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Recent results on Chlodovsky operators

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Abstract. We take a view on the results concerning the Bernstein–Chlodovsky operators obtained especially in the last five years. The list presented in this paper is not exhaustive. We apologise all authors possessing papers on the Chlodovsky operators and are not referred in this paper.

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1. Introduction

For a function f defined on $[0,\infty)$ and bounded on every finite interval $[0,b]\subset[0,\infty)$, the classical Bernstein-Chlodovsky operators are defined by

$$(C_n f)(x) := \sum_{k=0}^{n} f\left(\frac{b_n}{n}k\right) p_{k,n}\left(\frac{x}{b_n}\right), \tag{1.1}$$

where $p_{k,n}$ denotes as usual

$$p_{k,n}(x) = \left(\begin{array}{c} n \\ k \end{array}\right) x^k (1-x)^{n-k}, \quad 0 \le x \le 1,$$

and $(b_n)_{n=1}^{\infty}$ is a positive increasing sequence of reals with the properties

$$\lim_{n\to\infty} b_n = \infty \quad , \quad \lim_{n\to\infty} \frac{b_n}{n} = 0. \tag{1.2}$$

These polynomials were introduced by I. Chlodovsky [11] in **1937** to generalize the Bernstein polynomials $(B_n f)(x)$, for the case $b_n = 1$, $n \in \mathbb{N}_0$, which approximate the function f on the interval $[0,1]$ (or, suitably modified on any fixed finite interval $[-b,b]$). His main result is the following:

Chlodovsky's Theorem. *Let* (b_n) *satisfy* (1.2) *and, for* $b>0$*, let* $M(b;f) := \sup\limits_{0\le t\le b} |f(t)|$. *If*

$$\lim_{n\to\infty} M(b_n;f)\exp(-\sigma n/b_n) = 0 \qquad \textit{for each } \sigma > 0, \tag{1.3}$$

then

$$\lim_{n\to\infty} (C_n f)(x) = f(x)$$

at each point x of continuity of the function f.

As a corollary he states that if a function f belonging to $C[0,\infty)$ is of order $f(x) = \mathcal{O}(\exp x^p)$ for some $p > 0$, and if the sequence $\{b_n\}$ satisfies the condition

$$b_n \leq n^{\frac{1}{p+1+\eta}},$$

where $\eta > 0$, no matter how small, then $(C_n f)(x)$ converges to $f(x)$ at each point $x \in \mathbb{R}^+$.

The first part of the next and very important lemma is due to Chlodovsky [11].

For $t \in [0,1]$ the inequality

$$0 \leq z \leq \frac{3}{2}\sqrt{nt(1-t)}$$

implies

$$\sum_{|k-nt|\geq 2z\sqrt{nt(1-t)}} p_{k,n}(t) \leq 2\exp\left(-z^2\right).$$

In particular, for $0 < \delta \leq x < b_n$ and sufficiently large n,

$$\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} p_{k,n}\left(\frac{x}{b_n}\right) \leq 2\exp\left(-\frac{\delta^2}{4x}\frac{n}{b_n}\right). \tag{1.4}$$

The proof of (1.4) is given in the **1960** by Albrycht and Radecki [2].

Chlodovsky showed more, namely the simultaneous convergence of the derivative $(C_n f)'(x)$ to $f'(x)$ at points x where it exists, a result taken up by Butzer [6].

Next question concerning Chlodovsky operators was the rate of approximation by $(C_n f)(x)$ to $f(x)$, which is the counterpart of the classical questions for Bernstein polynomials answered by Voronovskaya [29] in **1932**. She showed that for bounded f on $[0,1]$, one has the asymptotic formula

$$\lim_{n\to\infty} n[(B_n f)(x_0) - f(x_0)] = \frac{x_0(1-x_0)}{2}f''(x_0) \tag{1.5}$$

at each fixed point $x_0 \in [0,1]$ for which there exists $f''(x_0) \neq 0$.

The following relations of the Voronovskaya-type for the Chlodovsky operators and their derivatives are presented in [2].

If a function f satisfies

$$\lim_{n\to\infty} \frac{n}{b_n}\exp\left(-\sigma\frac{n}{b_n}\right)M(b_n;f) = 0 \quad \textit{for each } \sigma > 0,$$

then the Voronovskaya-type theorems for Chlodovsky operators read

$$\lim_{n\to\infty} \frac{n}{b_n}[C_n f(x) - f(x)] = \frac{x}{2}f''(x)$$

at each point $x \geq 0$ for which $f''(x)$ exists.

