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Stochastic simulation of the gradient process in semi-discrete approximations of diffusion problems

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Abstract. We analyze a stochastic version of the so-called diffusion-velocity method. For moving particles with velocities depending on the gradient of their density function we introduce a stochastic scheme based on the simulation of the gradient process where the values of the density are recovered by a numerical integration method. We apply this method to the diffusion equation and show a convergence result.

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1. Introduction

Numerical particle methods are suited to approximate the time evolution of a density function by simulating trajectories of the (interacting) particles. We assume that the velocity vector depends (linearly or nonlinearly) on the gradient of this density function. For example in the so-called *deterministic diffusion* of particles one assigns to every particle a velocity vector which is proportional to the logarithmic derivative of the density function. This principle can be derived as follows: consider in a volume element $U \subset \mathbb{R}^d$ the *continuity equation*: $u_t = -\nabla \cdot (uv)$, which describes the motion with velocity v of a quantity with density function u. Pure formally, if we put $v = -\frac{\nabla u}{u}$ we obtain nothing else than the diffusion equation: $u_t = \Delta u$. This type of velocity is often called in the physical literature as *osmotic velocity.* A discussion of this deterministic particle method together with several applications is presented in [4] and [5] and a rigorous convergence result is proved in [3].

Our goal is to construct an analogous scheme in a stochastic framework. The main motivation is that this scheme may be used in combination with usual Monte Carlo methods for kinetic equations in a spatially inhomogeneous setting, with spatial motion and local interaction.

After moving a particle from a location to another the density configuration changes, so one has also to update the corresponding values of the gradient. In a stochastic framework for particle simulation, the use of standard discretization schemes for computing the gradient can lead to strong oscillations in the density values. In order to avoid this problem we consider the simulation of the gradient process. Given an initial data in terms of the density, one computes the gradient by a usual discretization scheme (e.g. finite differences). The scheme follows then only the dynamics of the gradient process which are derived from the original dynamics of the particle system. The density is recovered by a numerical integration method.

In this paper we present an application of the principle described above to the diffusion equation. By this example we intend also to develop a formalism and a methodology which can be applied in more general cases.

In Section 2 introduce the stochastic counterpart of the diffusion-velocity method. We point out that this direct approach in the stochastic framework leads to strong fluctuations of the density profile which we want to approximate. In order to overcome this problem we introduce in Section 3 a stochastic scheme for the gradient process. Convergence results for the density computed by numerical integration are presented in Section 4.

2. The diffusion-velocity method in a stochastic framework

In this section we will present an approach to approximate the diffusion equation in the one-dimensional case, which can be extended easily to higher dimensions. The goal is to approximate on the interval $(0,1)$ the solution of:

$$u_t = u_{xx}, \text{ with the boundary condition: } \begin{cases} u(0) = u(1) = 0 \ \ (D) \\ \text{or} \\ u_x(0) = u_x(1) = 0 \ \ (N) \end{cases} \tag{2.1}$$

and initial condition $u(0,x) = u_0(x) \in H^1(0,1)$ for all $x \in [0,1]$.

In this section we will assume that $u_0 \geq 0$. Let M be an integer, denote $\varepsilon = M^{-1}$ and consider the discrete set of sites $G_\varepsilon = \{k\varepsilon, k = \overline{1, M-1}\}$. Assume that we have N particles distributed in the locations of G_ε, and denote by $n^k(t)$ the number of particles present at the moment t in the location $k\varepsilon$. We introduce the scaling parameter $h = M/N = \varepsilon^{-1}N^{-1}$, which means that h^{-1} is the average number of particles per site. The density function corresponding to this particle system is defined in the points $k\varepsilon$ of the discretization grid by $u^k(t) = hn^k(t)$. In analogy to the formula $v = -\frac{\nabla u}{u}$ we assign to the particles located at $k\varepsilon$, $k = \overline{1, M-1}$, the velocity

$$v^k(t) = \frac{u^{k-1}(t) - u^{k+1}(t)}{2\varepsilon u^k(t)}.$$

Sometimes we will consider formal function values at the *boundary sites* 0 and $M\varepsilon = 1$ (which contain no particles), in order to model the boundary conditions. These values influence the transitions in the sites ε and $(M-1)\varepsilon$.

Every particle situated at $k\varepsilon$ can jump one site to the left or to the right (depending on the sign of the velocity). Our interest is however to follow the time evolution of the density function and from this viewpoint all particles present in the same location are indistinguishable. That is, if any particle from the site $k\varepsilon$ jumps with the rate $Mv^k = \varepsilon^{-1}|v^k|$, the density function in the new state will be the same, independently on which particle jumped. We can thus consider a single transition of this type and multiply the rate with $n^k(t)$, i.e. with the number of particles present at time t at the site $k\varepsilon$.

2.1. Construction of the Markov jump process

Based on the previous considerations, we will construct two $\mathbb{R}^{M-1}$-valued Markov jump processes as follows. Given the time moment t and the state $u(t) = (u^k(t))_{k=1}^{M-1}$, we define

$$w^k(t) = hn^k(t) \cdot v^k(t) = \frac{u^{k-1}(t) - u^{k+1}(t)}{2\varepsilon} =: -\nabla_\varepsilon u^k(t) \tag{2.2}$$

for $k = \overline{1, M-1}$.

The transitions in the interior sites are given by:

$$u(t) \to u(t) - he_k + he_{k+\zeta(w^k)} \text{ at rate } h^{-1}\varepsilon^{-1}|w^k(t)| \tag{2.3}$$

where e_i denotes the i-th unit vector in $\mathbb{R}^{M-1}$, while $\zeta(\cdot)$ denotes the *signum* function. This corresponds to the jump of a particle from the site k in the site $k + \zeta(w^k)$, for $k = \overline{1, M-1}$.

The quantity $w^k(t)$ defined in (2.2) represents the discrete derivative of the density function $u^k(t)$ and the transition (2.3) changes this function in the locations $k, k+\zeta(w^k), k+2\zeta(w^k), k-\zeta(w^k)$ as follows:

$$\begin{aligned}
w^k &\to -\zeta(w^k)\frac{u^{k+\zeta(w^k)} + h - u^{k-\zeta(w^k)}}{2\varepsilon} = w^k - \frac{h}{2\varepsilon}\zeta(w^k) \\
w^{k+\zeta(w^k)} &\to -\zeta(w^k)\frac{u^{k+2\zeta(w^k)} - u^k + h}{2\varepsilon} = w^{k+\zeta(w^k)} - \frac{h}{2\varepsilon}\zeta(w^k) \\
w^{k+2\zeta(w^k)} &\to -\zeta(w^k)\frac{u^{k+3\zeta(w^k)} - u^{k+\zeta(w^k)} - h}{2\varepsilon} = w^{k+2\zeta(w^k)} + \frac{h}{2\varepsilon}\zeta(w^k) \\
w^{k-\zeta(w^k)} &\to -\zeta(w^k)\frac{u^k - h - u^{k-2\zeta(w^k)}}{2\varepsilon} = w^{k-\zeta(w^k)} + \frac{h}{2\varepsilon}\zeta(w^k)
\end{aligned} \tag{2.4}$$

at rate $h^{-1}\varepsilon^{-1}|w^k(t)|$.

We will discuss next the **situation at the boundary**.

In the case of *zero boundary conditions* (D), we consider formally $u^i(t) = 0$ for all $i \notin \{1, \ldots, M-1\}$ in all expressions from (2.3). This value does not change after any possible transition, that is, the particle which leaves the interior of the domain is 'killed'. Outside the range $\overline{1, M-1}$ the function w is not defined. Consider $w^1(t) = -\frac{1}{2\varepsilon}u^2(t)$ and $w^{M-1}(t) = \frac{1}{2\varepsilon}u^{M-2}(t)$. Note that always holds $w^1 \leq 0$ and $w^{M-1} \geq 0$. This implies that in the case $k = 1$ or $k = M-1$, the changes of $w^{k-\zeta(w^k)}$ are well defined in (2.4), while $w^{k+\zeta(w^k)}$ and $w^{k+2\zeta(w^k)}$ are not present, the indices being out of range.

