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On some quadrature formulas on the real line with the higher degree of accuracy and its applications

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Abstract. In this paper we study quadrature formulas with the higher degree of accuracy. We study the quasi-orthogonality of orthogonal polynomials and we give some results on the location of their zeros.

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1. Introduction

Let P_n be a polynomial of degree n such that

$$\int_a^b x^k P_n(x)w(x)dx = 0, \quad k = 0, 1, \dots, n-1,$$

where w is a positive weight function on the finite or infinite interval $[a, b]$. P_n is the polynomial of degree n belonging to the family of orthogonal polynomials on $[a, b]$ with respect to the weight function w. It is well known that the zeros of P_n are all real and distinct and lie in (a, b).

Definition 1.1. *Let R_n be a polynomial of exact degree n, $n \geq r$, r being a fixed natural number. If R_n satisfies the conditions*

$$\int_a^b x^k P_n(x)w(x)dx = \begin{cases} 0, & \textit{for} \quad k = 0, 1, \dots, n-r-1 \\ \neq 0, & \textit{for} \quad k = n-r \end{cases} \tag{1.1}$$

where w is a positive weight function on $[a, b]$, then R_n is a quasi-orthogonal polynomial of order r on $[a, b]$ with respect to w.

Remark 1.2. The quasi-orthogonal polynomials R_n are only defined for $n \geq r$.

If $r = 0$ then $R_n = \lambda P_n$ where λ is a real constant.
The following result can be found in [1].

Theorem 1.3. *Let $\{P_n\}$ be the family of orthogonal polynomials on $[a,b]$ with respect to a positive weight function w. A necessary and sufficient condition for a polynomial R_n of degree n to be quasi-orthogonal of order r on $[a,b]$ with respect to w is that*

$$R_n(x) = c_0 P_n(x) + c_1 P_{n-1}(x) + \ldots + c_r P_{n-r}(x) \tag{1.2}$$

where c_i's are numbers which can depend on n and $c_0 c_r \neq 0$.

If R_n is quasi-orthogonal of order r on $[a,b]$, then at least $n-r$ distinct zeros of R_n lie in the interval (a,b).

In [1] C. Brezinski, K. A. Driver, M. Redino-Zaglia consider quasi-orthogonal polynomials of degree $n-1$, $n-2$:

$$R_n(x) = P_n(x) + a_n P_{n-1}(x), \quad a_n \neq 0 \tag{1.3}$$

and

$$R_n(x) = P_n(x) + a_n P_{n-1}(x) + b_n P_{n-2}(x), \quad b_n \neq 0 \tag{1.4}$$

and make a study of its zeros.

The following result is well known.

Theorem 1.4. *The quadrature formula*

$$\int_a^b f(x)w(x)dx = \sum_{i=1}^{n} A_{i,n} f(x_{i,n}) + R(f) \tag{1.5}$$

has the degree of exactness $n+k$ if and only if it is of interpolatory type and the nodal polynomial

$$\Pi_n(x) = \prod_{i=1}^{n} (x - x_{i,n})$$

is quasi-orthogonal of order $n-k-1$ in $[a,b]$ with respect to w.

A. Bultheel, R. Cruz-Barroso and Marc Van Borel ([2]) consider an n point quadrature formula of Gauss-Radon type:

$$\int_a^b f(x)w(x)dx = A_\alpha f(\alpha) + \sum_{k=1}^{n-1} A_{k,n} f(x_{k,n}) + R(f) \tag{1.6}$$

where $\alpha \in [a,b]$ is a fixed point and the degree of exactness is $2n-2$.

Remark 1.5. If $P_n(\alpha) = 0$ then (1.6) is actually a Gaussian quadrature formula.

Remark 1.6. The coefficients of the quadrature formula (1.6) are positive.

