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Almost greedy uniformly bounded orthonormal bases in rearrangement invariant Banach function spaces

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Abstract. We construct uniformly bounded orthogonal almost greedy bases in rearrangement invariant Banach spaces.

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1. Introduction

Let $\{x_n\}_{n\in\mathbb{N}}$ be a semi-normalized basis in a Banach space X. This means that $\{x_n\}_{n\in\mathbb{N}}$ is a Schauder basis and is semi-normalized i.e. $0 < \inf_{n\in\mathbb{N}} \|x_n\| \leq \sup_{n\in\mathbb{N}} \|x_n\| < \infty$. For an element $x \in X$ we define the error of the best $m-$term approximation as follows

$$\sigma_m(x) = \inf\{\|x - \sum_{n\in A} \alpha_n x_n\|\},$$

where the inf is taken over all subsets $A \subset \mathbb{N}$ of cardinality at most m and all possible scalars α_n. The main question in approximation theory concerns the construction of efficient algorithms for m-term approximation. A computationally efficient method to produce m-term approximations, which has been widely investigated in recent years, is the so called greedy algorithm. We define the greedy approximation of $x = \sum_n a_n x_n \in X$ as

$$\mathcal{G}_m(x) = \sum_{n\in A} a_n x_n,$$

where $A \subset \mathbb{N}$ is any set of the cardinality m in such a way that $|a_n| \geq |a_l|$ whenever $n \in A$ and $l \overline{\in} A$. We say that a semi-normalized basis $\{x_n\}_{n\in\mathbb{N}}$ is

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greedy if there exists a constant C such that for all $m = 1, 2, ...$ and all $x \in X$ we have

$$\|x - \mathcal{G}_m(x)\| \leq C\sigma_m(x).$$

This notion evolved in theory of non-linear approximation (see e.g.[1],[2]). A result of Konyagin and Temlyakov [3] characterizes greedy bases in a Banach spaces X as those which are unconditional and democratic, the latter meaning that for some constant $C > 0$

$$\left\| \sum_{\alpha \in A} \frac{x_\alpha}{\|x_\alpha\|} \right\| \leq C \left\| \sum_{\alpha \in A'} \frac{x_\alpha}{\|x_\alpha\|} \right\|$$

holds for all finite sets of indices A, $A' \subset \mathbb{N}$ with the same cardinality.

Wavelet systems are well known examples of greedy bases for many function and distribution spaces. Indeed, Temlyakov showed in [1] that the Haar system is greedy in the Lebesgye spaces $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. When wavelets have sufficient smoothness and decay, they are also greedy bases for the more general Sobolev and Tribel-Lizorkin classes (see e.g.[4-5]).

A bounded Schauder basis for a Banach space X is called quasi-greedy if there exists a constant C such that for $x \in X$ $\|\mathcal{G}_m(x)\| \leq C\|x\|$ for $m \geq 1$.

Wojtaszczyk [2] proved the following result which gives a more intuitive interpretation of quasi-greedy bases.

Theorem 1.1. *A bounded Schauder basis for a Banach space X is quasi-greedy if and only if* $\lim_{m\to\infty} \|x - \mathcal{G}_m(x)\|_X = 0$ *for every element $x \in X$.*

A bounded Schauder basis for a Banach space X is almost greedy if there exists a constant C such that for $x \in X$, $\|x - \mathcal{G}_m(x)\| \leq C \inf\{\|x - \sum_{n\in A} < x, x_n > x_n\| : A \subset \mathbb{N},\ |A| = m\}$.

It was proved in [6] that a basis is almost greedy if and only if it is quasi-greedy and democratic.

A Banach function space on $[0, 1]$ is said to be a rearrangement invariant (r.i) space provided $f^*(t) \leq g^*(t)$ for every $t \in [0, 1]$ and $g \in X$ imply $f \in X$ and $\|f\|_X \leq \|g\|_X$, where $f^*(t)$ denotes the decreasing rearrangement of $|f|$.

An r.i. space X with a norm $\|\cdot\|_X$ has the Fatou property if for any increasing positive sequence f_n in X with $\sup_n \|f_n\|_X < \infty$ we have that $\sup_n f_n \in X$ and $\|\sup_n f_n\|_X = \sup_n \|f_n\|_X$. We will assume that the r.i. space X has the Fatou property.