2002 [3] In their introduction the authors write that " as far as we know, [a Voronovskaya-type formula] cannot be stated for the classical C_n". For this purpose they introduced the "more flexible" polynomials

$$C_n^* f(x) = \sum_{k=0}^{n} f\left(\frac{c_n}{n}k\right)\begin{pmatrix} n \\ k \end{pmatrix}\left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k}$$

for which $b_n \leq c_n$ for all $n \geq 1$, and $b_n \to 0$, $b_n/n \to 0$, with $b_n - c_n \to 0$, all as $n \to \infty$. They worked in the weighted (polynomial) space. Their main theorem stated that

$$\lim_{n\to\infty} \rho_n[C_n^* f(x) - f(x)] = axf''(x) + bxf'(x),$$

where $\{\rho_n\}$ is a divergent increasing sequence of reals such that $\rho_n c_n/n \to 2a$ and $\rho_n(c_n/b_n - 1) \to b$ as $n \to \infty$, $a, b \geq 0$.

It is a pity these authors were not aware of the paper [2].

2003 [4] In this paper it is presented the extension of (1.5) to derivatives of the Bernstein polynomials. The result states that for bounded f for which $f'''(x)$ exists at $x \in [0, 1]$, one has

$$\lim_{n\to\infty} n[(B_n f)'(x) - f'(x)] = \frac{1-2x}{2} f''(x) + \frac{x(1-x)}{2} f'''(x). \tag{1.6}$$

2. A brief history of the recent results on Chlodovsky operators (2005-...)

We present below, in chronological order, a list of papers dealing with the Bernstein-Chlodovsky Polynomials.

2005 [13] We introduce a Chlodovsky Type Durrmeyer operator as follows: $D_n : BV[0, \infty) \to \mathcal{P}$,

$$\left(D_n f\right)(x) = \frac{(n+1)}{b_n} \sum_{k=0}^{n} p_{k,n}\left(\frac{x}{b_n}\right) \int_0^{b_n} f(t) p_{k,n}\left(\frac{t}{b_n}\right) \, dt, \quad 0 \leq x \leq b_n \tag{2.1}$$

where $\mathcal{P} := \{P : [0, \infty) \to R\}$, is a polynomial functions set, and $p_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ is the Bernstein basis. We estimated the rate of convergence of operators D_n, for functions of bounded variation at the points which one sided limit exist, for functions of bounded variation on the interval $[0, \infty)$, by means of the techniques of probability theory.

2006 [8] The authors establish two inverse theorems for Bernstein-Chlodovsky type polynomials of two variables in a rectangular and a triangular domain.

2006 [15] The aim of this paper is to study the problem of the approximation of functions of two variables by means of Bernstein-Chlodovsky polynomials in a rectangular domain.

2006 [1] The concern of this note is to introduce a general class of linear positive operators of discrete type acting on the space of real valued functions defined on a plane domain. These operators preserve some test functions of Bohman-Korovkin theorem. As a particular class, a modified variant of the bivariate Bernstein-Chlodovsky operators is presented.

2007 [17] We estimate the rate of pointwise convergence of the Chlodovsky-type Bernstein operators $(C_n f)(x)$ for functions defined on the interval $[0, b_n]$, for $b_n \to \infty$ as $n \to \infty$, which are of bounded variation on $[0, \infty)$. At those points for which one-sided limits exists, we shall prove that the operators $(C_n f)(x)$ converge to the limit $\dfrac{f(x+)+f(x-)}{2}$.

2007 [18] Denote by $DBV(I)$, the class of differentiable functions defined on a set $I \subset R$, whose derivatives are with bounded variation on I:

$$DBV(I) = \{f : f' \in BV(I)\}.$$

The aim of this paper is to estimate the rate of convergence of $D_n f$ defined in (2.1) toward f, which is a function that has a derivative with bounded variation on $[0, b_n]$, where $b_n \to \infty$ as n goes to infinity. $(D_n f)(x)$ converges to $f(x)$ in every point x of discontinuity of the first kind of the derivative of f.