In the case of *Neumann boundary conditions* (N) we take

$$w^1(t) = w^{M-1}(t) = 0.$$

2.2. Remarks

As the following considerations will show, the density function u may exhibit strong oscillations. Suppose that we have an even number of interior locations $\{1, \dots, 2M\}$, while the sites 0 and $2M+1$ correspond to the boundary. Expressing u in terms of w in the case of 0 boundary conditions we obtain

$$u^{2k} - u^{2k+1} = -2\varepsilon \left(\sum_{i=1}^{k} w^{2i-1} + \sum_{i=k+1}^{M} w^{2i} \right). \tag{2.5}$$

The value of the difference in (2.5) approaches the integral $-\int_0^1 w dx$. Since the expected limits satisfy $w = -u_x$, the difference will be close to $u(1) - u(0)$, up to a factor of $O(\varepsilon)$. The same statement holds for Neumann boundary conditions, where the computations are similar, but the expressions for the values of u in the sites $\overline{2, 2M-1}$ involve also the values of u^1 and u^{2M} (which cannot be computed directly from w). In this situation, especially in the case of asymmetric initial data, the value of the difference can be of $O(1)$. If the integral is nonzero, the difference $u^{2k} - u^{2k+1}$ will have basically a constant sign, which means that one can observe a strongly oscillating pattern. Only in the case that the integral vanishes (in our setting only for zero boundary conditions or symmetric data) the oscillations have a smaller amplitude. In this case, after some elementary computations, the difference (2.5) can be estimated by $\varepsilon(\|w\|_\infty + \frac{1}{2}\|w'\|_\infty) + O(\varepsilon^2)$.

3. The particle scheme for the gradient process

Based on the previous considerations, we will present next a particle scheme for the one-dimensional diffusion equation. We consider a discretized version of the initial condition u_0 of the equation (2.1), from which we derive the values of $w(0)$ according to (2.2) and the settings at the boundary. In the interior of the domain we simulate the time evolution of the gradient process w according to (2.4). The state changes of the process which affect the values of w at the 'near boundary' sites are chosen such that for given ε, the deterministic difference equation obtained in the limit proves to be consistent with the corresponding diffusion equation for $w = u_x$, where u is a sufficiently smooth solution of (2.1).

For *zero boundary conditions* (D) we construct the dynamics at the 'near-boundary' sites ε and $(M-1)\varepsilon$ according to the following natural conservation principle. Since we expect $w = -u_x$ and thus

$$\int_0^1 w(x)dx = u(0) - u(1) = 0,$$

we impose that $\sum_{i=1}^{M-1} w^i(t) = 0$ for all t. That is, the total sum of the changes after each transition should vanish. In the interior this condition is fulfilled in all situations, as one can see from (2.4).

In order to construct the approximate solution for equation (2.1) we have to perform a numerical integration. An accurate result is delivered for example by the computation of

$$u(t,x) := \frac{1}{2}\left(-\int_0^x w(t,y)dy + \int_x^1 w(t,y)dy + u(t,0) + u(t,1)\right) \qquad (3.1)$$

with the trapezoidal rule.

For *zero boundary conditions* (D) we can compute u by knowing only the gradient process w, since the density function vanishes at the boundary. In order to perform the integration, we need the values of w in the boundary sites 0 and $M\varepsilon = 1$. For zero boundary conditions we have (formally): $w_x(t,0) \approx -u_{xx}(t,0) \approx -\frac{d}{dt}u(t,0) = 0$. We take thus $w(0) = w(\varepsilon)$ and $w(1) = w((M-1)\varepsilon)$.

In the case of *Neumann boundary conditions* (N) we do not know the values $u(t,0)$ and $u(t,1)$ (except if we simulate also the density process). But we can recover u by using the conservation property $\int_0^1 u(t,x)dx = \int_0^1 u_0(x)dx$. If we let $f(t,x) = -\int_0^x w(t,y)dy + \int_x^1 w(t,y)dy$, we then have

$$u(t,x) := \frac{1}{2}\left(f(t,x) - \int_0^1 f(t,y)dy\right) + \int_0^1 u_0(x)dx.$$

By a discrete version of the above formula (computed by the trapezoidal rule) we can recover the desired approximation for the solution of (2.1) by knowing only the initial data and the time evolution of the gradient process w.

3.1. Dynamics in terms of the infinitesimal generator

We will express the dynamics of the Markov process w given by the transitions (2.4) in terms of its generator, by using the characterization from [1], p.162 f. If we have an E-valued Markov jump process with a set of transitions $\{x(\cdot) \to y(\cdot)\}$ and the corresponding rates $r_{x\to y}$, the *waiting time parameter function* $\lambda(t) = \sum r_{x\to y}$ is given by the sum of all possible transition rates. The *infinitesimal generator* Λ is an operator acting on the bounded, measurable functions on E and is given by $(\Lambda f)(x) = \sum_{x\to y}(f(y) - f(x))r_{x\to y}$.

We note that for fixed M and N the process w has bounded components, i.e. there exists a constant $L_{M,N}$ such that $max|w^k| \leq L_{M,N}$ for all times. The waiting time parameter function λ is also bounded, which implies that the process is well-defined for all t, i.e. the jumps do not accumulate.

For a vector $w \in \mathbb{R}^{M-1}$ and $3 \leq k \leq M-3$ denote $\eta_k^w := e_{k-\zeta(w^k)} - e_k - e_{k+\zeta(w^k)} + e_{k+2\zeta(w^k)}$. In order to define the values in the sites near the boundary, we take into account the requirements of conservation and consistency.

In the case of *zero boundary condition* (D) we define:

$$\eta_1^w = \begin{cases} -e_1 + e_2 & \text{if } w^1 < 0 \\ -e_2 + e_3 & \text{if } w^1 > 0 \end{cases} \tag{3.2}$$

$$\eta_2^w = \begin{cases} -e_2 + e_3 & \text{if } w^2 < 0 \\ e_1 - e_2 - e_3 + e_4 & \text{if } w^2 > 0. \end{cases}$$

At the other end of the interval we define similarly η_{M-i}^w for $i = 1, 2$ by replacing w^i with $-w^{M-i}$ and e_j with e_{M-j} for $j = 1 \dots 4$.

For *Neumann boundary condition* (N) we consider $\eta_1^w = \eta_{M-1}^w = 0$ and for $k = 2, 3$ (or $k = M-3, M-2$) we suppress the term e_1 (respectively e_{M-1}) if it appears in the formula $\eta_k^w = e_{k-\zeta(w^k)} - e_k - e_{k+\zeta(w^k)} + e_{k+2\zeta(w^k)}$.

Taking in account (2.4), the transitions of the process w are therefore given by

$$w \longrightarrow w + \zeta(w^k)\frac{h}{2\varepsilon}\eta_k^w$$

at rate $h^{-1}\varepsilon^{-1}|w^k(t)|$.

Define the linear operator $\Delta_\varepsilon^{\zeta(w)}\phi$ by:

$$(\Delta_\varepsilon^{\zeta(w)}\phi)^k := \frac{1}{2\varepsilon^2}\langle \eta_k^w, \phi\rangle \tag{3.3}$$

for all $\phi \in \mathbb{R}^{M-1}$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product on $\mathbb{R}^{M-1}$.

For $3 \le k \le M-3$ we thus have

$$(\Delta_\varepsilon^{\zeta(w)}\phi)^k = \frac{1}{2\varepsilon^2}[\phi^{k-\zeta(w^k)} - \phi^k - \phi^{k+\zeta(w^k)} + \phi^{k+2\zeta(w^k)}]. \tag{3.4}$$

For a fixed element $\phi \in \mathbb{R}^{M-1}$ consider on $\mathbb{R}^{M-1}$ a bounded smooth function f_ϕ which on the set $\{x : max|x^k| \le L_{M,N}\}$ has the form $f_\phi(x) = \langle x, \phi\rangle = \sum_{i=1}^{M-1} x^i\phi^i$. Outside this set the values of the function are in our case not of interest, only the boundedness is essential. From [1], p.162 we have that the process w satisfies the identity

$$f_\phi(w(t)) = f_\phi(w(0)) + \int_0^t (\Lambda^w f_\phi)(w(s))ds + M_\phi(t) \tag{3.5}$$

where $M_\phi(\cdot)$ is a martingale with respect to the filtration generated by the process w and Λ^w is the infinitesimal generator. The value $\Lambda^w f_\phi$ is given by

$$\begin{aligned} (\Lambda^w f_\phi)(w(t)) &= \frac{h}{2\varepsilon}\sum_k \langle \zeta(w^k)\eta_k^w, \phi\rangle h^{-1}\varepsilon^{-1}|w^k(t)| \\ &= \frac{1}{2\varepsilon^2}\sum_k [\phi^{k-\zeta(w^k)} - \phi^k - \phi^{k+\zeta(w^k)} + \phi^{k+2\zeta(w^k)}]w^k(t) \\ &= \langle \Delta_\varepsilon^{\zeta(w)}\phi, w(t)\rangle. \end{aligned} \tag{3.6}$$