In [2] the authors studied also Gauss-Lobatto-type quadrature formulas with two arbitrary prefixed nodes, α and β:

$$\int_a^b f(x)w(x)dx = A_\alpha f(\alpha) + A_\beta f(\beta) + \sum_{k=1}^{n-2} A_{k,n} f(x_{k,n}) + R_n(f) \tag{1.7}$$

the degree of exactness being $2n-3$.

From Theorem 1.3, the nodes of such a rule will be the zeros of

$$R_n(x) = P_n(x) + a_n P_{n-1}(x) + b_n P_{n-2}(x).$$

2. $P_{n,k}$-polynomials and its properties

Let w be a positive weight function on $[a, b]$ ($a > -\infty$), $n \in \mathbb{N}^*$, $k \in \mathbb{N}$ such that $k \leq n$.

We denote by $P_{n,k}$ the polynomial of degree n which satisfies the following conditions:

$$\int_a^b (x-a)^i P_{n,k}(x) w(x) dx = \delta_{k,i}, \quad i = 0, 1, \ldots, n. \tag{2.1}$$

In the following, without loss of generality, we will consider $a = 0$.

Remark 2.1. By (2.1) it follows that $P_{n,k}$ is a quasi-orthogonal polynomial of order $n - k$ with respect to the weight function w.

Theorem 2.2. *The zeros of $P_{n,k}$ are all real, distinct and lie in $(0, b)$.*

Proof. Let us denote by $0 < x_1 < \ldots < x_i < b$ the zeros of $P_{n,k}$ where it changes the sign. Obviously $i \geq k$. Suppose $i < n$. We have

$$\int_0^b (x - x_1) \ldots (x - x_i) P_{n,k}(x) w(x) dx > 0. \tag{2.2}$$

Using the definition of $P_{n,k}$, from (2.2) we obtain

$$(-1)^{i-k} \sigma_{i-k} > 0, \tag{2.3}$$

where $(-1)^{i-k}\sigma_{i-k}$ is the coefficient of x^k of the polynomial

$$(x - x_1) \ldots (x - x_i), \quad \sigma_{i-k} > 0.$$

On the other hand we have:

$$\int_0^b x(x - x_1) \ldots (x - x_i) P_{n,k}(x) w(x) dx > 0$$

or

$$(-1)^{i-k-1} \sigma_{i-k+1} > 0. \tag{2.4}$$

The relations (2.3) and (2.4) are contradictory. □

It is easy to see that the set $\{P_{n,k}\}_{k=0}^n$ forms a base in Π_n and for every $P \in \Pi_n$ we have:

$$\begin{aligned} P &= \sum_{k=0}^n \langle e_k, P \rangle P_{n,k} \\ &= \sum_{k=0}^n e_k \langle P_{n,k} P \rangle, \end{aligned}$$

where $e_k : \mathbb{R} \to \mathbb{R}$, $e_k(x) = x^k$, and

$$\langle f, g \rangle = \int_a^b f(x)g(x)w(x)dx.$$

We denote by $K_n(x, y)$ the Christoffel-Darboux kernel

$$K_n(x, y) = \sum_{k=0}^{n} p_k(x)p_k(y)$$

where the set $\{p_k\}_{k=0}^n$ is an orthonormal set

$$\int_0^b p_k(x)p_i(x)w(x)dx = \delta_{k,i}, \quad k, i \in \{0, 1, \ldots, n\}.$$

The result from the following Theorem is easily verified.

Theorem 2.3. *The following relations hold:*

$$\begin{aligned} K_n(x, y) &= \sum_{k=0}^{n} x^k P_{n,k}(y) \\ &= \sum_{k=0}^{n} y^k P_{n,k}(x) \\ &= \frac{1}{a_{n+1,n+1}} \cdot \frac{P_{n+1,n+1}(x)P_{n,n}(y) - P_{n,n}(x)P_{n+1,n+1}(y)}{x - y} \end{aligned} \tag{2.5}$$

where $a_{n+1,n+1}$ is the coefficient of x^{n+1} from $P_{n+1,n+1}$.