2008 [19] We define a new kind of MKZD operators for functions defined on $[0, b_n]$, named Chlodovsky-type MKZD operators as

$$(M_n^* f)(x) = \sum_{k=0}^{\infty} \frac{n+k}{b_n} m_{n,k}\left(\frac{x}{b_n}\right) \int_0^{b_n} f(t) b_{n,k}\left(\frac{t}{b_n}\right) dt, \quad 0 \le x \le b_n,$$

where $m_{n,k}(x) = \binom{n+k-1}{k} x^k (1-x)^n$ and $b_{n,k}(t) = n\binom{n+k}{k} t^k (1-t)^{n-1}$. The aim of this paper is to study the behavior of the M_n^* operators for functions of bounded variation and give an estimate, by means of the techniques of probability theory, of the rate of convergence of the operators on the interval $[0, b_n]$, $(n \to \infty)$ extending infinity.

2008 [20] The concern of this paper is to study the rate of convergence of $C_n f$ to f for $f \in DBV([0, b_n])$, $(n \to \infty)$ extending infinity. At the point x, which is a discontinuity of the first kind of the derivative, we shall prove that $(C_n f)(x)$ converge to the limit $f(x)$.

2008 [23] For $\alpha \geq 1$, we now introduce Chlodovsky-Bézier operators $C_{n,\alpha}$ as follows:

$$(C_{n,\alpha}f)(x) = \sum_{k=0}^{n} f\left(\frac{k\,b_n}{n}\right) Q_{n,k}^{(\alpha)}\left(\frac{x}{b_n}\right) \quad , (\, 0 \leq x \leq b_n\,), \tag{2.2}$$

where $Q_{n,k}^{(\alpha)}(\frac{x}{b_n}) = \left(J_{n,k}(\frac{x}{b_n})\right)^{\alpha} - \left(J_{n,k+1}(\frac{x}{b_n})\right)^{\alpha}$ and $J_{n,k}(\frac{x}{b_n}) = \sum\limits_{j=k}^{n} p_{j,n}(\frac{x}{b_n})$ be the Bézier basis functions. Obviously, $C_{n,\alpha}$ is a positive linear operator and $C_{n,\alpha}(1,x) = 1$. In particular when $\alpha = 1$, the operators (2.2) reduce to the operators (1.1) In this paper, we estimate the rate of pointwise convergence of the Bézier Variant of Chlodovsky operators $C_{n,\alpha}$ for functions, defined on the interval extending infinity, of bounded variation.

2008 [24] We introduce the following q-Chlodovsky polynomials defined as

$$(C_{n,q}f)(x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q} b_n\right) \left[\begin{array}{c} n \\ k \end{array}\right]_q \left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n-k-1}\left(1 - q^s \frac{x}{b_n}\right), 0 \leq x \leq b_n$$

where (b_n) is a positive increasing sequence with the property (1.2). We study some approximation properties of these new operators, which include the well-known Bohman-Korovkin-type theorem, degree of pointwise and uniform convergence and investigation of the monotonocity property of $q-$Chlodovsky operators.

2009 [9] The author introduce the positive linear operators q-Bernstein-Chlodovsky polynomials on a rectangular domain and obtain their Korovkin type approximation properties. The rate of convergence of this generalization is obtained by means of the modulus of continuity, and also by using the K-functional of Peetre. He obtains weighted approximation properties for these positive linear operators and their generalizations.

2009 [16] Approximation on an unbounded interval is studied in this work by means of a new-defined two-parameter polynomial operator based on Chlodovsky polynomials. The operator's properties including convergence rate are investigated using the weighted modulus of continuity.

2009 [7] This paper is first of all devoted to the counterpart of (1.6) for the Chlodovsky polynomials, namely the Voronovskaya-type theorem for $(C_n f)'(x)$. The Theorem states that:

For a function f, defined on $[0, \infty)$

$$\lim_{n\to\infty} \frac{n}{b_n}[(C_n f)'(x) - f'(x)] = \frac{f''(x) + xf'''(x)}{2} \tag{2.3}$$

at each fixed point $x \geq 0$ for which $f'''(x)$ exists, provided that the growth condition (1.3) is satisfied.

The second aim of this paper is to study Voronovskaya-type theorems for the derivatives of this operator and to compare the effectiveness of the Szász-Mirakyan operator with the Bernstein-Chlodovsky polynomials in general.