Equation (3.5) becomes thus

$$\langle w(t), \phi\rangle = \langle w(0), \phi\rangle + \int_0^t \langle \Delta_\varepsilon^{\zeta(w)}\phi, w(s)\rangle ds + M_\phi^w(t). \tag{3.7}$$

3.2. The deterministic scheme as limit of the family of stochastic processes

By standard techniques as in [1] or [2] one can show that for fixed ε the stochastic processes converge in probability for $N \to \infty$ to the solution of the ODE-system obtained by suppressing the martingale term in (3.7). We may note that the stochastic method proposed here delivers for fixed ε an approximation of the solution of the ODE-system by computing all transitions of the stochastic process at the 'microscopic level'. However, this ODE-system is not meant to be approximated by a deterministic time-discretization scheme, but its solution is approximated directly by the stochastic simulations. The convergence for $\varepsilon \to 0$ of the difference scheme provided by the spatially discretized system to the solution of the corresponding spatially continuous diffusion equation will be analyzed subsequently. The system of ODE's which is obtained for $N \to \infty$ and fixed ε is therefore given by:

$$\langle v_\varepsilon(t), \phi\rangle = \langle v_\varepsilon(0), \phi\rangle + \int_0^t \langle \Delta_\varepsilon^{\zeta(v_\varepsilon)}\phi, v_\varepsilon(s)\rangle ds \tag{3.8}$$

for all $\phi \in \mathbb{R}^{M-1}$, where for any $w \in \mathbb{R}^{M-1}$, $\Delta_\varepsilon^{\zeta(w)}\phi$ was defined in (3.3) as

$$(\Delta_\varepsilon^{\zeta(w)}\phi)^k := \frac{1}{2\varepsilon^2}\langle \eta_k^w, \phi\rangle.$$

The vectors η_k^w can be written in the more convenient form:

$$\eta_k^w = -e_{k-2}\zeta(w^k \wedge 0) + e_{k-1}\zeta(w^k) - e_k - e_{k+1}\zeta(w^k) + e_{k+2}\zeta(w^k \vee 0) \tag{3.9}$$

for $3 \le k \le M-3$ and, for zero boundary conditions,

$$\begin{aligned} \eta_2^w &= e_1\zeta(w^2 \vee 0) - e_2 - e_3\zeta(w^2) + e_4\zeta(w^2 \vee 0) \\ \eta_1^w &= e_1\zeta(w^1 \wedge 0) - e_2\zeta(w^1) + e_3\zeta(w^1 \vee 0). \end{aligned}$$

By $x \wedge 0$, $x \vee 0$ we denote respectively the minimum between x and 0 and the maximum between x and 0. The terms $\eta_{M-1}^w, \eta_{M-2}^w$ are defined analogous to η_1^w and η_2^w. With this form we obtain:

$$\begin{aligned} (\Delta_\varepsilon^{\zeta(w)}\phi)^k = \frac{1}{2\varepsilon^2}\big[&-\phi^{k-2}\zeta(w^k \wedge 0) + \phi^{k-1}\zeta(w^k) - \phi^k - \phi^{k+1}\zeta(w^k) \\ &+ \phi^{k+2}\zeta(w^k \vee 0)\big] \end{aligned}$$

for $3 \le k \le M-3$ and

$$(\Delta_\varepsilon^{\zeta(w)}\phi)^2 = \frac{1}{2\varepsilon^2}\left[\phi^1\zeta(w^2 \vee 0) - \phi^2 - \phi^3\zeta(w^2) + \phi^4\zeta(w^2 \vee 0)\right]$$

$$(\Delta_\varepsilon^{\zeta(w)}\phi)^1 = \frac{1}{2\varepsilon^2}\left[\phi^1\zeta(w^1 \wedge 0) - \phi^2\zeta(w^1) + \phi^3\zeta(w^1 \vee 0)\right].$$

An explicit form of the deterministic equations is given by letting $\phi = e_i$ in (3.8), for $i = \overline{1, M-1}$. We obtain then the system

$$\begin{aligned} v_\varepsilon^i(t) &= v_\varepsilon^i(0) + \int_0^t \langle \Delta_\varepsilon^{\zeta(v)} e_i, v_\varepsilon(s)\rangle ds \\ &= v_\varepsilon^i(0) + \int_0^t F_\varepsilon^i(v_\varepsilon(s)) ds \end{aligned} \tag{3.10}$$

for $i = \overline{1, M-1}$, where for $3 \le i \le M-3$ we have the explicit form:

$$F_\varepsilon^i(v_\varepsilon) = \frac{1}{2\varepsilon^2}\left[v_\varepsilon^{i-2} \vee 0 - |v_\varepsilon^{i-1}| - v_\varepsilon^i + |v_\varepsilon^{i+1}| + v_\varepsilon^{i+2} \wedge 0\right].$$

The terms corresponding to the sites near the boundary are computed similarly by using the corresponding values of η_k^w.

4. Convergence results

In this section we analyze the approximation properties of the diffusion equation in the case of *zero boundary conditions* (D) by the ODE system (3.10).

Set $u_\varepsilon^0 = u_\varepsilon^M = 0$ and for $i \in \{1, \dots, M-1\}$ define

$$u_\varepsilon^i(t) = -\varepsilon \sum_{k=1}^{i} v_\varepsilon^k(t). \tag{4.1}$$

Note that this corresponds to a discrete integration scheme for computing $u(t,x) = -\int_0^x v(t,x)dx$. For the sake of computations we will treat the theoretical estimates with this construction, but in practice we will use a scheme for computing the integrals in (3.1), that is we integrate in both directions.

Define the piecewise linear function $u_\varepsilon(t,\cdot) : [0,1] \to \mathbb{R}$ by:

$$u_\varepsilon(t,x) := -v_\varepsilon^{i+1}(t)(x - i\varepsilon) + u_\varepsilon^i(t) \tag{4.2}$$

for $x \in [i\varepsilon, (i+1)\varepsilon]$. We take $u_\varepsilon^0 = u_\varepsilon^M = v_\varepsilon^M = 0$. This is the linear interpolant between the values u_ε^i at the sites $i\varepsilon$. Let us first show some properties of the solutions v_ε of (3.10) and of u_ε defined in (4.1).

Lemma 4.1. *(i) We have for all $T > 0$:*

$$\sup_{t\in[0,T]} \sum_{i=1}^{M-1} (u_\varepsilon^i(t))^2 \le \sum_{i=1}^{M-1} (u_\varepsilon^i(0))^2.$$

(ii) Suppose $v_\varepsilon(t)$ has the following properties for all $t \in [0,T]$: $v_\varepsilon^1(t) \le 0$, $v_\varepsilon^{M-1}(t) \ge 0$ and if $v_\varepsilon^i(t)\cdot v_\varepsilon^{i+2}(t) \ge 0$ then we have also $v_\varepsilon^i(t)\cdot v_\varepsilon^{i+1}(t) \ge 0$. Under these assumptions we have:

$$\sup_{t\in[0,T]} \sum_{i=1}^{M-1} (v_\varepsilon^i(t))^2 \le \sum_{i=1}^{M-1} (v_\varepsilon^i(0))^2.$$

Proof. We recall the form (3.8) of the deterministic equation system:

$$\langle \frac{d}{dt} v_\varepsilon(t), \phi\rangle = \langle \Delta_\varepsilon^{\zeta(v_\varepsilon)}\phi, v_\varepsilon(t)\rangle \tag{4.3}$$

which holds for all vectors $\phi = (\phi^i)_{i=1}^{M-1} \in \mathbb{R}^{M-1}$. In order to derive a similar equation for u_ε, we take as test vectors $\phi = -\varepsilon\phi_{(i)} := -\varepsilon(\phi^i, \phi^i, \dots \phi^i, 0, \dots 0)$ where the first i components are equal to ϕ^i and the rest are 0. We obtain then:

$$\frac{d}{dt} u_\varepsilon^i(t)\phi^i = -\varepsilon\langle \Delta_\varepsilon^{\zeta(v_\varepsilon)}\phi_{(i)}, v_\varepsilon(t)\rangle. \tag{4.4}$$