3. Main results

Let P be a polynomial of degree n and let m_k be the moment of order k with respect to the weight function w,

$$m_k = \langle e_k, P \rangle = \int_0^b x^k P(x)w(x)dx, \quad k = 0, 1, \ldots, n.$$

Then P can be written as

$$P(x) = \sum_{k=0}^{n} m_k P_{n,k}(x).$$

Theorem 3.1. *If*

$$(-1)^k m_k \geq 0, \quad k = 0, 1, 2, \ldots, n \tag{3.1}$$

then the zeros of P are all real, distinct and lie in $(0, b)$.

Proof. By (3.1) it follows that there exist at least a point x_1 where P changes the sign.

Let $x_1, \ldots, x_p$ be all the zeros where P changes its sign in the interval $(0, b)$ and suppose that $p < n$.

So, the polynomial $(x - x_1) \ldots (x - x_p)P(x)$ doesn't change the sign.

Suppose that

$$(x-x_1)\dots(x-x_p)P(x)\geq 0. \tag{3.2}$$

From (3.2) we get:

$$\int_0^b (x-x_1)\dots(x-x_p)P(x)w(x)dx > 0 \tag{3.3}$$

$$\int_0^b (x-x_1)\dots(x-x_p)P(x)w(x)dx = (-1)^p \sum_{i=0}^{p} (-1)^{p-i} m_{p-i}\sigma_i \tag{3.4}$$

where σ_i are Vieta's sum of order i of the numbers $x_1,\dots,x_p$.

On the other hand we have:

$$\int_0^b x(x-x_1)\dots(x-x_p)P(x)w(x)dx > 0 \tag{3.5}$$

$$\int_0^b x(x-x_1)\dots(x-x_p)P(x)w(x)dx = (-1)^{p+1} \sum_{i=0}^{p} (-1)^{p-i+1} m_{p-i+1}\sigma_i. \tag{3.6}$$

From (3.4) and (3.6) it follows that the inequalities (3.3) and (3.4) are contradictory and so $p=n$. □

Corollary 3.2. *Let R_n be a quasi-orthogonal polynomial of order* 1*,*

$$R_n(x) = P_{n,n-1}(x) - a_n P_{n,n}(x).$$

If $a_n > 0$ then the zeros of R_n are all real and distinct and lie in $(0,b)$.

Remark 3.3. The condition $a_n > 0$ is only sufficient.

A necessary and sufficient condition is given by

$$(-1)^n (P_{n,n-1}(0) - a_n P_{n,n}(0))(P_{n,n-1}(b) - a_n P_{n,n}(b)) > 0.$$

Let $\alpha \in [0,b]$ be a fixed point and let us consider the quadrature formula

$$\int_0^b f(x)w(x)x = A_\alpha f(\alpha) + \sum_{k=1}^{n-1} A_{k,n} f(x_{k,n}) + R(f) \tag{3.7}$$

having the degree of exactness $2n-2$.

This means that α is a root of polynomial R_n which is of the form

$$R_n(x) = P_{n,n-1}(x) + aP_{n,n}(x).$$

The coefficients A_α, $A_{k,n}$, $k=1,2,\dots,n-1$ are positive and are given by

$$A_{k,n} = \frac{\displaystyle\int_0^b (x-\alpha)^2 l_k^2(x) w(x)dx}{(x_{k,n}-\alpha)^2}, \quad A_\alpha = \frac{\displaystyle\int_0^b l^2(x)w(x)dx}{l^2(\alpha)}$$

where

$$l(x) = \prod_{k=1}^{n-1}(x-x_{k,n}), \quad l_k(x) = \frac{l(x)}{(x-x_{k,n})l'(x_{k,n})}.$$