The only way to fully match the assertion of (2.3) is to work with the Szász-Chlodovsky operator

$$\exp\left(-\frac{nx}{b_n}\right)\sum_{k=0}^{\infty} f\left(\frac{kb_n}{n}\right)\left(\frac{nx}{b_n}\right)^k \frac{1}{k!} := (L_n f)(x),$$

defined and studied by Stypinski [28].

In the same paper, given a function f locally integrable on the interval $[0,\infty)$ we define the Kantorovich variant of the Chlodovsky-Bernstein polynomials as

$$(K_n f)(x) := \frac{n+1}{b_{n+1}}\sum_{k=0}^{n} p_{k,n}\left(\frac{x}{b_{n+1}}\right)\int_{\frac{kb_{n+1}}{n+1}}^{\frac{(k+1)b_{n+1}}{n+1}} f(u)du \quad \text{if } 0 \le x \le b_{n+1}, \tag{2.4}$$

where (b_n) satisfies conditions (1.2).

If F denotes the indefinite integral of f, i.e., $F(x) = \int_0^x f(u)du$, then we have $(C_{n+1}F)'(x) = (K_n f)(x)$ for almost all $x \in [0, b_{n+1}]$, in particular for every $x \in [0, b_{n+1}]$ at which f is continuous.

We set

$$M^I(b; f) := \sqrt{\int_0^b |f(u)|^2 \, du}.$$

The following result is a corollary of our Theorem on the Voronovskaya-type theorems for the derivatives of $(C_n f)(x)$.

If one has

$$\lim_{n\to\infty} \frac{n}{\sqrt{b_n}} \exp\left(-\alpha\frac{n}{b_n}\right) M^I(b_n; f) = 0$$

for every $\alpha > 0$, *then*

$$\lim_{n\to\infty} \frac{n+1}{b_{n+1}} \left[(K_n f)(x) - f(x)\right] = \frac{f'(x) + xf''(x)}{2}$$

at each fixed point $x \ge 0$ *for which* $f''(x)$ *exists.*

2009 [26] In this paper we introduce the Bézier variant of the Chlodovsky-Kantorovich operators (2.4) of order $(n-1)$ for $f \in L_{loc}[0,\infty)$ as

$$K_{n-1,\alpha} f(x) := \frac{n}{b_n}\sum_{k=0}^{n-1} Q_{n-1,k}^{(\alpha)}\left(\frac{x}{b_n}\right)\int_{\frac{kb_n}{n}}^{\frac{(k+1)b_n}{n}} f(u)du \quad \text{if } 0 \le x \le b_n, \tag{2.5}$$

where $\alpha > 0$, $Q_{n-1,k}^{(\alpha)}(t) = J_{n-1,k}^{\alpha}(t) - J_{n-1,k+1}^{\alpha}(t)$ and $J_{n-1,k}(t)$ are the Bézier basis functions defined for $t \in [0,1]$ as

$$J_{n-1,k}(t) = \sum_{j=k}^{n-1} p_{j,n-1}(t) \quad \text{if } \; k = 0,1,\ldots,n-1,$$

$J_{n-1,n}(t) = 0$. Clearly, if $\alpha = 1$ then $K_{n-1,\alpha}f$ reduce to operators (2.4) with n replaced by $n-1$. Our paper is concerned with the rate of pointwise convergence of operators (2.5) when $f \in M_{loc}[0,\infty)$,i.e. f is measurable and locally bounded on $[0,\infty)$. By using the Chanturiya modulus of variation we present estimations for the rate of convergence of $K_{n-1,\alpha}f(x)$ at the points x of continuity of f and at the discontinuity points of the first kind of f. We will formulate our results for $K_{n-1,\alpha}f$ with $\alpha > 0$. The corresponding estimations for the Chlodovsky-Kantorovich polynomials $K_{n-1}f$ follow immediately as a special case $\alpha = 1$.

2009 [27] The author estimate the rates of convergence of Chlodovsky-Kantorovich polynomials in classes of locally integrable functions. Namely, *if* $f \in L_{loc}[0,\infty)$ *and if*

$$\lim_{n\to\infty} \int_0^{b_n} |f(u)|\, du \exp(-\sigma \frac{n}{b_n}) = 0 \quad \text{for each } \sigma > 0,$$

then

$$\lim_{n\to\infty} (K_n f)(x) = f(x) \;\; \text{almost everywhere on } [0,\infty),$$

i.e. at every $x > 0$ *at which* $F'(x) = f(x)$.

