi(\omega)), \quad \text{where} \quad \hat{u}_\alpha(x) = \mathbf{E}_{\mathbf{Q}} \hat{u}(x) q_\alpha(\xi).$$

Furthermore, the projection $\Pi_{\mathring{U}^1_{\mathbf{Q},N,K}} \hat{u}$ is given by the truncated expansion

$$\Pi_{\mathring{U}^1_{\mathbf{Q},N,K}} \hat{u}(x,\omega) = \sum_{\alpha\in\Lambda_{N,K}} \hat{u}_\alpha(x) q_\alpha(\xi(\omega)).$$

Corollary 4.7. *If the polynomials* $\{q_\alpha(\xi),\ \alpha \in \Lambda\}$ *are dense in* $L^2(\Omega, \sigma(\xi), \mathbf{Q})$ *and* $\hat{u} \in \mathring{U}^1_{\mathbf{Q}}$ *then the approximation error*

$$|\hat{u} - \Pi_{\mathring{U}^1_{\mathbf{Q},N,K}} \hat{u}|_{U^1_{\mathbf{Q}}} \to 0 \quad (K, N \to \infty).$$

Proof. The multivariate polynomials $\{q_\alpha(\xi),\ \alpha \in \Lambda\}$ form an orthonormal basis of $\mathrm{L}^2(\Omega, \sigma(\xi), \mathbf{Q})$ because they are dense in $\mathrm{L}^2(\Omega, \sigma(\xi), \mathbf{P})$. Since the weak solution $\hat{u}$ is $\sigma(\xi)$-measurable (according to Theorem 3.3) and

$$\bigcup_{N\geq 0,\, K\geq 1} \Lambda_{N,K} = \Lambda$$

there holds that $\quad \Pi_{\mathring{U}^1_{\mathbf{Q},N,K}} \hat{u} \to \hat{u} \quad$ in $\mathring{U}^1_{\mathbf{Q}}$ for $K \to \infty,\ N \to \infty.$ □

Hence in view of Corollary 4.6 and Corollary 4.7 the approximation error $|\hat{u} - \hat{u}_{N,K,p}|_{U^1}$ converges to zero if the solution $\hat{u} \in U^2_{\frac{\kappa^2_{max}}{\kappa^3_{min}}} \cap \mathring{U}^1_{\mathbf{Q}}$ and the orthonormal polynomials $\{q_\alpha(\xi),\ \alpha \in \Lambda\}$ are complete in $\mathrm{L}^2(\Omega, \sigma(\xi), \mathbf{Q})$.

5. Numerical example

Now, we turn to a specific application, namely the approximation of the solution of an one-dimensional differential equation with random data. Consider the boundary value problem

$$\begin{aligned} -(\kappa(x,\omega) u'(x,\omega))' &= f(x), \quad x \in (0,1),\ \omega \in \Omega \\ u(0,\omega) &= 0, \quad \omega \in \Omega \\ \kappa(1,\omega) u'(1,\omega) &= F, \quad \omega \in \Omega \end{aligned}$$

where forcing $f \in H^{-1}(D)$ is a deterministic function, F a given constant and κ a strongly measurable random variable in $\mathrm{L}^\infty(D)$ satisfying

$$0 < \kappa_{min}(\omega) \leq \kappa(x,\omega) \leq \kappa_{max}(\omega) < \infty \qquad \text{a.e. and a.s.}$$

for some real-valued random variables κ_{min} and κ_{max}. Then the exact solution is given by

$$u(x,\omega) = \int_0^x \frac{1}{\kappa(y,\omega)} \left(F + \int_y^1 f(z)\, dz \right) dy.$$

If the coefficient κ is modeled as an exponential function of the absolute value of one standard Gaussian distributed random variable, that is,

$$\kappa(x,\omega) := \exp(|\zeta(\omega)|x) \quad \text{with } \zeta \sim \mathcal{N}(0,1)$$

then κ is bounded by

$$0 < 1 \leq \kappa(x,\omega) \leq \exp(|\zeta(\omega)|) < \infty \qquad \text{a.e. and a.s.}$$