Theorem 3.4. *The coefficients* $A_{k,n}$, $k = 1, \dots, n-1$ *and* A_α *are given by*

$$A_{k,n} = \frac{1}{K_{n-1}(x_{k,n}, x_{k,n})}, \quad k = 1, 2, \dots, n-1$$

$$A_\alpha = \frac{1}{K_{n-1}(\alpha, \alpha)}.$$

Proof. Let us denote by:

$$M_i = \int_0^b x^i (x-\alpha) l_k(x) w(x) dx.$$

We have

$$\begin{aligned} M_1 &= x_{k,n} M_0 \\ M_2 &= x_{k,n}^2 M_0 \\ &\dots \\ M_{n-1} &= x_{k,n}^{n-1} M_0. \end{aligned} \tag{3.8}$$

From (3.8) we get

$$(x-\alpha) l_k(x) = M_0 \sum_{i=0}^{n-1} x_{k,n}^i P_{n-1,i}(x). \tag{3.9}$$

By (3.9) we obtain

$$M_0 = \frac{x_{k,n} - \alpha}{K_{n-1}(x_{k,n}, x_{k,n})}$$

and so

$$A_{k,n} = \frac{1}{K_{n-1}(x_{k,n}, x_{k,n})}, \quad k = \overline{1, n-1}.$$

Similarly we get

$$A_\alpha = \frac{1}{K_{n-1}(\alpha, \alpha)}.$$

The proof of the theorem is finished. □

Corollary 3.5. *Let* $P \in \Pi_{2n-2}$, $P(x) > 0$, $\forall\ x \in \mathbb{R}$. *Then*

$$\int_0^b P(x) w(x) dx \geq \frac{1}{K_{n-1}(\alpha, \alpha)} P(\alpha), \ \forall\ \alpha \in \mathbb{R}.$$

Theorem 3.6. *Let* R_n *be a quasi-orthogonal polynomial of order* 1 *with the weight function* w *having all its zeros lie in* $[0, b)$. *Suppose that*

$$R_n(x) = a_n x^n + \dots$$

Then for every continuous function f, $f : [a, b] \to \mathbb{R}$, *the following equality holds:*

$$\int_0^b w(x) f(x) dx - \sum_{k=1}^n A_k f(x_k) = \frac{1}{a_n} [x_1, x_2, \dots, x_n; [x, x_1, \dots, x_n; f]] \tag{3.10}$$

$$+\frac{1}{a_n^2}\int_0^b [x,x_1,x_2,\ldots,x_n;[\cdot,x_1,\ldots,x_n;f]]R_n^2(x)w(x)dx$$

where x_k*,* $k=1,2,\ldots,n$*, are the zeros of* R_n *and* $A_k = \dfrac{1}{K_{n-1}(x_k,x_k)}$.

Proof. The quadrature formula

$$\int_0^b w(x)f(x)dx = \sum_{k=1}^n A_k f(x_k) + R(f) \tag{3.11}$$

having degree of exactness $2n-2$ is a quadrature formula of interpolatory type, coefficients A_k, $k=1,2,\ldots,n$ being given by

$$\begin{aligned} A_k &= \int_0^b l_k(x)w(x)dx \\ &= \frac{1}{K_{n-1}(x_k,x_k)}. \end{aligned}$$

We have

$$f(x) - L_{n-1}(f;x_1,\ldots,x_n)(x) = \frac{1}{a_n}R_n(x)[x,x_1,\ldots,x_n;f] \tag{3.12}$$

where $L_{n-1}(f;x_1,\ldots,x_n)$ is Lagrange's polynomial of degree $n-1$ which interpolates the function f at the points x_k, $k=\overline{1,n}$.