Some modified Chlodovsky-Kantorovich operators are considered also in [14].

2009 [22] For $f \in X_{loc}[0,\infty)$ and $\alpha \geq 1$, we introduce the Bézier variant of Chlodovsky-Durrmeyer operators $D_{n,\alpha}$ as follows:

$$(D_{n,\alpha}f)(x) = \frac{n+1}{b_n} \sum_{k=0}^{n} Q_{n,k}^{(\alpha)}\left(\frac{x}{b_n}\right) \int_0^{b_n} f(t) p_{k,n}\left(\frac{t}{b_n}\right) dt, \;\; 0 \leq x \leq b_n, \quad (2.6)$$

Obviously, $D_{n,\alpha}$ is a positive linear operator and $D_{n,\alpha}(1,x) = 1$. Particularly, when $\alpha = 1$ the operators (2.6) reduce to the operators (2.1).

The paper is concerned with the rate of pointwise convergence of the operators (2.6) when f belong to $X_{loc}[0,\infty)$. By using the Chanturiya modulus of variation we examine the rate of pointwise convergence of $(D_{n,\alpha}f)(x)$ at the points of continuity and at the discontinuity points of the first kind of f.

It is necessary to point out that in the present paper we extend and improve the earlier result of [13] for Chlodovsky-Durrmeyer operators.

At first, we give the following definition.

Definition. *Let* f *be a bounded function on a compact interval* $I = [a,b]$. *The* *modulus of variation* $\nu_n(f;[a,b])$ *of the function* f *is defined for nonnegative integers* n *as follows:*

$$\nu_0(f;[a,b])=0$$

and for $n\geq 1$

$$\nu_n(f;[a,b])=\sup_{\Pi_n}\sum_{k=0}^{n-1}\left|f(x_{2k+1})-f(x_{2k})\right|,$$

where Π_n *is an arbitrary system of* n *disjoint intervals* (x_{2k},x_{2k+1})*, where* $k=0,1,...,n-1$*, i.e.,* $a\leq x_0<x_1\leq x_2<x_3...\leq x_{2n-2}<x_{2n-1}\leq b$.

If $f\in BV_p(I),\ p\geq 1, i.e.,$ if f is of bounded *pth* power variation on I, then for every $k\in N$,

$$\nu_k(f;I)\leq k^{1-1/p}V_p(f,I),$$

where $V_p(f,I)$ denotes the total *pth* power variation of f on I, defined as the upper bound of the set of numbers $\left(\sum\limits_j\left|f(k_j)-f(l_j)\right|^p\right)^{1/p}$ over all finite systems of non-overlapping intervals $(k_j,l_j)\subset I$. We also consider the class $BV^p_{loc}[0,\infty),\ p\geq 1$, consisting of all functions of bounded pth power variation on every compact interval $I\subset[0,\infty)$.

Theorem 2.1. *Let* $f\in X_{loc}[0,\infty)$ *and assume that the one-sided limits* $f(x+)$, $f(x-)$ *exist at a fixed point* $x\in(0,\infty)$*. Then, for all integers* n *such that* $b_n>2x$ *and* $4b_n\leq n$ *one has*

$$\left|D_{n,\alpha}(f;x)-\frac{f(x+)+\alpha f(x-)}{\alpha+1}\right|\leq 2\nu_1(g_x;H_x(x\sqrt{b_n/n}))$$

$$+\frac{32\alpha}{x^2}\left(x\left(1-\frac{x}{b_n}\right)+\frac{b_n}{n}\right)\left[\sum_{j=1}^{m-1}\frac{\nu_j(g_x;H_x(jx\sqrt{b_n/n}))}{j^3}+\frac{\nu_m(g_x;H_x(x))}{m^2}\right]$$

$$+\frac{2\alpha c_q}{x^{2q}}\mu(b_n;f)\left(\frac{b_n}{n}\right)^q\left(x\left(1-\frac{x}{b_n}\right)+\frac{b_n}{n}\right)^q+\frac{2\alpha\left|f(x+)-f(x-)\right|}{\sqrt{\frac{nx}{b_n}\left(1-\frac{x}{b_n}\right)}},$$

where $m:=[\sqrt{n/b_n}]$, $H_x(u)=[x-u,x+u]$ *for* $0\leq u\leq x$, $\mu(b;f):=\sup\limits_{0\leq t\leq b}\left|f(t)\right|$,

$$g_x(t):=\begin{cases}f(t)-f(x+) & if\ \ t>x,\\ 0 & if\ \ t=x,\\ f(t)-f(x-) & if\ \ 0\leq t<x,\end{cases}$$

q *is an arbitrary positive integer and* c_q *is a positive constant depending only on* q.

