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Approximation by max-product Lagrange interpolation operators

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Abstract. The aim of this note is to associate to the Lagrange interpolatory polynomials on various systems of nodes (including the equidistant and the Jacobi nodes), continuous piecewise rational interpolatory operators of the so-called max-product kind, uniformly convergent to the function f, with Jackson-type rates of approximation.

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1. Introduction

Based on the Open Problem 5.5.4, pp. 324-326 in [12], in a series of recent papers we have introduced and studied the so-called max-product operators attached to the Bernstein polynomials and to other linear Bernstein-type operators, like those of Favard-Szász-Mirakjan operators (truncated and non-truncated case), see [3], Baskakov operators (truncated and nontruncated case), Meyer-König and Zeller operators, see [4] and Bleimann-Butzer-Hahn operators, see [5].

For example, in the two recent papers [1], [2], starting from the linear Bernstein operators

$$B_n(f)(x) = \sum_{k=0}^{n} b_{n,k}(x) f(k/n),$$

where $b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, written in the equivalent form

$$B_n(f)(x) = \frac{\sum_{k=0}^{n} b_{n,k}(x) f(k/n)}{\sum_{k=0}^{n} b_{n,k}(x)}$$

and then replacing the sum operator Σ by the maximum operator $\bigvee$, one obtains the nonlinear Bernstein operator of max-product kind

$$B_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{n} b_{n,k}(x)},$$

where the notation $\bigvee_{k=0}^{n} b_{n,k}(x)$ means $max\{b_{n,k}(x); k \in \{0, ..., n\}\}$ and similarly for the numerator.

For this max-product operator nice approximation and shape preserving properties were found in e.g. [2].

In other two recent papers [9] and [10], this idea is applied to the Lagrange interpolation based on the Chebyshev nodes of second kind plus the endpoints, and to the Hermite-Fejér interpolation based on the Chebyshev nodes of first kind respectively, obtaining max-product interpolation operators which, in general, (for example, in the class of positive Lipschitz functions) approximates essentially better than the corresponding Lagrange and Hermite-Fejér interpolation polynomials.

The aim of the present paper is to use the same idea (but slightly modified to simplify the calculation) in the case of the linear interpolation polynomials of Lagrange type on general nodes. Applications to Lagrange interpolation based on equidistant knots and on the roots of orthogonal polynomials, including the Jacobi roots, are obtained.

Thus, let $I \subset \mathbb{R}$ be a bounded or unbounded interval, $f : I \to \mathbb{R}$, $x_{n,k} \in I$, $k \in \{0, ..., n\}$, $x_{n,0} < x_{n,1} < ... < x_{n,n}$, and consider the Lagrange interpolation polynomial of degree $\leq n$ attached to f and to the nodes $(x_{n,k})_k$,

$$P_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f(x_{n,k}),$$

with

$$p_{n,k}(x) = \frac{(x - x_{n,0})...(x - x_{n,k-1})(x - x_{n,k+1})...(x - x_{n,n})}{(x_{n,k} - x_{n,0})...(x_{n,k} - x_{n,k-1})(x_{n,k} - x_{n,k+1})...(x_{n,k} - x_{n,n})}.$$

It is well known that $\sum_{k=0}^{n} p_{n,k}(x) = 1$, for all $x \in \mathbb{R}$, which allows us to write

$$P_n(f)(x) = \frac{\sum_{k=0}^{n} p_{n,k}(x) f(x_{n,k})}{\sum_{k=0}^{n} p_{n,k}(x)}, \text{ for all } x \in I.$$

Therefore, its corresponding max-product interpolation operator will be obtained by replacing the sum operator Σ, by the maximum operator $\bigvee$, that is

$$P_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} p_{n,k}(x) f(x_{n,k})}{\bigvee_{k=0}^{n} p_{n,k}(x)}, x \in I.$$

By the property $p_{n,k}(x_{n,j}) = 1$ if $k = j$ and $p_{n,k}(x_{n,j}) = 0$ if $k \neq j$, we immediately obtain that $P_n^{(M)}(f)(x_{n,j}) = f(x_{n,j})$, for all $j \in \{0, ..., n\}$.

But because this max-product operator seems to present some difficulties in calculations, in this paper we deal with a simplified max-product operator with good approximation properties and which keeps the interpolation properties, given by

$$L_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} l_{n,k}(x) f(x_{n,k})}{\bigvee_{k=0}^{n} l_{n,k}(x)}, x \in I,$$

where

$$l_{n,k}(x) = c_{n,k} \cdot p_{n,k}(x) = (-1)^{n-k} \Pi_{i=0}^{n} (x - x_{n,i})/(x - x_{n,k}) \tag{1.1}$$

and

$$c_{n,k} = (x_{n,k} - x_{n,0})...(x_{n,k} - x_{n,k-1})(x_{n,k+1} - x_{n,k})...(x_{n,n} - x_{n,k}) > 0.$$

The plan of the paper goes as follows. In Section 2 we present some auxiliary results while in Section 3 we prove the approximation results for the max-product Lagrange interpolation operators on equidistant and Jacobi nodes.

