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Remark on Voronovskaja theorem for q-Bernstein operators

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Abstract. We establish quantitative Voronovskaja type theorems for the q-Bernstein operators introduced by Phillips in 1997. Our estimates are given with the aid of the first order Ditzian-Totik modulus of smoothness.

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1. Introduction

Let $q > 0$ and n be a non-negative integer. Then the q-integers $[n]_q$ and the q-factorials $[n]_q!$ are defined by

$$[n]_q = \begin{cases} 1 + q + \ldots + q^{n-1}, & \text{if} \quad n \geq 1 \\ 0, & \text{if} \quad n = 0 \end{cases}$$

and

$$[n]_q! = \begin{cases} [1]_q [2]_q \ldots [n]_q, & \text{if} \quad n \geq 1 \\ 1, & \text{if} \quad n = 0. \end{cases}$$

For integers $0 \leq k \leq n$, the q-binomial coefficients are defined by

$$\left[\begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

The so-called q-Bernstein operators were introduced by G.M. Phillips [3] and they are defined by $B_{n,q} : C[0,1] \to C[0,1]$,

$$(B_{n,q}f)(x) \equiv B_{n,q}(f,x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) p_{n,k}(q,x),$$

where

$$p_{n,k}(q,x) = \left[\begin{array}{c} n \\ k \end{array} \right]_q x^k(1-x)(1-qx)\dots(1-q^{n-k-1}x), \quad x \in [0,1],$$

and an empty product denotes 1. Note that for $q = 1$, we recover the classical Bernstein operators. It is well-known that Voronovskaja's theorem [5] deals with the asymptotic behaviour of Bernstein operators. Then naturally raises the following question: can we state a similar Voronovskaja theorem for the q-Bernstein operators? The positive answer was given in [3] as follows.

Theorem 1.1. *Let $q = q_n$ satisfy $0 < q_n < 1$ and let $q_n \to 1$ as $n \to \infty$. If f is bounded on $[0,1]$, differentiable in some neighborhood of x and has second derivative $f''(x)$ for some $x \in [0,1]$, then the rate of convergence of the sequence $\{(B_{n,q_n}f)(x)\}$ is governed by*

$$\lim_{n\to\infty} [n]_{q_n} \{(B_{n,q_n}f)(x) - f(x)\} = \frac{1}{2}x(1-x)f''(x). \tag{1.1}$$

In [4], the convergence (1.1) was given in quantitative form as follows.

Theorem 1.2. *Let $q = q_n$ satisfy $0 < q_n < 1$ and let $q_n \to 1$ as $n \to \infty$. Then for any $f \in C^2[0,1]$ the following inequality holds*

$$\left| [n]_{q_n} \{(B_{n,q_n}f)(x) - f(x)\} - \frac{1}{2}x(1-x)f''(x) \right| \le c\,x(1-x)\,\omega\left(f'', [n]_{q_n}^{-1/2}\right),$$

where c is an absolute positive constant, $x \in [0,1]$, $n = 1,2,\dots$ and ω is the first order modulus of continuity.

The goal of this note is to obtain new quantitative Voronovskaja type theorems for the q-Bernstein operators. Our results will be formulated with the aid of the first order Ditzian-Totik modulus of smoothness (see [1]), which is given for $f \in C[0,1]$ by

$$\omega_\varphi^1(f,\delta) = \sup_{0<h\le\delta} \|\Delta^1_{h\varphi(\cdot)}f(\cdot)\|, \tag{1.2}$$

where $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$, $\|\cdot\|$ is the uniform norm and

$$\Delta^1_{h\varphi(x)}f(x) = \begin{cases} f\left(x + \frac{1}{2}h\varphi(x)\right) - f\left(x - \frac{1}{2}h\varphi(x)\right), & \text{if} \quad x \pm \frac{1}{2}h\varphi(x) \in [0,1] \\ 0, & \text{otherwise.} \end{cases}$$

Further, the corresponding $K-$functional to (1.2) is defined by

$$K_{1,\varphi}(f,\delta) = \inf\{\|f-g\| + \delta\|\varphi g'\| : g \in W^1(\varphi)\},$$

where $W^1(\varphi)$ is the set of all $g \in C[0,1]$ such that g is absolutely continuous on every interval $[a,b] \subset [0,1]$ and $\|\varphi g'\| < +\infty$. Then, in view of [1, p.11], there exists $C > 0$ such that

$$K_{1,\varphi}(f,\delta) \le C\omega_\varphi^1(f,\delta). \tag{1.3}$$

Here we mention that throughout this paper C denotes a positive constant independent of n and x, but it is not necessarily the same in different cases.