In particular, if we take $\phi = -\varepsilon\theta_{(i)} := -\varepsilon(1,1,\dots 1,0,\dots 0)$ (the first i components are equal to 1) we obtain:

$$\frac{d}{dt}u_\varepsilon^i(t) = -\varepsilon\langle \Delta_\varepsilon^{\zeta(v_\varepsilon)}\theta_{(i)}, v_\varepsilon(t)\rangle. \tag{4.5}$$

For $3 \le i \le M-3$ we have:

$$\frac{d}{dt}u_\varepsilon^i(t) = -\frac{1}{2\varepsilon}\sum_{k=1}^{M-1}[\theta_{(i)}^{k-\zeta(v_\varepsilon^k)} - \theta_{(i)}^k - \theta_{(i)}^{k+\zeta(v_\varepsilon^k)} + \theta_{(i)}^{k+2\zeta(v_\varepsilon^k)}]v_\varepsilon^k. \tag{4.6}$$

Taking in account the structure of $\theta_{(i)}$ we can note that the terms in the brackets vanish, except in the situation that $i-2 \le k \le i+2$. In this case the values depend on the sign of the corresponding v_ε^k, and we obtain easily:

$$\frac{d}{dt}u_\varepsilon^i(t) = \frac{1}{2\varepsilon}\left[(v_\varepsilon^{i-1}\vee 0) + (v_\varepsilon^i \wedge 0) - (v_\varepsilon^{i+1}\vee 0) - (v_\varepsilon^{i+2}\wedge 0)\right]. \tag{4.7}$$

Let us consider further a general form for ϕ. By summing up the equations (4.4) with respect to i we obtain:

$$\langle \frac{d}{dt}u_\varepsilon(t), \phi\rangle = -\varepsilon\sum_{i=1}^{M-1}\langle \Delta_\varepsilon^{\zeta(v_\varepsilon)}\phi_{(i)}, v_\varepsilon(t)\rangle. \tag{4.8}$$

Let us compute the r.h.s. by rearranging the terms in a convenient form. We have:

$$\sum_{i=1}^{M-1}\langle \Delta_\varepsilon^{\zeta(v_\varepsilon)}\phi_{(i)}, v_\varepsilon(t)\rangle = \sum_{i=1}^{M-1}\sum_{k=1}^{M-1}(\Delta_\varepsilon^{\zeta(v_\varepsilon)}\phi_{(i)})^k\cdot v_\varepsilon^k = \sum_{k=1}^{M-1}v_\varepsilon^k\cdot\sum_{i=1}^{M-1}(\Delta_\varepsilon^{\zeta(v_\varepsilon)}\phi_{(i)})^k.$$

Taking in account formula (3.10) we thus have for $3 \le k \le M-3$:

$$(\Delta_\varepsilon^{\zeta(v_\varepsilon)}\phi_{(i)})^k =$$
$$= \frac{1}{2\varepsilon^2}\left[-\phi_{(i)}^{k-2}\zeta(v_\varepsilon^k\wedge 0) + \phi_{(i)}^{k-1}\zeta(v_\varepsilon^k) - \phi_{(i)}^k - \phi_{(i)}^{k+1}\zeta(v_\varepsilon^k) + \phi_{(i)}^{k+2}\zeta(v_\varepsilon^k\vee 0)\right].$$

Since $\phi_{(i)} = (\phi^i, \phi^i, \dots \phi^i, 0, \dots 0)$, it follows imediately that the expression vanishes for $i < k-2$ and $i \ge k+2$. By analyzing all possibilities with respect to the sign of v_ε^k we obtain for $3 \le k \le M-3$ the expression:

$$\sum_{i=1}^{M-1}(\Delta_\varepsilon^{\zeta(v_\varepsilon)}\phi_{(i)})^k = \frac{1}{2\varepsilon^2}\left[(\phi^{k-1}-\phi^{k+1})\zeta(v_\varepsilon^k\vee 0) + (\phi^k - \phi^{k-2})\zeta(v_\varepsilon^k\wedge 0)\right]. \tag{4.9}$$

In the case of the the terms corresponding to the sites near the boundary we obtain similarly

$$\sum_{i=1}^{M-1}(\Delta_\varepsilon^{\zeta(v_\varepsilon)}\phi_{(i)})^1 = \frac{1}{2\varepsilon^2}[-\phi^2\zeta(v_\varepsilon^1\vee 0) + \phi^1\zeta(v_\varepsilon^1\wedge 0)]$$

and

$$\sum_{i=1}^{M-1}(\Delta_\varepsilon^{\zeta(v_\varepsilon)}\phi_{(i)})^2 = \frac{1}{2\varepsilon^2}[(\phi^1-\phi^3)\zeta(v_\varepsilon^2\vee 0) + \phi^2\zeta(v_\varepsilon^2\wedge 0)].$$

Similar equations are derived at the other end of the interval. We note that these equations can be also reduced to the form (4.9) by setting $\phi^j = 0$ if $j \notin \{1, \dots, M-1\}$.

Since all computations are nothing more than rearrangements of the terms, we can obtain the same results by multiplying the explicit equations with time dependent test functions $\phi(t)$. We can take now $\phi = u_\varepsilon(t)$ and use the corresponding $\phi_{(i)}$. Note that $u_\varepsilon^{j-1} - u_\varepsilon^{j+1} = \varepsilon(v_\varepsilon^{j+1} + v_\varepsilon^j)$ and that we can set $u_\varepsilon^j = v_\varepsilon^j = 0$ if $j \notin \{1, \dots, M-1\}$, due to the considered boundary conditions. We then have:

$$\sum_{k=1}^{M-1} v_\varepsilon^k \sum_{i=1}^{M-1} (\Delta_\varepsilon^{\zeta(v_\varepsilon)} \phi_{(i)})^k =$$

$$= \frac{1}{2\varepsilon} \sum_{k=1}^{M-1} v_\varepsilon^k [(v_\varepsilon^k + v_\varepsilon^{k+1})\zeta(v_\varepsilon^k \vee 0) - (v_\varepsilon^k + v_\varepsilon^{k-1})\zeta(v_\varepsilon^k \wedge 0)]$$

$$= \frac{1}{2\varepsilon} \sum_{k=1}^{M-1} [(v_\varepsilon^k)^2 + v_\varepsilon^k v_\varepsilon^{k+1} \zeta(v_\varepsilon^k \vee 0) - v_\varepsilon^k v_\varepsilon^{k-1} \zeta(v_\varepsilon^k \wedge 0)] =: \frac{1}{2\varepsilon} \sum_{k=1}^{M-1} a_k.$$

The terms a_k have the form:

$$a_k = \begin{cases} (v_\varepsilon^k)^2 + v_\varepsilon^k v_\varepsilon^{k+1} & \text{if } v_\varepsilon^k > 0 \\ (v_\varepsilon^k)^2 + v_\varepsilon^k v_\varepsilon^{k-1} & \text{if } v_\varepsilon^k \le 0. \end{cases}$$

We claim that $\sum_k a_k \ge 0$. In order to show this, we proceed inductively. Denote $S_m = \sum_{k=1}^m a_k$. We have $S_1 = a_1 = (v_\varepsilon^1)^2 + v_\varepsilon^1 v_\varepsilon^2 \zeta(v_\varepsilon^1 \vee 0)$. If $v_\varepsilon^1 \le 0$ then $S_1 = (v_\varepsilon^1)^2 \ge 0$. If $v_\varepsilon^1 \ge 0$ and $v_\varepsilon^2 \le 0$ then $S_2 = (v_\varepsilon^1)^2 + 2v_\varepsilon^1 v_\varepsilon^2 + (v_\varepsilon^2)^2 \ge 0$. If $v_\varepsilon^1 > 0, v_\varepsilon^2 > 0, \dots, v_\varepsilon^{p-1} > 0, v_\varepsilon^p \le 0$, then we have $a_1 + a_2 + \dots + a_p = (v_\varepsilon^1)^2 + v_\varepsilon^1 v_\varepsilon^2 + \dots + (v_\varepsilon^{p-1})^2 + 2v_\varepsilon^p v_\varepsilon^{p-1} + (v_\varepsilon^p)^2 \ge 0$. We have thus ($S_p \ge 0$ and $v_\varepsilon^p \le 0$).

The first step leads thus to a situation on the type ($S_p \ge 0$ and $v_\varepsilon^p \le 0$).

If $p = M-1$ we are done. If not, we repeat the procedure. Suppose that we have shown that ($S_{k-1} \ge 0$ and $v_\varepsilon^{k-1} \le 0$). If $v_\varepsilon^k \le 0$, then we have $a_k = (v_\varepsilon^k)^2 + v_\varepsilon^k v_\varepsilon^{k-1} \ge 0$ and thus ($S_k \ge 0$ and $v_\varepsilon^k \le 0$).