R_n is of the form:

$$R_n = P_{n,n-1} + \alpha P_{n,n}, \quad \alpha \in \mathbb{R}.$$

From (3.12) we obtain

$$\begin{aligned} &\int_0^b f(x)R_n(x)w(x)dx - [x_1,x_2,\ldots,x_n;f] \\ &= \frac{1}{a_n}\int_0^b R_n^2(x)[x,x_1,x_2,\ldots,x_n;f]w(x)dx \end{aligned} \tag{3.13}$$

and

$$\int_0^b f(x)w(x)dx - \sum_{k=1}^n A_k f(x_k) = \frac{1}{a_n}\int_0^b R_n(x)[x,x_1,\ldots,x_n;f]w(x)dx \tag{3.14}$$

From (3.13) and (3.14) we get (3.10). □

Corollary 3.7. *Let* $f \in C^1[0,b]$*. Then there exists* $\theta \in [0,b]$ *such that* $R(f)$ *from (*3.11*) can be written in the following form*

$$\begin{aligned} R(f) &= \frac{1}{a_n}[x_1,x_2,\ldots,x_n;[x,x_1,\ldots,x_n;f]] \\ &+ \frac{k_n}{a_n^2}[\theta,x_1,\ldots,x_n;[x,x_1,\ldots,x_n;f]] \end{aligned} \tag{3.15}$$

where

$$k_n = \int_0^b R_n^2(x)w(x)dx.$$

Proof. Equation (3.15) follows from (3.13) if we put instead of f the divided difference $[x, x_1, \ldots, x_n; f]$. □

Theorem 3.8. *Let x_k, $k = 1, 2, \ldots, n$ be the zeros of $P_{n,0}$ and w a positive weight such that*

$$\int_0^b w(x)dx = 1.$$

Then, for every $P \in \Pi_{n-1}$ we have:

$$\int_0^b P(x)w(x)dx = \sum_{k=1}^n \frac{P(x_k)}{K_n(x_k, x_k)} - \frac{1}{a_n}\left[x_1, \ldots, x_n; \frac{P(x)}{x}\right]$$

where a_n is the coefficient of x^n from $P_{n,0}$.

Proof. Let us consider the quadrature formula

$$\int_0^b f(x)w(x)dx = \sum_{k=1}^n A_k f(x_k) + R(f). \tag{3.16}$$

The quadrature formula (3.16) has the degree of exactness $n-1$ and A_k, $k = 1, 2, \ldots, n$ are given by

$$A_k = \int_0^b \frac{P_{n,0}(x)w(x)}{(x - x_k)P'_{n,0}(x_k)}dx.$$

Let us denote by M_i the moment of order i, $i = 0, 1, \ldots, n$ of the polynomial

$$\frac{P_{n,0}(x)}{(x - x_k)P'_{n,0}(x_k)}.$$

We get

$$M_1 - x_k M_0 = \frac{1}{P'_{n,0}(x_k)} \tag{3.17}$$

$$M_i = x_k^{i-1} M_1, \quad i = 2, 3, \ldots, n.$$

So

$$\frac{P_{n,0}(x)}{(x - x_k)P'_{n,0}(x)} = M_0 P_{n,0}(x) + M_1 P_{n,1}(x) \tag{3.18}$$

$$+\frac{M_1}{x_k}(K_n(x, x_k) - P_{n,0}(x) - x_k P_{n,1}(x)).$$

For $x = x_k$ we get

$$1 = \frac{M_1}{x_k} K_n(x_k, x_k). \tag{3.19}$$

From (3.17) and (3.19) we obtain

$$M_0 = \frac{1}{K_n(x_k, x_k)} - \frac{1}{x_k P'_{n,0}(x_k)}.$$

On the other hand $M_0 = A_k$ and the quadrature formula (3.16) becomes:

$$\int_0^b f(x)w(x)dx = \sum_{k=1}^n \frac{f(x_k)}{K_n(x_k, x_k)} - \frac{1}{a_n}\left[x_1, \ldots, x_n; \frac{f(x)}{x}\right] + R(f).$$

If $f \in \Pi_{n-1}$, $R(f) = 0$ and the theorem is proved. □

Corollary 3.9. *If* $P(0) = 0$ *and* $P \in \Pi_{n-1}$ *then*

$$\int_0^b P(x)w(x)dx = \sum_{k=1}^{n} \frac{P(x_k)}{K_n(x_k, x_k)}.$$

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