Given $s > 0$, the dilation operator σ_s given by

$$\sigma_s f(t) = f(t/s)\chi_{[0,1]}(t/s),\ t \in [0, 1]$$

(χ_A denotes the characteristic function of a measurable set $A \subset [0, 1]$) is well defined in every r.i. space X. The classical Boyd indices of X are defined by

$$p_X = \lim_{s\to\infty} \frac{\ln s}{\ln \|\sigma_s\|_{X\to X}},\quad q_X = \lim_{s\to 0+} \frac{\ln s}{\ln \|\sigma_s\|_{X\to X}}.$$

In general, $1 \leq p_X \leq q_X \leq \infty$.

Any r.i. function space X on $[0,1]$ satisfies $L^\infty([0,1]) \subset X \subset L^1([0,1])$. If we have information on the Boyd indices of X then a stronger assertion is valid. Indeed for every $1 \le p < p_X$ and $q_X < q < \infty$, we have

$$L^q([0,1]) \subset X \subset L^p([0,1]) \tag{1.1}$$

with the inclusion maps being continuous. Let X' denote the associate Banach function space of X. Then X' is a r.i. Banach function space whose Boyd indices are defined as $1/p_X + 1/q_{X'} = 1$ and $1/q_X + 1/p_{X'} = 1$ (see [7]).

M. Nielsen in [8] proved that there exists a uniformly bounded orthonormal almost greedy basis in $L^p([0,1])$, $1 < p < \infty$, that shows that it is not possible to extend Orlicz's theorem, stating that there are no uniformly bounded orthonormal unconditional bases for $L^p([0,1])$, $p \neq 2$, to the class of almost greedy bases.

The purpose of this paper is to study these problems in the r.i. function spaces. Namely, the following theorem is obtained.

Theorem 1.2. *Let X be a separable r.i. Banach function space on $[0,1]$ and $1 < p_X \le q_X < 2$ or $2 < p_X \le q_X < \infty$. Then there exists a uniformly bounded orthogonal almost greedy basis in X.*

2. Proof of theorem

Let us construct some system in the following way. For $k = 1,2,...$, we define the $2^k \times 2^k$ Olevskii matrix $A^k = (a_{ij}^{(k)})_{i,j=1}^{2^k}$ by the following formulas

$$a_{i1}^k = 2^{-\frac{k}{2}} \quad \text{for} \quad i = 1,2,...,2^k,$$

and for $j = 2^s + \nu$, with $1 \le \nu \le 2^s$ and $s = 0,1,...,k-1$, we let

$$a_{ij}^{(k)} = \begin{cases} 2^{\frac{s-k}{2}} & \text{for} \quad (\nu-1)2^{k-s} < i \le (2\nu-1)2^{k-s-1} \\ -2^{\frac{s-k}{2}} & \text{for} \quad (2\nu-1)2^{k-s-1} < i \le \nu 2^{k-s} \\ 0 & \text{otherwise.} \end{cases}$$

It is known [16] that A^k are orthogonal matrices and there exists a finite constant C such that for all i,k we have

$$\sum_{j=1}^{2^k} |a_{i,j}^{(k)}| \le C.$$

Put $N_k = 2^{10^k}$ and define F_k such that $F_0 = 0$, $F_1 = N_1 - 1$ and $F_k - F_{k-1} = N_k - 1$, $k = 1,2,....$ We consider the Walsh system $\mathcal{W} = \{W_n\}_{n=0}^\infty$ on $[0,1]$. We split $\mathcal{W}$ into two subsystems. The first subsystem $\mathcal{W}_1 = \{r_k\}_{k=1}^\infty$ is Rademacher functions with their natural ordering. The second subsystem $\mathcal{W}_2 = \{\phi_k\}_{k=1}^\infty$ is the collection of Walsh functions not in $\mathcal{W}_1$ with the ordering from $\mathcal{W}$. We now impose the ordering

$$\phi_1, r_1, r_2, ..., r_{F_1}, \phi_2, r_{F_1+1}, ..., r_{F_2}, \phi_3, r_{F_2+1}, ..., r_{F_3}, \phi_4, ...$$

The block $\mathcal{B}_k := \{\phi_k, r_{F_{k-1}+1}, ..., r_{F_k}\}$ has length N_k and we apply A^{10^k} to $\mathcal{B}_k$ to obtain a new orthonormal system $\{\psi_i^{(k)}\}_{i=1}^{N_k}$ given by

$$\psi_i^{(k)} = \frac{\phi_k}{\sqrt{N_k}} + \sum_{j=2}^{N_k} a_{ij}^{(10^k)} r_{F_{k-1}+j-1}.$$

The system ordered $\psi_1^{(1)}, ..., \psi_{N_1}^{(1)}, \psi_1^{(2)}, ..., \psi_{N_2}^{(2)}, ...$ will be denoted by $\mathcal{B} = \{\psi_k\}_{k=1}^{\infty}$. It is easy to verify that $\mathcal{B}$ is an orthonormal basis for L_2 since each matrix A^{10^k} is orthogonal and it is uniformly bounded also.