If $v_\varepsilon^k > 0$ and $v_\varepsilon^{k+1} \le 0$, then we have $a_k + a_{k+1} = (v_\varepsilon^k)^2 + 2v_\varepsilon^k v_\varepsilon^{k+1} + (v_\varepsilon^{k+1})^2 \ge 0$. This implies ($S_{k+1} \ge 0$ and $v_\varepsilon^{k+1} \le 0$).

If $v_\varepsilon^k > 0$ and $v_\varepsilon^{k+1} > 0, \dots, v_\varepsilon^{p-1} > 0, v_\varepsilon^p \le 0$, then we have $a_k + a_{k+1} + \dots + a_p = (v_\varepsilon^k)^2 + v_\varepsilon^k v_\varepsilon^{k+1} + \dots + (v_\varepsilon^{p-1})^2 + 2v_\varepsilon^p v_\varepsilon^{p-1} + (v_\varepsilon^p)^2 \ge 0$. We have thus ($S_p \ge 0$ and $v_\varepsilon^p \le 0$).

If starting with some index j we have $v_\varepsilon^k \ge 0$ for $k \ge j$, then we are done, since we add only positive terms a_k for $k \ge j$. In the case of the last term we have then only $(v_\varepsilon^{M-1})^2$. The other alternative is to obtain the situation ($S_{M-1} \ge 0$ and $v_\varepsilon^{M-1} \le 0$), when we are also done.

The fact that for $\phi = u_\varepsilon$ we have $S_{M-1} \ge 0$, together with equation (4.8) imply $\frac{d}{dt}\langle u_\varepsilon(t), u_\varepsilon(t)\rangle = 2\langle \frac{d}{dt} u_\varepsilon(t), u_\varepsilon(t)\rangle = -S_{M-1} \le 0$ which proves the first part of the lemma.

For the second part we take $\phi = v_\varepsilon(t)$ in (4.3) and we have to show that $\langle \Delta_\varepsilon^{\zeta(v_\varepsilon)} v_\varepsilon(t), v_\varepsilon(t)\rangle \le 0$. This can be written also as $\sum_{k=1}^{M-1} T_k \le 0$, where

$$T_k = [-v_\varepsilon^{k-2}\zeta(v_\varepsilon^k \wedge 0) + v_\varepsilon^{k-1}\zeta(v_\varepsilon^k) - v_\varepsilon^k - v_\varepsilon^{k+1}\zeta(v_\varepsilon^k) + v^{k+2}\zeta(v_\varepsilon^k \vee 0)] \cdot v_\varepsilon^k$$

if $3 \le k \le M-3$, and

$$\begin{aligned} T_1 &= [v_\varepsilon^1\zeta(v_\varepsilon^1 \wedge 0) - v_\varepsilon^2\zeta(v_\varepsilon^1) + v_\varepsilon^3\zeta(v_\varepsilon^1 \vee 0)] \cdot v_\varepsilon^1 \\ T_2 &= [v_\varepsilon^1\zeta(v_\varepsilon^2 \vee 0) - v_\varepsilon^2 - v_\varepsilon^3\zeta(v_\varepsilon^2) + v^4\zeta(v_\varepsilon^2 \vee 0)] \cdot v_\varepsilon^2, \end{aligned}$$

while T_{M-1}, T_{M-2} are computed analogous to T_1, T_2. We will structure the proof in an algorithmic fashion.

From the hypothesis we have that $v_\varepsilon^1 \le 0$. Define $T := 0$.

0. If we have $v_\varepsilon^2 \le 0$ let $T := T_1 + T_2$. We have thus

$$T = -(v_\varepsilon^1)^2 + v_\varepsilon^1 v_\varepsilon^2 - (v_\varepsilon^2)^2 + v_\varepsilon^2 v_\varepsilon^3 \le -\frac{1}{2}(v_\varepsilon^2)^2 + v_\varepsilon^2 v_\varepsilon^3.$$

Let $q = 1$ and GOTO **2.**

1. Else, if we have $v_\varepsilon^2 > 0$, then the hypothesis implies that we have also $v_\varepsilon^3 \ge 0$. Let $T := T_1 + T_2$. We have thus

$$T = -(v_\varepsilon^1)^2 + 2v_\varepsilon^1 v_\varepsilon^2 - (v_\varepsilon^2)^2 - v_\varepsilon^2 v_\varepsilon^3 + v_\varepsilon^2 v_\varepsilon^4 \le -(v_\varepsilon^2)^2 + v_\varepsilon^2 v_\varepsilon^4.$$

Let $p = 2$ and GOTO **3.**

2. Suppose we have $v_\varepsilon^q \le 0, v_\varepsilon^{q+1} \le 0, \dots v_\varepsilon^{p-1} \le 0, v_\varepsilon^p > 0$ and

$$T \le -\frac{1}{2}(v_\varepsilon^{q+1})^2 + v_\varepsilon^{q+1} v_\varepsilon^{q+2}.$$

The hypothesis on v_ε implies that we must have also $v_\varepsilon^{p+1} \ge 0$. We observe that for $q+2 \le k \le p-1$ in the sum $T_{k-1} + T_k$ appear the cancelling terms $-v_\varepsilon^k v_\varepsilon^{k-1}\zeta(v_\varepsilon^{k-1}) + v_\varepsilon^k v_\varepsilon^{k-1}\zeta(v_\varepsilon^k)$, since v_ε^{k-1} and v_ε^k have the same sign. If $p = q+2$ we do not have such terms. We thus have:

$$\begin{aligned} \sum_{k=q+2}^{p} T_k &= -\sum_{k=q+2}^{p} (v_\varepsilon^k)^2 + \sum_{k=q+2}^{p-1} v_\varepsilon^k v_\varepsilon^{k-2} + 2v_\varepsilon^{p-1} v_\varepsilon^p - v_\varepsilon^p v_\varepsilon^{p+1} + v_\varepsilon^p v_\varepsilon^{p+2} \\ &\le -\frac{1}{2}(v_\varepsilon^{q+2})^2 \cdot \chi_{\{p>q+2\}} - (v_\varepsilon^p)^2 + v_\varepsilon^p v_\varepsilon^{p+2}. \end{aligned}$$

We grouped the terms in order to obtain nonpositive quantities like $[(-v_\varepsilon^k)^2 + 2v_\varepsilon^k v_\varepsilon^{k-2} - (v_\varepsilon^{k-2})^2]/2$, together with $2v_\varepsilon^{p-1} v_\varepsilon^p$ and $-v_\varepsilon^p v_\varepsilon^{p+1}$ which are also ≤ 0.

Let $T := T + \sum_{k=q+2}^{p} T_k$. If $p = q+2$ we obtain:

$$T \le -\frac{1}{2}(v_\varepsilon^{q+1})^2 + v_\varepsilon^{q+1} v_\varepsilon^{q+2} - (v_\varepsilon^{q+2})^2 + v_\varepsilon^{q+2} v_\varepsilon^{q+4} \le -(v_\varepsilon^{q+2})^2 + v_\varepsilon^{q+2} v_\varepsilon^{q+4}$$

since $v_\varepsilon^{q+1} v_\varepsilon^{q+2} \le 0$. If $p > q+2$ we have:

$$T \le -\frac{1}{2}(v_\varepsilon^{q+1})^2 + v_\varepsilon^{q+1} v_\varepsilon^{q+2} - \frac{1}{2}(v_\varepsilon^{q+2})^2 - (v_\varepsilon^p)^2 + v_\varepsilon^p v_\varepsilon^{p+2} \le -(v_\varepsilon^p)^2 + v_\varepsilon^p v_\varepsilon^{p+2}.$$

(stopping condition) If $p = M-2$ we are done, since the term $v_\varepsilon^p v_\varepsilon^{p+2}$ does not appear, while T_{M-1} equals $-(v_\varepsilon^{M-1})^2 - v_\varepsilon^{M-1} v_\varepsilon^{M-2}$, which can be grouped together with $-(v_\varepsilon^{M-2})^2$ in order to obtain a nonpositive quantity. If $p = M-1$ we are also done.