2. Auxiliary results

Let us define the space

$$CB_+(I) = \{f : I \to \mathbb{R}_+; f \text{ is continuous and bounded on } I\}.$$

Remark. Firstly, it is clear that $L_n^{(M)}(f)(x)$ is a well-defined function for all $x \in \mathbb{R}$ and it is continuous on $\mathbb{R}$. Indeed, by $\sum_{k=0}^{n} p_{n,k}(x) = 1$, for all $x \in \mathbb{R}$, for any x there exists an index $k \in \{0, ..., n\}$ such that $p_{n,k}(x) > 0$ (which implies that $\bigvee_{k=0}^{n} p_{n,k}(x) > 0$), because contrariwise would follow that $p_{n,k}(x) \leq 0$ for all k and therefore we would obtain the contradiction $\sum_{k=0}^{n} p_{n,k}(x) \leq 0$. Therefore, as $l_{n,k}(x) = c_{n,k} \cdot p_{n,k}(x)$ with $c_{n,k} > 0$, for this k we also have $\bigvee_{k=0}^{n} l_{n,k}(x) > 0$.

Also, by the obvious property $l_{n,k}(x_{n,j}) = c_{n,j} > 0$ if $k = j$ and $l_{n,k}(x_{n,j}) = 0$ if $k \neq j$, we immediately obtain that $L_n^{(M)}(f)(x_{n,j}) = f(x_{n,j})$, for all $j \in \{0, ..., n\}$. In addition, clearly we have $L_n^{(M)}(e_0)(x) = 1$, where $e_0(x) = 1$, for all $x \in I$.

In what follows we will see that for $f \in CB_+[a, b]$, the $L_n^{(M)}(f)$ operator fulfils similar properties with those of the $B_n^{(M)}(f)$ operator in [1].

Lemma 2.1. *Let $I \subset \mathbb{R}$ be a bounded or unbounded interval.*

(i) Then $L_n^{(M)} : CB_+(I) \to CB_+(I)$, for all $n \in \mathbb{N}$:

(ii) If $f, g \in CB_+(I)$ satisfy $f \leq g$ then $L_n^{(M)}(f) \leq L_n^{(M)}(g)$ for all $n \in N$;

(iii) $L_n^{(M)}(f+g) \le L_n^{(M)}(f) + L_n^{(M)}(g)$ *for all* $f, g \in CB_+(I)$ *;*

(iv) For all $f, g \in CB_+(I)$, $n \in N$ *and* $x \in I$ *we have*

$$|L_n^{(M)}(f)(x) - L_n^{(M)}(g)(x)| \le L_n^{(M)}(|f-g|)(x);$$

(v) $L_n^{(M)}$ *is positive homogenous, that is* $L_n^{(M)}(\lambda f) = \lambda L_n^{(M)}(f)$ *for all* $\lambda \ge 0$ *and* $f \in CB_+(I)$.

Proof. (i) The continuity of $L_n^{(M)}(f)(x)$ on I follows from the previous Remark. Also, by the formula of definition for $L_n^{(M)}(f)(x)$, if f is bounded on I, then it easily follows that $L_n^{(M)}$ is bounded on I. It remains to prove the positivity of $L_n^{(M)}(f)$. So let $f : I \to \mathbb{R}_+$ and fix $x \in I$. Reasoning exactly as in the above Remark, there exists $k \in \{0, 1, ..., n\}$ such that $l_{n,k}(x) > 0$. Therefore, denoting $I_n^+(x) = \{k \in \{0, 1, ..., n\}; l_{n,k}(x) > 0\}$, clearly $I_n^+(x)$ is nonempty and for $f \in CB_+(I)$ we get that

$$L_n^{(M)}(f)(x) = \frac{\bigvee_{k \in I_n^+(x)} l_{n,k}(x) f(x_{n,k})}{\bigvee_{k \in I_n^+(x)} l_{n,k}(x)} \ge 0. \tag{2.1}$$

(ii) Let $f, g \in CB_+(I)$ be with $f \le g$ and fix $x \in I$. Since $I_n^+(x)$ is independent of f and g, by (2.1) we immediately obtain $L_n^{(M)}(f)(x) \le L_n^{(M)}(g)(x)$.

(iii) By (2.1) and by the sublinearity of $\bigvee$, it is immediate.