2. Main result

Our result is the following.

Theorem 2.1. *Let* $\{q_n\}$ *be a sequence such that* $0 < q_n < 1$ *and* $q_n \to 1$ *as* $n \to \infty$*. Then for every* $f \in C^2[0,1]$ *the following inequalities hold*

$$\begin{aligned} & \left| \, [n]_{q_n} \{(B_{n,q_n} f)(x) - f(x)\} - \frac{1}{2} x(1-x) f''(x) \, \right| \\ \leq & \;\; C\, \omega_\varphi^1 \left(f'', \sqrt{[n]_{q_n}^{-1} x(1-x)} \right), \end{aligned} \tag{2.1}$$

$$\begin{aligned} & \left| \, [n]_{q_n} \{(B_{n,q_n} f)(x) - f(x)\} - \frac{1}{2} x(1-x) f''(x) \, \right| \\ \leq & \;\; C \sqrt{x(1-x)}\, \omega_\varphi^1 \left(f'', \sqrt{[n]_{q_n}^{-1}} \right), \end{aligned} \tag{2.2}$$

where $x \in [0,1]$ *and* $n = 1, 2, \ldots$

Proof. We recall some properties of the q-Bernstein operators (see [3]):

$$B_{n,q_n}(1,x) = 1,\ B_{n,q_n}(t,x) = x,\ B_{n,q_n}(t^2,x) = x^2 + [n]_{q_n}^{-1} x(1-x) \tag{2.3}$$

and B_{n,q_n} are positive.

Let $f \in C^2[0,1]$ be given and $t, x \in [0,1]$. Then, by Taylor's formula, $f(t) = f(x) + f'(x)(t-x) + \int_x^t f''(u)(t-u)\,du$. Hence

$$\begin{aligned} & f(t) - f(x) - f'(x)(t-x) - \frac{1}{2} f''(x)(t-x)^2 \\ = & \int_x^t f''(u)(t-u)\,du - \int_x^t f''(x)(t-u)\,du \\ = & \int_x^t [f''(u) - f''(x)](t-u)\,du. \end{aligned}$$

In view of (2.3), we obtain

$$\begin{aligned} & \left| \, B_{n,q_n}(f,x) - f(x) - \frac{1}{2} [n]_{q_n}^{-1} x(1-x) f''(x) \, \right| \\ = & \left| \, B_{n,q_n} \left(\int_x^t [f''(u) - f''(x)]\,(t-u)\,du, x \right) \right| \\ \leq & \;\; B_{n,q_n} \left(\left| \int_x^t |f''(u) - f''(x)|\,|t-u|\,du \right|, x \right). \end{aligned} \tag{2.4}$$

In what follows we estimate $\left| \int_x^t |f''(u) - f''(x)|\,|t-u|\,du \right|$. For $g \in W^1(\varphi)$, we have

$$
\begin{aligned}
& \Big| \int_x^t |f''(u)-f''(x)|\,|t-u|\,du \Big| \\
\le\ & \Big| \int_x^t |f''(u)-g(u)|\,|t-u|\,du \Big| + \Big| \int_x^t |g(u)-g(x)|\,|t-u|\,du \Big| \\
& + \Big| \int_x^t |g(x)-f''(x)|\,|t-u|\,du \Big| \\
\le\ & 2\|f''-g\|(t-x)^2 + \left| \int_x^t \Big| \int_x^u |g'(v)|\,dv \Big| \,|t-u|\,du \right| \\
\le\ & 2\|f''-g\|(t-x)^2 + \|\varphi g'\| \left| \int_x^t \Big| \int_x^u \frac{dv}{\varphi(v)} \Big| \,|t-u|\,du \right| \\
\le\ & 2\|f''-g\|(t-x)^2 \\
& + \|\varphi g'\| \left| \int_x^t \Big| \int_x^u \frac{|u-x|^{1/2}}{\varphi(x)} \frac{dv}{|u-v|^{1/2}} \Big| \,|t-u|\,du \right| \\
=\ & 2\|f''-g\|(t-x)^2 + 2\|\varphi g'\|\varphi^{-1}(x) \left| \int_x^t |u-x|\,|t-u|\,du \right| \\
\le\ & 2\|f''-g\|(t-x)^2 + 2\|\varphi g'\|\varphi^{-1}(x)|t-x|^3, \qquad (2.5)
\end{aligned}
$$