3. Suppose we have $v_\varepsilon^p \geq 0, v_\varepsilon^{p+1} \geq 0, \dots v_\varepsilon^{q-1} \geq 0, v_\varepsilon^q \leq 0$ and

$$T \leq -(v_\varepsilon^p)^2 + v_\varepsilon^p v_\varepsilon^{p+2}.$$

The hypothesis implies that we have $v_\varepsilon^{q+1} \leq 0$. Similarly as in **2.** (dropping nonpositive terms which are not needed and using the cancelling property) we compute:

$$\sum_{k=p+1}^{q+1} T_k =$$

$$= -\sum_{k=p+1}^{q+1} (v_\varepsilon^k)^2 + \sum_{k=p+1}^{q-1} v_\varepsilon^k v_\varepsilon^{k+2} + v_\varepsilon^q v_\varepsilon^{q-2} + 2v_\varepsilon^q v_\varepsilon^{q-1} + v_\varepsilon^{q+1} v_\varepsilon^{q-1} + v_\varepsilon^{q+1} v_\varepsilon^{q+2}$$

$$\leq -\frac{1}{2}(v_\varepsilon^{p+2})^2 - \frac{1}{2}(v_\varepsilon^q)^2 - \frac{1}{2}(v_\varepsilon^{q+1})^2 + v_\varepsilon^q v_\varepsilon^{q-2} + v_\varepsilon^{q+1} v_\varepsilon^{q+2}.$$

Let $T := T + \sum_{k=p+1}^{q+1} T_k$. If $q = p+2$ we obtain:

$$T \leq -(v_\varepsilon^p)^2 + v_\varepsilon^p v_\varepsilon^{p+2} - (v_\varepsilon^{p+2})^2 - \frac{1}{2}(v_\varepsilon^{p+3})^2 + v_\varepsilon^p v_\varepsilon^{p+2} + v_\varepsilon^{p+3} v_\varepsilon^{p+4}$$

$$\leq -\frac{1}{2}(v_\varepsilon^{p+3})^2 + v_\varepsilon^{p+3} v_\varepsilon^{p+4},$$

since $v_\varepsilon^p v_\varepsilon^{p+2} \leq 0$.

If $q > p+2$ we have:

$$T \leq -(v_\varepsilon^p)^2 + v_\varepsilon^p v_\varepsilon^{p+2} - \frac{1}{2}(v_\varepsilon^{p+2})^2 - \frac{1}{2}(v_\varepsilon^q)^2 - \frac{1}{2}(v_\varepsilon^{q+1})^2 + v_\varepsilon^q v_\varepsilon^{q-2} + v_\varepsilon^{q+1} v_\varepsilon^{q+2}$$

$$\leq -\frac{1}{2}(v_\varepsilon^{q+1})^2 + v_\varepsilon^{q+1} v_\varepsilon^{q+2},$$

since $v_\varepsilon^q v_\varepsilon^{q-2} \leq 0$. If $q+1 = M-1$ and $v_\varepsilon^{M-1} = 0$ we don't have the term $v_\varepsilon^{q+1} v_\varepsilon^{q+2}$ and we are done. Otherwise, since $v_\varepsilon^{M-1} \geq 0$, we cannot end in the situation $q = M-2$. Thus, GOTO **2.**

The above algorithm clearly stops in step **2.** arriving in the final situation with $T = \sum_{k=1}^{M-1} T_k \leq 0$. The proof is thus completed. □

Remark. The assumptions on v_ε in (ii) hold true, at least on a given time interval, if $v_\varepsilon(0)$ is constructed by taking the finite differences of a positive, piecewise Lipschitz continuous function u_0 and ε is chosen small enough. As it will be shown further, for the convergence of the method we will need bounds for $\sum \varepsilon (v_\varepsilon^k(t))^2$ independent on ε. This condition is not fulfilled if we choose an arbitrary initial data $v_\varepsilon(0)$. Numerical computations show that the sum will blow up in a short time if we take e.g. $v_\varepsilon^k(0) = (-1)^k$ if $k \in \{1, \dots, 2M\} \setminus \{M\}$ and $v_\varepsilon^M(0) = 2(-1)^M$. In general we cannot expect for v_ε a similar inequality as for u_ε in (i), but in most practically relevant situations this property holds true, as shown for example in (ii).

Lemma 4.2. *(Energy estimates). Assume that for all ε we have*

$$\|u_\varepsilon(0,\cdot)\|_{H_0^1(0,1)} \le C_0\|u_0\|_{H_0^1(0,1)}$$

for a given function $u_0 \in H_0^1(0,1)$, with a positive constant C_0. Suppose further that the functions $v_\varepsilon(t)$ satisfy the inequality $\sup_{t\le T}\sum_k (v_\varepsilon^k(t))^2 \le C_1\sum_k (v_\varepsilon^k(0))^2$ with a positive constant C_1, independent on ε (in particular, if the assumption from Lemma 4.1 (ii) holds). Then there exists a constant $C>0$, independent on ε, such that:

$$\sup_{t\in[0,T]}\big[\|u_\varepsilon(t,\cdot)\|_{H_0^1(0,1)} + \|\frac{d}{dt}u_\varepsilon(t,\cdot)\|_{H^{-1}(0,1)}\big] \le C\|u_0\|_{H_0^1(0,1)}.$$

Proof. We have:

$$\begin{aligned}
&\sup_{t\in[0,T]}\big[\int_0^1 u_\varepsilon^2(t,x)dx\big]\\
&= \sup_{t\in[0,T]}\big[\sum_{i=0}^{M-1}\int_{i\varepsilon}^{(i+1)\varepsilon} \big(-v_\varepsilon^{i+1}(t)(x-i\varepsilon)+u_\varepsilon^i(t)\big)^2\,dx\big]\\
&\le 2\sup_{t\in[0,T]}\big[\sum_{i=0}^{M-1}\int_{i\varepsilon}^{(i+1)\varepsilon} \big((v_\varepsilon^{i+1}(t))^2(x-i\varepsilon)^2+(u_\varepsilon^i(t))^2\big)dx\big]\\
&= 2\sup_{t\in[0,T]}\big[\sum_{i=0}^{M-1}\big(\frac{(v_\varepsilon^{i+1}(t))^2}{3}(x-i\varepsilon)^3\Big|_{i\varepsilon}^{(i+1)\varepsilon}+\varepsilon(u_\varepsilon^i(t))^2\big)\big]\\
&= 2\sup_{t\in[0,T]}\big[\sum_{i=0}^{M-1}\big(\frac{\varepsilon^3}{3}(v_\varepsilon^{i+1}(t))^2+\varepsilon(u_\varepsilon^i(t))^2\big)\big]\\
&\le 2C_1\sum_{i=0}^{M-1}\big(\frac{\varepsilon^3}{3}(v_\varepsilon^{i+1}(0))^2+\varepsilon(u_\varepsilon^i(0))^2\big)\\
&= 2C_1\frac{\varepsilon^2}{3}\|(u_\varepsilon(0,\cdot))_x\|_{L^2(0,1)}^2 + 2C_1\|u_\varepsilon(0,\cdot)\|_{L^2(0,1)}^2.
\end{aligned}$$

We made use of Lemma 4.1 and on the hypothesis on v_ε. Using now the estimate for $u_\varepsilon(0,\cdot)$ from the hypothesis we obtain: $\sup_{t\in[0,T]}\|u_\varepsilon(t)\|_{L^2(0,1)}^2 \le C'\|u_0\|_{H^1}^2$. Note that the equation for v_ε implies that we always have

$$\sup_{t\in[0,T]}\sum_{i=0}^{M-1}\varepsilon^3(v_\varepsilon^{i+1}(t))^2 \le C(T)\sum_{i=0}^{M-1}\varepsilon(v_\varepsilon^{i+1}(0))^2.$$

We can show thus that $\sup_{t\in[0,T]}\|u_\varepsilon(t)\|_{L^2(0,1)} \le C'\|u_0\|_{H^1}$ by using only the H^1-bounds for $u_\varepsilon(0)$, independent on any additional assumptions on $v_\varepsilon(t)$.