From Theorem 2.1 we get

Theorem 2.2. *Let* $f \in BV^p_{loc}[0,\infty)$, $p \geq 1$, *and let* $x \in (0,\infty)$. *Then, for all integers* n *such that* $b_n > 2x$ *and* $4b_n \leq n$ *we have*

$$\left|D_{n,\alpha}(f;x) - \frac{f(x+) + \alpha f(x-)}{\alpha+1}\right| \leq 2V_p(g_x; H_x(x\sqrt{b_n/n}))$$

$$+\frac{2^{7+1/p}\alpha}{x^2 m^{1+1/p}}\left(x\left(1-\frac{x}{b_n}\right)+\frac{b_n}{n}\right)\sum_{k=1}^{(m+1)^2-1}\frac{V_p(g_x; H_x(\frac{x}{\sqrt{k}}))}{\left(\sqrt{k}\right)^{1-1/p}}$$

$$+\frac{2\alpha c_q}{x^{2q}}\mu(b_n;f)\left(\frac{b_n}{n}\right)^q\left(x\left(1-\frac{x}{b_n}\right)+\frac{b_n}{n}\right)^q+\frac{2\alpha\,|f(x+)-f(x-)|}{\sqrt{\frac{nx}{b_n}\left(1-\frac{x}{b_n}\right)}}.$$

So, we get the following approximation theorem.

Corollary 2.3. *Suppose that* $f \in X_{loc}[0,\infty)$ (*in particular,* $f \in BV^p_{loc}[0,\infty)$, $p \geq 1$) *and that there exists a positive integer* q *such that*

$$\lim_{n\to\infty}\left(\frac{b_n}{n}\right)^q \mu(b_n;f) = 0.$$

Then, at every point $x \in (0,\infty)$ *at which the limits* $f(x+)$, $f(x-)$ *exist, we have*

$$\lim_{n\to\infty} D_{n,\alpha}(f;x) = \frac{f(x+)+\alpha f(x-)}{\alpha+1}.$$

Obviously, the above relations hold true for every measurable function f bounded on $[0,\infty)$, in particular for every function f of bounded *pth* power variation ($p \geq 1$) on the whole interval $[0,\infty)$.

2010 [10] In this paper, the author investigates convergence and approximation properties of a Chlodovsky type generalization of Stancu polynomials.

2010 [25] The authors estimate the rates of pointwise approximation of certain King-type positive linear operators for functions with derivative of bounded variation. We also extend our results to the statistical approximation process via the concept of statistical convergence.

2010 [5] In this work, they state a Chlodovsky variant of a multivariate beta operator to be called hereafter the multivariate beta-Chlodovsky operator. They show that the multivariate beta-Chlodovsky operator can preserve properties of a general function of modulus of continuity and also the Lipschitz constant of a Lipschitz continuous function.

2010 [12] Another recent result concerning uniform approximation by the Chlodovsky operators is due to A. Holhoş.

2010 [21] Let $J_{n,k}(t) = \sum_{j=k}^{n} p_{j,n}(t)$, $t \in [0,1]$, be the Bézier basis functions. For $f \in X_{loc}[0,\infty)$ and $\alpha > 0$, the Bézier modification $C_{n,\alpha}f$ of operators (1.1) is defined as

$$C_{n,\alpha}f(x) = \sum_{k=0}^{n} f\left(\frac{kb_n}{n}\right) Q_{n,k}^{(\alpha)}\left(\frac{x}{b_n}\right) \qquad for \quad x \in [0, b_n], \tag{2.7}$$

where $Q_{n,k}^{(\alpha)}(t) = J_{n,k}^{\alpha}(t) - J_{n,k+1}^{\alpha}(t)$ for $t \in [0,1]$ ($J_{n,l}(x) \equiv 0$ if $l > n$).

If $\alpha = 1$, then $C_{n,\alpha}f$ reduce to the operators (1.1).