The random variable $\kappa_{max}^2/\kappa_{min}^2 = \exp(2|\zeta|)$ is in $\mathrm{L}^r(\Omega,\mathfrak{A},\mathbf{P})$ for all $r \geq 1$. As basic random variable we choose the standard Gaussian distributed random variable ζ, i.e., $\xi = \zeta$, and employ the Stochastic Galerkin Method using orthonormal polynomials, i.e. polynomial chaos, in ξ. Figure 1 shows the rel-

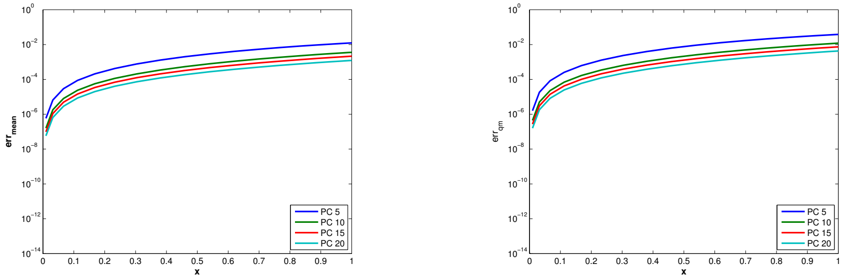

FIGURE 1. Relative errors of mean (left) and second moment (right) of the Stochastic Galerkin approximation to the solution with $f \equiv 1$ and $F = 1$ using polynomials of different orders in ξ.

ative errors of the mean and second-order moment of the Stochastic Galerkin approximation to the exact solution as a function of the spatial variable x. Thereby we have chosen the forcing $f \equiv 1$ and the boundary value $F = 1$ and we use a p version of the Finite Element Method, precisely, a single Gauss-Lobatto-Legendre spectral finite element of degree $p = 20$ for the spatial discretization. In the stochastic dimension we use orthonormal polynomials in ξ up to degree 5, 10, 15 and 20. Obviously, the error decays, which agrees with the theory developed in Section 4. On the other hand it is also possible to choose as basic random variable $\eta = |\zeta|$, a chi-distributed random variable with one degree of freedom. Thus, we can use orthonormal polynomials, i.e. generalized polynomial chaos, in η within the Stochastic Galerkin Method, in particular orthonormal polynomials in η up to degree 2 and 5. In the spatial dimension we again use a single Gauss-Lobatto-Legendre spectral finite element of degree $p = 20$. In Figure 2 we observe that the associated relative errors of the mean and second-order moment tend to zero much faster than for the standard Gaussian basic random variable ξ. Notably, we obtain much better approximation results by using polynomials up to order 2 and 5 in $\eta = |\zeta|$ as compared to polynomials up to order 20 in $\xi = \zeta$. Hence, the approximation error, more precisely the rate of convergence, and thus the

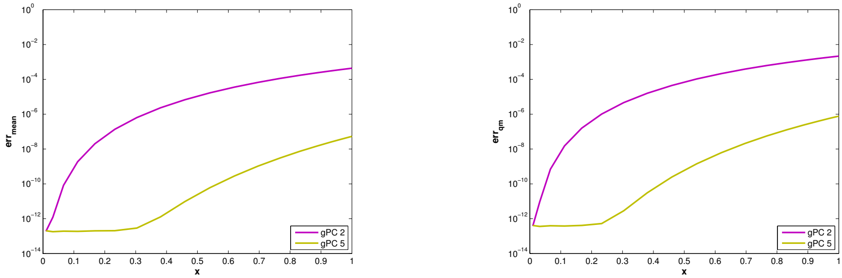

FIGURE 2. Relative errors of mean (left) and second moment (right) of the Stochastic Galerkin approximation to the solution with $f \equiv 1$ and $F = 1$ using polynomials of different orders in η.

approximation quality depends on the set of basic random variables. This relation is currently being investigated in ongoing research.

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