We further have:

$$\sup_{t\in[0,T]}\int_0^1 (u_\varepsilon(t,x)_x)^2dx = \sup_{t\in[0,T]}\sum_{i=0}^{M-1}\varepsilon(v_\varepsilon^{i+1}(t))^2$$
$$\leq C_1\sum_{i=0}^{M-1}\varepsilon(v_\varepsilon^{i+1}(0))^2 = C_1\|(u_\varepsilon(0,\cdot))_x\|^2_{L^2(0,1)} \leq C_1C_0\|u_0\|^2_{H^1}.$$

In the previously used notation, where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product on $\mathbb{R}^{M-1}$, we have for $\psi\in C_0^\infty(0,1)$ with $\|\psi\|_{H_0^1}=1$:

$$\sup_{t\in[0,T]}\int_0^1 \frac{d}{dt}u_\varepsilon(t,x)\psi(x)dx$$
$$= \sup_{t\in[0,T]}\sum_{i=0}^{M-1}\int_{i\varepsilon}^{(i+1)\varepsilon}\big(-\frac{d}{dt}v_\varepsilon^{i+1}(t)(x-i\varepsilon)+\frac{d}{dt}u_\varepsilon^i(t)\big)\psi(x)dx$$
$$= \sup_{t\in[0,T]}[-\langle\frac{d}{dt}v_\varepsilon(t),\tilde\Psi_1\rangle+\langle\frac{d}{dt}u_\varepsilon(t),\tilde\Psi_2\rangle]$$

where $\tilde\Psi_1,\tilde\Psi_2\in\mathbb{R}^{M-1}$ are given by

$$(\tilde\Psi_1)^k = \int_{(k-1)\varepsilon}^{k\varepsilon}(x-(k-1)\varepsilon)\psi(x)dx$$

respectively

$$(\tilde\Psi_2)^k = \int_{k\varepsilon}^{(k+1)\varepsilon}\psi(x)dx.$$

By (4.3) we have

$$\langle\frac{d}{dt}v_\varepsilon(t),\tilde\Psi_1\rangle = \langle\Delta_\varepsilon^{\zeta(v_\varepsilon)}\tilde\Psi_1, v_\varepsilon(t)\rangle$$

where $\Delta_\varepsilon^{\zeta(v_\varepsilon)}\tilde\Psi_1$ is computed like in (3.10) and has the form

$$(\Delta_\varepsilon^{\zeta(v_\varepsilon)}\tilde\Psi_1)^k = \frac{1}{2\varepsilon^2}\sum_{j\in I_k}\pm(\tilde\Psi_1^{j+1}-\tilde\Psi_1^j), \tag{4.10}$$

since we always can group the terms which arise in the r.h.s. of (3.10) in pairs with opposite signs. The index set I_k has two elements, except for sites $k\varepsilon$ near the boundary, when we have only one element.

By partial integration we have:

$$\tilde\Psi_1^j = \int_{(j-1)\varepsilon}^{j\varepsilon}(x-(j-1)\varepsilon)\psi(x)dx = \frac{\varepsilon^2}{2}\psi(j\varepsilon)-\frac{1}{2}\int_{(j-1)\varepsilon}^{j\varepsilon}(x-(j-1)\varepsilon)^2\psi'(x)dx.$$

A similar formula holds also for $\tilde\Psi_1^{j+1}$. By substraction we obtain easily the estimate

$$\frac{1}{2\varepsilon^2}|\tilde\Psi_1^{j+1}-\tilde\Psi_1^j| \leq \frac{1}{4}(2\int_{j\varepsilon}^{(j+1)\varepsilon}|\psi'(x)|dx+\int_{(j-1)\varepsilon}^{j\varepsilon}|\psi'(x)|dx)$$

$$\leq \frac{1}{2} \int_{(j-1)\varepsilon}^{(j+1)\varepsilon} |\psi'(x)| dx,$$

which in conjunction with the Cauchy-Schwartz inequality implies:

$$T_j := \frac{1}{4\varepsilon^4} |\tilde{\Psi}_1^{j+1} - \tilde{\Psi}_1^j|^2 \leq \frac{\varepsilon}{2} \int_{(j-1)\varepsilon}^{(j+1)\varepsilon} |\psi'(x)|^2 dx. \tag{4.11}$$

Returning to (4.10), where the index j has at most two values, we thus have the estimate: $[(\Delta_\varepsilon^{\zeta(v_\varepsilon)} \tilde{\Psi}_1)^k]^2 \leq \varepsilon \sum_{j \in I_k} T_j$ with T_j given in (4.11). It can be readily seen that there exists a constant $C' > 0$ such that

$$\sum_k \sum_{j \in I_k} T_j \leq C' \|\psi'\|^2_{L^2(0,1)} \leq C',$$

since every index j appears in the above sums maximally a given number of times. We thus have:

$$|\langle \frac{d}{dt} v_\varepsilon(t), \tilde{\Psi}_1 \rangle|^2 = |\langle \Delta_\varepsilon^{\zeta(v_\varepsilon)} \tilde{\Psi}_1, v_\varepsilon(t) \rangle|^2 \leq \sum_k (v_\varepsilon^k(t))^2 \sum_k [(\Delta_\varepsilon^{\zeta(v_\varepsilon)} \tilde{\Psi}_1)^k]^2$$
$$\leq C' \sum_k \varepsilon (v_\varepsilon^k(t))^2 \leq C' C_1 \sum_k \varepsilon (v_\varepsilon^k(0))^2 \leq C' C_1 C_0 \|u_0\|^2_{H^1} \tag{4.12}$$

where the constants C', C_1, C_0 do not depend on ψ and on ε.

By (4.8) and the subsequent computations we have:

$$\langle \frac{d}{dt} u_\varepsilon(t), \tilde{\Psi}_2 \rangle = -\varepsilon \sum_{i=1}^{M-1} \langle \Delta_\varepsilon^{\zeta(v_\varepsilon)} (\tilde{\Psi}_2)_{(i)}, v_\varepsilon(t) \rangle = -\varepsilon \sum_{k=1}^{M-1} v_\varepsilon^k(t) T'_k$$

where from (4.9), we have that

$$T'_k := \sum_{i=1}^{M-1} (\Delta_\varepsilon^{\zeta(v_\varepsilon)} (\tilde{\Psi}_2)_{(i)})^k = \pm \frac{1}{2\varepsilon^2} (\tilde{\Psi}_2^{j_k} - \tilde{\Psi}_2^{j_k - 2})$$

with $j_k = k$ or $j_k = k+1$, depending on the sign on $v_\varepsilon^k(t)$.

By partial integration we have:

$$\tilde{\Psi}_2^{j_k} = \int_{j_k \varepsilon}^{(j_k+1)\varepsilon} \psi(x) dx = \varepsilon \psi((j_k+1)\varepsilon) - \int_{j_k \varepsilon}^{(j_k+1)\varepsilon} (x - j_k \varepsilon) \psi'(x) dx$$

and for $\tilde{\Psi}_2^{j_k-2}$ we obtain a similar equation. Proceeding similarly as in the case of $\tilde{\Psi}_1$ we arrive at

$$\varepsilon (T'_k)^2 \leq 3 \int_{(j_k-2)\varepsilon}^{(j_k+1)\varepsilon} |\psi'(x)|^2 dx$$

with $\sum_k \varepsilon (T'_k)^2 \leq C'' \|\psi'\|^2_{L^2(0,1)} \leq C''$.

Similarly as in the previous computations we have:

$$|\langle \frac{d}{dt} u_\varepsilon(t), \tilde{\Psi}_2 \rangle|^2 \leq$$

$$\leq (\sum_{k=1}^{M-1} |\varepsilon^{\frac{1}{2}} v_\varepsilon^k(t)||\varepsilon^{\frac{1}{2}} T_k'|)^2 \leq \sum_{k=1}^{M-1} \varepsilon |v_\varepsilon^k(t)|^2 \cdot \sum_{k=1}^{M-1} \varepsilon (T_k')^2 \leq C_1 C'' C_0 \|u_0\|_{H^1}^2.$$

This completes the proof of the lemma. □

Theorem 4.3. *If on the time interval $[0,T]$ the hypotheses of Lemma 4.2 hold, then the family $u_\varepsilon(\cdot,\cdot)$ has a weakly convergent subsequence in $L^2(0,T;H_0^1(0,1)) \cap H^1(0,T;H^{-1}(0,1))$ and the limit function, denoted by u, lies in $C(0,T;L^2(0,1))$. Moreover, if we have $\|u_\varepsilon(0)-u_0\|_{L^2(0,1)} \to 0$ as $\varepsilon \to 0$, then u satisfies for all test functions $\phi \in C([0,T]\times[0,1]) \cap C^\infty((0,T)\times(0,1))$ with supp$\phi \subset [0,T)\times(0,1)$ the equation:*

$$-\int_0^T (u(t), \frac{d}{dt}\phi(t))dt = (u_0, \phi(0)) - \int_0^T (u_x(t), \phi_x(t))dt.$$

By $(\cdot,\cdot)$ we denote the usual duality pairing in the corresponding function spaces.