Recently, Karsli and Ibikli [17],[23] gave some estimates for the rates of convergence of operators (1.1) and (2.7) (with $\alpha \geq 1$) for functions $f \in BV[0,\infty)$. In this paper:

1$-$ we essentially improve those estimates,

2$-$ we extend those results to some wider classes of functions, in particular for classes $BV^p[0,\infty)$ with $p > 1$,

3$-$ we extend them to all parameters $\alpha > 0$.

If $x \in (0,\infty)$, the following intervals $H_x(u) := [x-u, x+u]$ for $0 < u \leq x$ will be used.

Theorem 2.4. *Let $f \in X_{loc}[0,\infty)$ and assume that the one-sided limits $f(x+)$, $f(x-)$ exist at a fixed point $x \in (0,\infty)$. Then, for all integers n such that $b_n > 2x$ and $n/b_n \geq \max\{4, 21/x\}$ we have*

$$\left|C_{n,\alpha}f(x) - \frac{1}{2^\alpha}f(x+) - \left(1 - \frac{1}{2^\alpha}\right)f(x-)\right| \leq 2\nu_1(g_x; H_x(x\sqrt{b_n/n}))$$

$$+\frac{16\lambda_\alpha}{x^2}\left(x\left(1-\frac{x}{b_n}\right)+\frac{b_n}{n}\right)\left[\sum_{j=1}^{m-1}\frac{\nu_j(g_x; H_x(jx\sqrt{b_n/n}))}{j^3} + \frac{\nu_m(g_x; H_x(x))}{m^2}\right]$$

$$+\kappa_\alpha\sqrt{\frac{b_n}{n}}\sqrt{\frac{b_n}{x(b_n-x)}}\left(|f(x+)-f(x-)| + |f(x)-f(x-)|\,e_n\left(\frac{x}{b_n}\right)\right)$$

$$+4\kappa_\alpha M(b_n; f)\exp\left(-\rho_\alpha\frac{nx}{4b_n}\right),$$

where $m := [\sqrt{n/b_n}]$, $\kappa_\alpha = \max\{1,\alpha\}$, $\rho_\alpha = \min\{1,\alpha\}$, λ_α is a positive constant depending only on α (if $\alpha \geq 1$ then $\lambda_\alpha = \alpha$), $e_n(x/b_n) = 1$ if there exists a $k' \in \{0,1,...,n\}$ such that $nx = k'b_n$, $e_n(x/b_n) = 0$ otherwise, $M(b;f) := \sup_{0\leq t\leq b}|f(t)|$.

Here we note that, under the Chlodovsky condition (1.3), Theorem 2.4 is also an approximation theorem. To see this we must verify that the right-hand side of the inequality given in this theorem converges to zero as $n \to \infty$. In view of (1.2) we have $b_n/n \to 0$ and $m = \left[\sqrt{n/b_n}\right] \to \infty$ as $n \to \infty$. Clearly,

$$\frac{\nu_m(g_x; H_x(x))}{m^2} \leq \frac{2}{m} M(2x; f) \to 0 \quad as \quad n \to \infty.$$

Therefore it is enough to consider only the term

$$\Lambda_m(x) := \sum_{j=1}^{m-1} \frac{\nu_j(g_x; H_x(jxd_n))}{j^3} \qquad where \qquad d_n = \sqrt{b_n/n}.$$

It is easy to see that

$$\begin{aligned} \Lambda_m(x) \quad &\leq \quad \sum_{j=1}^{m-1} \frac{\nu_1(g_x; H_x(jxd_n))}{j^2} \leq 4d_n \int_{d_n}^{md_n} \frac{\nu_1(g_x; H_x(xt))}{t^2} dt \\ &\leq \quad 4d_n \int_1^{m+1} \nu_1\left(g_x; H_x\left(\frac{x}{s}\right)\right) ds \leq \frac{4}{m} \sum_{k=1}^{m} \nu_1\left(g_x; H_x\left(\frac{x}{k}\right)\right). \end{aligned}$$

Since the function g_x is continuous at x and $\nu_1(g_x; H_x(x/k))$ denotes the oscillation of g_x on the interval $[x - x/k, x + x/k]$, we have $\lim_{k\to\infty} \nu_1\left(g_x; H_x\left(x/k\right)\right) = 0$. Consequently $\lim_{m\to\infty} \Lambda_m(x) = 0$, by the well-known theorem on the limit of the sequence of arithmetic means. Hence we get the following

Corollary 2.5. *Suppose that $f \in X_{loc}[0, \infty)$ and that the Chlodovsky condition (1.3) is satisfied. Then*

$$\lim_{n\to\infty} C_{n,\alpha} f(x) = \frac{1}{2^\alpha} f(x+) + \left(1 - \frac{1}{2^\alpha}\right) f(x-) \tag{2.8}$$

at every point $x \in (0, \infty)$ at which the limits $f(x+)$, $f(x-)$ exist.