Lemma 2.1. *Let X be a r.i. Banach function space on $[0,1]$ and $1 < p_X \leq q_X < \infty$. The system $\mathcal{B} = \{\psi_k\}_{k=1}^{\infty}$ is democratic in X with*

$$\|\sum_{k\in A} \psi_k\|_X \asymp |A|^{\frac{1}{2}}.$$

Proof. Taking into account that fact that $B\|\cdot\|_{p_X} \leq \|\cdot\|_X \leq C\|\cdot\|_{q_X}$ and the estimate (see [8])

$$\|\sum_{k\in A} \psi_k\|_p \asymp |A|^{\frac{1}{2}} \text{ for any } 1 < p < \infty$$

we obtain our result. □

Lemma 2.2. *(Khintchine's inequality)Suppose that X is a r.i. Banach function space on $[0,1]$, $1 < p_X \leq q_X < \infty$,and $r_k(t), k \geq 1$, are the Rademacher functions. Then there exist A, B such that for any sequence $\{a_k\}_{k\geq 1}$,*

$$A(\sum_k |a_k|^2)^{\frac{1}{2}} \leq \|\sum_k a_k r_k(t)\|_X \leq B(\sum_k |a_k|^2)^{\frac{1}{2}}.$$

Proof. It is known that (see [10]) for $1 \leq p < \infty$ there exist A_p, B_p such that for any sequence $\{a_k\}_{k\geq 1}$,

$$A_p(\sum_k |a_k|^2)^{\frac{1}{2}} \leq \|\sum_k a_k r_k(t)\|_p \leq B_p(\sum_k |a_k|^2)^{\frac{1}{2}}.$$

Taking into account that fact that $B\|\cdot\|_{q_X} \leq \|\cdot\|_X \leq C\|\cdot\|_{p_X}$ and the above inequality we obtain Lemma 2.2. □

Lemma 2.3. *Suppose that X is a r.i. Banach function space on $[0,1]$, $1 < p_X \leq q_X < \infty$, and $r_k(t), k \geq 1$, are the Rademacher functions. Then for $f \in X$ we have*

$$(\sum_{k=1}^{\infty} |< f, r_k >|^2)^{\frac{1}{2}} \leq C\|f\|_X.$$

Proof. For any $n \geq 1$ by the Hölder inequality and Khintchine's inequality we obtain

$$\sum_{k=1}^{n} |< f, r_k >|^2 = \int_0^1 f(x)(\sum_{k=1}^{n} r_k(x) < f, r_k >)dx \leq$$
$$2\|\sum_{k=1}^{n} < f, r_k > r_k\|_{X'}\|f\|_X \leq C(\sum_{k=1}^{n} |< f, r_k >|^2)^{1/2}\|f\|_X.$$

This implies

$$(\sum_{k=1}^{n} |< f, r_k >|^2)^{\frac{1}{2}} \leq B\|f\|_X.$$

Now taking the limit when $n \to \infty$ we obtain our result. □

Lemma 2.4. *Let X be a separable r.i. Banach function space on $[0,1]$ and $1 < p_X \leq q_X < \infty$. Then the system $\mathcal{B} = \{\psi_k\}_{k=1}^{\infty}$ is a Schauder basis for X.*

Proof. Notice that $span(\mathcal{B}) = span(\mathcal{W})$ by construction, so $span(\mathcal{B})$ is dense in X, since $\mathcal{W}$ is a Schauder basis for X (see [11]).