Proof. The weak convergence property is implied by the apriori estimates from Lemma 4.2. By a result from [6], p.379 we have further $u \in C(0,T;L^2(0,1))$. We denote the convergent subsequence again by u_ε. For a test function ϕ like in the hypothesis we have:

$$-\int_0^T (u(t), \frac{d}{dt}\phi(t))dt - (u_0, \phi(0)) + \int_0^T (u_x(t), \phi_x(t))dt =$$
$$= -\int_0^T (u(t)-u_\varepsilon(t), \frac{d}{dt}\phi(t))dt - (u_0 - u_\varepsilon(0), \phi(0)) +$$
$$+ \int_0^T (u_x(t) - (u_\varepsilon)_x(t), \phi_x(t))dt - \int_0^T (u_\varepsilon(t), \frac{d}{dt}\phi(t))dt$$
$$-(u_\varepsilon(0), \phi(0)) + \int_0^T ((u_\varepsilon)_x(t), \phi_x(t))dt.$$

The first three terms converge to 0 as $\varepsilon \to 0$ due to the weak convergence property. In order to prove the statement of the theorem, we will show that the rest of the sum can be made arbitrarily small for ε small enough. For this it suffices to show that the term

$$\sup_{t\leq T}[(\frac{d}{dt}u_\varepsilon(t), \phi(t)) + ((u_\varepsilon)_x(t), \phi_x(t))] \tag{4.13}$$

can be proved to be arbitrarily small by taking ε small enough.

Similarly like in Lemma 4.2 we obtain:

$$(\frac{d}{dt}u_\varepsilon(t), \phi(t)) + ((u_\varepsilon)_x(t), \phi_x(t)) = \tag{4.14}$$
$$= -\langle \frac{d}{dt}v_\varepsilon(t), \tilde{\Phi}_1(t)\rangle + \langle \frac{d}{dt}u_\varepsilon(t), \tilde{\Phi}_2(t)\rangle + \langle v_\varepsilon(t), \tilde{\Phi}_3(t)\rangle$$
$$= \langle v_\varepsilon(t), \Delta_\varepsilon^{\zeta(v_\varepsilon)}\tilde{\Phi}_1(t)\rangle + \langle v_\varepsilon(t), \nabla_\varepsilon^{\zeta(v_\varepsilon)}\tilde{\Phi}_2(t)\rangle + \langle v_\varepsilon(t), \tilde{\Phi}_3(t)\rangle$$

where the vectors $\tilde{\Phi}_i(t) \in \mathbb{R}^{M-1}$ have the components

$$\tilde{\Phi}_1^k(t) = \int_{(k-1)\varepsilon}^{k\varepsilon} (y-(k-1)\varepsilon)\phi(t,y)dy,$$

$$\tilde{\Phi}_2^k(t) = \int_{k\varepsilon}^{(k+1)\varepsilon} \phi(t,y)dy,$$

$$\tilde{\Phi}_3^k(t) = \int_{(k-1)\varepsilon}^{k\varepsilon} \phi_x(t,y)dy$$

and where

$$\begin{aligned}(\Delta_\varepsilon^{\zeta(v_\varepsilon)}\tilde{\Phi}_1(t))^k &= \frac{1}{2\varepsilon^2}[-\tilde{\Phi}_1^{k-2}\zeta(v_\varepsilon^k \wedge 0) + \tilde{\Phi}_1^{k-1}\zeta(v_\varepsilon^k) - \tilde{\Phi}_1^k \\ &\qquad -\tilde{\Phi}_1^{k+1}\zeta(v_\varepsilon^k) + \tilde{\Phi}_1^{k+2}\zeta(v_\varepsilon^k \vee 0)] \\ (\nabla_\varepsilon^{\zeta(v_\varepsilon)}\tilde{\Phi}_2(t))^k &= \frac{1}{2\varepsilon}[(\tilde{\Phi}_2^{k-1} - \tilde{\Phi}_2^{k+1})\zeta(v_\varepsilon^k \vee 0) + (\tilde{\Phi}_2^k - \tilde{\Phi}_2^{k-2})\zeta(v_\varepsilon^k \wedge 0)].\end{aligned}$$

By taking ε small enough, since ϕ has compact support in $[0,T) \times (0,1)$, we may consider only indices k for which the above formulas hold for all t, disregarding the sites near the boundary where ϕ vanishes. By the same reason (neglecting the sites close to the boundary), we may note that for $i = 1,2,3$ we can write $(\tilde{\Phi}_i(t))^k = \Phi_i(t,k\varepsilon)$ where the functions Φ_i are defined on $[0,T] \times (0,1)$ by

$$\Phi_1(t,x) = \int_{x-\varepsilon}^{x} (y-x+\varepsilon)\phi(t,y)dy,$$

$$\Phi_2(t,x) = \int_{x}^{x+\varepsilon} \phi(t,y)dy,$$

$$\Phi_3(t,x) = \int_{x-\varepsilon}^{x} \phi_x(t,y)dy = \phi(t,x) - \phi(t,x-\varepsilon).$$

The derivatives with respect to x of these functions are given by

$$\Phi_{1,x}(t,x) = \varepsilon\phi(t,x) - \int_{x-\varepsilon}^{x} \phi(t,y)dy,$$

$$\Phi_{1,xx}(t,x) = \varepsilon\phi_x(t,x) - \phi(t,x) + \phi(t,x-\varepsilon),$$

$$\Phi_{2,x}(t,x) = \phi(t,x+\varepsilon) - \phi(t,x).$$

By the Taylor formula, using the form of Φ_i and the bounds of the derivatives of second and third order of $\phi(t,\cdot)$ it is easy to see that if $v_\varepsilon^k > 0$ we have:

$$\begin{aligned}(\Delta_\varepsilon^{\zeta(v_\varepsilon)}\tilde{\Phi}_1(t))^k &= \Phi_{1,xx}(t,k\varepsilon) + O(\varepsilon^2) \\ (\nabla_\varepsilon^{\zeta(v_\varepsilon)}\tilde{\Phi}_2(t))^k &= -\Phi_{2,x}(t,k\varepsilon) + O(\varepsilon^2)\end{aligned} \tag{4.15}$$

while for $v_\varepsilon^k < 0$ we have:

$$\begin{aligned}(\Delta_\varepsilon^{\zeta(v_\varepsilon)}\tilde{\Phi}_1(t))^k &= \Phi_{1,xx}(t,(k-1)\varepsilon) + O(\varepsilon^2) \\ (\nabla_\varepsilon^{\zeta(v_\varepsilon)}\tilde{\Phi}_2(t))^k &= -\Phi_{2,x}(t,(k-1)\varepsilon) + O(\varepsilon^2).\end{aligned} \tag{4.16}$$

Plugging (4.15), (4.16) together with the formulae for Φ_3, $\Phi_{1,x}$, $\Phi_{1,xx}$, $\Phi_{2,x}$ into (4.14), we obtain:

$$(\frac{d}{dt}u_\varepsilon(t), \phi(t)) + ((u_\varepsilon)_x(t), \phi_x(t)) = \sum_k v_\varepsilon^k(t) U_k(t) \tag{4.17}$$

where for $v_\varepsilon^k(t) > 0$ we have:

$$\begin{aligned} U_k(t) &= \varepsilon\phi_x(t, k\varepsilon) - \phi(t, k\varepsilon) + \phi(t, (k-1)\varepsilon) \\ &\quad -\phi(t, (k+1)\varepsilon) + \phi(t, k\varepsilon) + \phi(t, k\varepsilon) - \phi(t, (k-1)\varepsilon) + O(\varepsilon^2) \\ &= \varepsilon\phi_x(t, k\varepsilon) - \phi(t, (k+1)\varepsilon) + \phi(t, k\varepsilon) + O(\varepsilon^2) \\ &= -\frac{\varepsilon^2}{2}\phi_{xx}(t, \xi_k) + O(\varepsilon^2) \end{aligned}$$

while for $v_\varepsilon^k(t) < 0$ we have similarly $U_k(t) = -\frac{\varepsilon^2}{2}\phi_{xx}(t, \eta_k) + O(\varepsilon^2)$.

The regularity of ϕ implies that if $v_\varepsilon^k(t) \neq 0$ we have $U_k(t)$ of magnitude $O(\varepsilon^2)$, otherwise we have $v_\varepsilon^k(t)U_k(t) = 0$. Using the L^2-boundedness property of $v_\varepsilon(t)$ we conclude that the expression in (4.13) can be made thus arbitrary small for ε small enough. The proof is completed. □

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