where we have used the inequality $\frac{|u-v|}{\varphi^2(v)} \le \frac{|u-x|}{\varphi^2(x)}$, v is between u and x (see [1, p. 141]).

On the other hand, by [2, p. 440], we have the following property: for any $m = 1, 2, \ldots$ and $0 < q < 1$, there exists a constant $C(m) > 0$ such that

$$
|B_{n,q}((t-x)^m, x)| \le C(m)\, \frac{\varphi^2(x)}{[n]_q^{\lfloor (m+1)/2 \rfloor}}, \qquad (2.6)
$$

where $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$ and $\lfloor a \rfloor$ is the integer part of $a \ge 0$ (see also [4, (4.2) and (5.6)]).

Now combining (2.4), (2.5), (2.6) and the Cauchy-Schwarz inequality, we find that

$$
\begin{aligned}
& \Big| (B_{n,q_n} f)(x) - f(x) - \frac{1}{2}[n]_{q_n}^{-1} x(1-x) f''(x) \Big| \\
\le\ & 2\|f''-g\|\, B_{n,q_n}((t-x)^2, x) + 2\|\varphi g'\|\, \varphi^{-1}(x)\, B_{n,q_n}(|t-x|^3, x) \\
\le\ & 2\|f''-g\|\, B_{n,q_n}((t-x)^2, x) \\
& + 2\|\varphi g'\|\, \varphi^{-1}(x) (B_{n,q_n}((t-x)^2, x))^{1/2} (B_{n,q_n}((t-x)^4, x))^{1/2} \\
\le\ & C \left\{ \|f''-g\| \frac{1}{[n]_{q_n}}\, \varphi^2(x) + \|\varphi g'\| \varphi^{-1}(x)\, \frac{\varphi(x)}{[n]_{q_n}^{1/2}}\, \frac{\varphi(x)}{[n]_{q_n}} \right\} \\
=\ & \frac{C}{[n]_{q_n}} \left\{ \|f''-g\|\, \varphi^2(x) + \|\varphi g'\|\, \frac{\varphi(x)}{[n]_{q_n}^{1/2}} \right\}. \qquad (2.7)
\end{aligned}
$$

Because $\varphi^2(x) \le \varphi(x) \le 1$, $x \in [0,1]$, we obtain, in view of (2.7),

$$\begin{aligned} &\left| \, [n]_{q_n} \left\{ (B_{n,q_n} f)(x) - f(x) \right\} - \frac{1}{2} x(1-x) f''(x) \, \right| \\ &\quad \le \quad C \left\{ \| f'' - g \| + \frac{\varphi(x)}{[n]_{q_n}^{1/2}} \, \| \varphi g' \| \right\} \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} &\left| \, [n]_{q_n} \left\{ (B_{n,q_n} f)(x) - f(x) \right\} - \frac{1}{2} x(1-x) f''(x) \, \right| \\ &\quad \le \quad C \, \varphi(x) \left\{ \| f'' - g \| + \frac{1}{[n]_{q_n}^{1/2}} \, \| \varphi g' \| \right\}, \end{aligned} \tag{2.9}$$

respectively. Taking the infimum on the right hand side of (2.8) and (2.9) over all $g \in W^1(\varphi)$, we obtain

$$\left| \, [n]_{q_n} \left\{ (B_{n,q_n} f)(x) - f(x) \right\} - \frac{1}{2} x(1-x) f''(x) \, \right| \le \begin{cases} C \, K_{1,\varphi}(f'', \varphi(x) [n]_{q_n}^{-1/2}) \\ C \, \varphi(x) K_{1,\varphi}(f'', [n]_{q_n}^{-1/2}). \end{cases}$$

Hence, by (1.3), we find the estimates (2.1) and (2.2). Thus the theorem is proved. □

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