Let $S_n(f) = \sum_{k=1}^{n} < f, \psi_k > \psi_k$ be the partial sum operator. We need to prove that the family of operators $\{S_n\}_{n=1}^{\infty}$ is uniformly bounded on X. Let $f \in L^\infty([0,1]) \subset L^2([0,1])$. For $n \in \mathbb{N}$ we can find $L \geq 1$ and $1 \leq m \leq N_L$ such that

$$S_n(f) = \sum_{k=1}^{n} < f, \psi_k > \psi_k = \sum_{k=1}^{L-1}\sum_{j=1}^{N_k} < f, \psi_j^{(k)} > \psi_j^{(k)} + \sum_{k=1}^{m} < f, \psi_k^{(L)} > \psi_k^{(L)}$$

$$:= T_1 + T_2.$$

Let us estimate T_1. If $L = 1$ then $T_1 = 0$, so we may assume $L > 1$. The construction of $\mathcal{B}$ shows that T_1 is the orthogonal projection of f onto

$$span\left(\cup_{k=1}^{L-1} \cup_{j=1}^{N_k} \psi_k^{(k)}\right) = span\{\{W_0, W_1, ..., W_{L-2}\} \cup \{r_{l_0}, r_{l_0+1}, ..., r_{F_{L-1}}\}\},$$

with $l_0 = [\log_2(L)]$. It follows that we can rewrite T_1 as

$$T_1 = \sum_{k=0}^{L-2} < f, W_k > W_k + P_R(f),$$

where $P_R(f)$ is the orthogonal projection of f onto $span\{r_{l_0}, r_{l_0+1}, ..., r_{F_{L-1}}\}$. Thus, using the fact that $\mathcal{W}$ is a Schauder basis for X, Khintchine's inequality and Lemma 2.3, we will have

$$\|T_1\|_X \leq C\|f\|_X.$$

Let us now estimate T_2.

$$T_2 = \sum_{k=1}^{m} < f, \psi_k^{(L)} > \psi_k^{(L)}$$

$$= \sum_{k=1}^{m} < f, \frac{\phi_L}{\sqrt{N_L}} + \sum_{j=2}^{N_L} a_{kj}^{(10^L)} r_{F_{L-1}+j-1} >= (\frac{\phi_L}{\sqrt{N_L}}\phi_L + \sum_{t=2}^{N_L} a_{kt}^{(10^L)} r_{F_{L-1}+t-1})$$

$$= \frac{m}{N_L} < f, \phi_L > + \frac{\phi_L}{\sqrt{N_L}} \sum_{j=2}^{N_L}(\sum_{k=1}^{m} a_{kj}^{(10^L)}) < f, r_{F_{L-1}+j-1} >$$

$$+ < f, \frac{\phi_L}{\sqrt{N_L}} > \sum_{j=2}^{N_L}(\sum_{k=1}^{m} a_{kj}^{(10^L)}) r_{F_{L-1}+j-1}$$

$$+\sum_{k=1}^{m}[\sum_{j=2}^{N_L} a_{kj}^{(10^L)} < f, r_{F_{L-1}+j-1} >][\sum_{t=2}^{N_L} a_{kt}^{(10^L)} r_{F_{L-1}+t-1}]$$

$$= G_1 + G_2 + G_3 + G_4.$$

Using that fact that $1 \le m \le N_L$ and Hölder inequality we obtain $\|G_1\|_X \le C\|f\|_X$. Using the Hölder and Khintchine's inequality, the fact that matrices A^k are orthonormal and Lemma 2.3 we obtain $\|G_i\|_X \le C\|f\|_X$ $i = 2, 3, 4$ for some constant C independent of $f \in L^\infty([0,1])$. Consequently for some constant C independent on $f \in L^\infty([0,1])$ we have $\|S_n f\|_X \le C\|f\|_X$. Since $L^\infty([0,1])$ is dense in X we deduce that $\{S_n\}_{n=1}^\infty$ is a uniformly bounded family of linear operators on X and the system $\mathcal{B}$ is a Schauder basis for X. □

Lemma 1.1 and Lemma 2.4 give the following

Theorem 2.5. *Let X be a separable r.i. Banach function space on $[0,1]$ and $1 < p_X \le q_X < \infty$. Then there exists a uniformly bounded orthonormal democratic basis in X.*

Lemma 2.6. *Let X be a separable r.i. Banach function space on $[0,1]$ and $1 < p_X \le q_X < 2$ or $2 < p_X \le q_X < \infty$. Then the system $\mathcal{B} = \{\psi_k\}_{k=1}^\infty$ is a quasi-greedy basis for X.*

Proof. First we consider $2 < p_X \le q_X < \infty$ case. Let $f \in X \subset L_2$. We have

$$f = \sum_{i=1}^{\infty} < f, \psi_i > \psi_i,$$

with $\|\{< f, \psi_i >\}\|_{l_2} \le \|f\|_2 \le C\|f\|_X$. We must prove that $\mathcal{G}_m(f)$ is convergent in X.