Of course, relation (2.8) holds true for every function f bounded on the interval $[0, \infty)$. In particular, if $\alpha = 1$ and x is the point of continuity of f, our Corollary 2.5 coincides with the above mentioned theorem of Chlodovsky.

Retaining the symbols used in Theorem 2.4 we also get

Theorem 2.6. *Let $f \in BV^p_{loc}[0, \infty)$, $p \geq 1$, and let $x \in (0, \infty)$. Then*

$$\left| C_{n,\alpha} f(x) - \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right| \leq 2V_p(g_x; H_x(x\sqrt{b_n/n}))$$

$$+\frac{2^{6+1/p}\lambda_\alpha}{x^2 m^{1+1/p}}\left(x\left(1-\frac{x}{b_n}\right)+\frac{b_n}{n}\right)\sum_{k=1}^{(m+1)^2-1}\frac{V_p\left(g_x;H_x\left(\frac{x}{\sqrt{k}}\right)\right)}{\left(\sqrt{k}\right)^{1-1/p}}$$

$$+\kappa_\alpha\sqrt{\frac{b_n}{n}}\sqrt{\frac{b_n}{x\left(b_n-x\right)}}\left(\left|f(x+)-f(x-)\right|+\left|f(x)-f(x-)\right|e_n\left(\frac{x}{b_n}\right)\right)$$

$$+4\kappa_\alpha M(b_n;f)\exp\left(-\rho_\alpha\frac{nx}{4b_n}\right),$$

for all integers n such that $b_n > 2x$ and $n/b_n \geq \max\{4, 21/x\}$.

It is easy to verify that, in view of continuity of g_x at x,

$$\lim_{m\to\infty}\frac{1}{m^{1+1/p}}\sum_{k=1}^{(m+1)^2-1}\frac{1}{\left(\sqrt{k}\right)^{1-1/p}}V_p\left(g_x;H_x\left(\frac{x}{\sqrt{k}}\right)\right)=0.$$

Hence from Theorem 2.6 we have

Corollary 2.7. *If f belongs to the class $BV^p_{loc}[0,\infty)$, $p \geq 1$, and if it satisfies condition (1.3), the relation (2.8) holds true at every $x \in (0,\infty)$. In particular, (2.8) remains valid for every function f of class $BV^p[0,\infty)$, $p \geq 1$.*

Corollary 2.8. *Let us consider now the special case $p = 1$, $\alpha \geq 1$, and let us suppose that $f \in BV[0,\infty)$. Then at every $x > 0$ and for all integers n such that $b_n > 2x$ and $n/b_n \geq 4$, we have*

$$\left|C_{n,\alpha}f(x)-\frac{1}{2^\alpha}f(x+)-\left(1-\frac{1}{2^\alpha}\right)f(x-)\right|\leq 2V(g_x;H_x(x\sqrt{b_n/n}))$$

$$+\frac{2^9\alpha b_n}{n}\left(\frac{1}{x}-\frac{1}{b_n}\right)\sum_{k=1}^{2[n/b_n]}V\left(g_x;H_x\left(\frac{x}{\sqrt{k}}\right)\right)+4\alpha M\exp\left(-\frac{nx}{4b_n}\right)$$

$$+\alpha\sqrt{\frac{b_n}{n}}\sqrt{\frac{b_n}{x\left(b_n-x\right)}}\left(\left|f(x+)-f(x-)\right|+\left|f(x)-f(x-)\right|e_n\left(\frac{x}{b_n}\right)\right),$$

where $M = \sup_{0\leq x<\infty}|f(x)|$ and $V(g_x;H)$ denotes the Jordan variation of g_x on the interval H.

The above estimate is essentially better than the estimates presented in [17] ($\alpha = 1$) and [23] ($\alpha \geq 1$).

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