Let us formally write

$$f = \sum_{k=1}^{\infty}\sum_{j=1}^{N_k} < f, \psi_j^{(k)} > \psi_j^{(k)}$$

$$= \sum_{k=1}^{\infty}\sum_{j=1}^{N_k} < f, \psi_j^{(k)} > \frac{\phi_k}{\sqrt{N_k}} + \sum_{k=1}^{\infty}\sum_{i=1}^{N_k} < f, \psi_i^{(k)} > \sum_{j=2}^{N_k} a_{ij}^{(10^k)} r_{F_{k-1}+j-1}$$

$$= S_1 + S_2.$$

Consider $\varepsilon_i^k \subset \{0,1\}$. By Kchintchine's inequality and the fact that each A^{10^k} is orthogonal we conclude that S_2 converges unconditionally in X. Indeed

$$\left\| \sum_{k=1}^{\infty}\sum_{j=2}^{N_k} \left(\sum_{i=1}^{N_k} \varepsilon_i^k < f, \psi_i^{(k)} > a_{ij}^{(10^k)} \right) r_{F_{k-1}+j-1} \right\|_X$$

$$\le C \left(\sum_k \sum_{i=1}^{N_k} \varepsilon_i^k | < f, \psi_i^{(k)} > |^2 \right)^{1/2}.$$

The series defining S_2 converges unconditionally, so it suffices to prove that the series defining S_1 converges in X when the coefficients $< f, \psi_>$ are arranged in decreasing order. Let us consider the sets

$$\Lambda_k^1 = \left\{ j : \frac{1}{N_k} < | < f, \psi_j^{(k)} > | < \frac{1}{N_k^{1/10}} \right\}$$
$$\Lambda_k^2 = \left\{ j : | < f, \psi_j^{(k)} > | \leq \frac{1}{N_k} \right\}$$
$$\Lambda_k^3 = \left\{ j : | < f, \psi_j^{(k)} > | \geq \frac{1}{N_k^{1/10}} \right\}.$$

Then

$$S_1 = \sum_{k=1}^{\infty} \sum_{j \in \Lambda_k^1} < f, \psi_j^{(k)} > \frac{\phi_k}{\sqrt{N_k}} + \sum_{k=1}^{\infty} \sum_{j \in \Lambda_k^2} < f, \psi_j^{(k)} > \frac{\phi_k}{\sqrt{N_k}} +$$
$$\sum_{k=1}^{\infty} \sum_{j \in \Lambda_k^3} < f, \psi_j^{(k)} > \frac{\phi_k}{\sqrt{N_k}} = T_1 + T_2 + T_3.$$

By the construction of sets Λ_k^i we can conclude that the series defining T_2 and T_3 converges absolutely in X.

From the definition of Λ_k^1 we get

$$| < f, \psi_i^{(k)} > | > \frac{1}{N_k} \geq \frac{1}{N_{k+1}^{1/10}} \geq | < f, \psi_j^{(k+1)} > |,$$

$i \in \Lambda_k^1$, $j \in \Lambda_{k+1}^1$, $k = 1, 2, ...$ so when we arrange T_1 by decreasing order the rearrangement can only take place inside the blocks. From the estimate

$$\sum_{j \in \Lambda_k^1} \left\| < f, \psi_j^{(k)} > \frac{\phi_k}{\sqrt{N_k}} \right\|_X \leq \left(\sum_{j \in \Lambda_k^1} | < f, \psi_j^{(k)} > |^2 \right)^{1/2} \frac{|\Lambda_k^1|^{1 \backslash 2}}{\sqrt{N_k}}, \ k \geq 1$$

we obtain that the rearrangements inside blocks are well-behaved, and

$$\sum_{j \in \Lambda_k^1} \left\| < f, \psi_j^{(k)} > \frac{\phi_k}{\sqrt{N_k}} \right\|_X \to 0, \ k \to \infty.$$

We can conclude that $\mathcal{G}_m(f)$ is convergent in X.

Using Theorem 1.1 we conclude that $\mathcal{B}$ is a quasi-greedy basis and consequently almost greedy in X.

Let $1 < p_X \leq q_X < 2$. By the results proved above it follows that the system $\mathcal{B}$ is almost greedy in X. From [6, Theorem 5.4] we conclude that $\mathcal{B}$ is quasi-greedy basis and consequently almost greedy in X This completes the proof. □

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