(iv) Let $f, g \in CB_+(I)$. We have $f = f - g + g \le |f-g| + g$, which by $(i) - (iii)$ successively implies $L_n^{(M)}(f)(x) \le L_n^{(M)}(|f-g|)(x) + L_n^{(M)}(g)(x)$, that is $L_n^{(M)}(f)(x) - L_n^{(M)}(g)(x) \le L_n^{(M)}(|f-g|)(x)$.

Writing now $g = g - f + f \le |f-g| + f$ and applying the above reasonings, it follows $L_n^{(M)}(g)(x) - L_n^{(M)}(f)(x) \le L_n^{(M)}(|f-g|)(x)$, which combined with the above inequality gives $|L_n^{(M)}(f)(x) - L_n^{(M)}(g)(x)| \le L_n^{(M)}(|f-g|)(x)$.

(v) By (2.1) it is immediate. □

Remark. By (2.1) it is easy to see that instead of (ii), $L_n^{(M)}$ satisfies the stronger condition

$$L_n(f \vee g)(x) = L_n(f)(x) \vee L_n(g)(x),\ \ f, g \in CB_+(I).$$

Corollary 2.2. *For all* $f \in CB_+(I)$, $n \in N$ *and* $x \in I$ *we have*

$$|f(x) - L_n^{(M)}(f)(x)| \le \left[\frac{1}{\delta} L_n^{(M)}(\varphi_x)(x) + 1\right] \omega_1(f; \delta)_I,$$

where $\delta > 0$, $\varphi_x(t) = |t - x|$ *for all* $t \in I$, $x \in I$ *and* $\omega_1(f;\delta)_I = \max\{|f(x) - f(y)|; x, y \in I, |x-y| \le \delta\}$.

Proof. Indeed, denoting $e_0(x) = 1$, from the identity

$$L_n^{(M)}(f)(x) - f(x) = [L_n^{(M)}(f)(x) - f(x) \cdot L_n^{(M)}(e_0)(x)] + f(x)[L_n^{(M)}(e_0)(x) - 1],$$

by Lemma 2.1 it easily follows

$$|f(x) - L_n^{(M)}(f)(x)| \le$$

$$|L_n^{(M)}(f(x))(x) - L_n^{(M)}(f(t))(x)| + |f(x)| \cdot |L_n^{(M)}(e_0)(x) - 1| \leq$$
$$L_n^{(M)}(|f(t) - f(x)|)(x) + |f(x)| \cdot |L_n^{(M)}(e_0)(x) - 1|.$$

Now, since for all $t, x \in I$ we have

$$|f(t) - f(x)| \leq \omega_1(f; |t-x|)_I \leq \left[\frac{1}{\delta}|t-x| + 1\right] \omega_1(f; \delta)_I,$$

replacing above and taking into account that $L_n^{(M)}(e_0) = 1$, for all $x \in I$, we immediately obtain the estimate in the statement. $\square$

Remark. The results in Lemma 2.1 and Corollary 2.2 remain valid if we replace the space $CB_+(I)$ by the space

$$C_+(I) = \{f : I \to \mathbb{R}_+; f \text{ is continuous on } I\}.$$

3. Approximation results for max-product Lagrange interpolation

In this section we study the approximation properties of the max-product operators $L_n^{(M)}$.

It is clear that for the approximation purpose, in the case of the operator $L_n^{(M)}$, from Corollary 2.2 it is enough to obtain a good estimate for the expression

$$E_n(x) := L_n^{(M)}(\varphi_x)(x) = \frac{\bigvee_{k=0}^{n} l_{n,k}(x)\,|x_{n,k} - x|}{\bigvee_{k=0}^{n} l_{n,k}(x)} = \frac{\bigvee_{k \in I_n^+(x)} l_{n,k}(x)|x_{n,k} - x|}{\bigvee_{k \in I_n^+(x)} l_{n,k}(x)}.$$

We present the first main approximation result.

Theorem 3.1. *Given the nodes $-\infty < a \leq x_{n,0} < x_{n,1} < ... < x_{n,n} \leq b < \infty$, $f \in C_+([a,b])$ and denoting*

$$d_n = \max\{x_{n,0} - a, \max\{x_{n,k+1} - x_{n,k}; k = 0, 1, ..., n-1\}, b - x_{n,n}\},$$

we have

$$|L_n^{(M)}(f)(x) - f(x)| \leq 2\omega_1(f; d_n)_{[a,b]}, \text{ for all } x \in [a,b],$$

where $\omega_1(f; \delta)_{[a,b]} = \sup\{|f(x) - f(y)|; x, y \in [a,b], |x-y| \leq \delta\}$.

Proof. Firstly, because $L_n^{(M)}(f)(x_{n,j}) = f(x_{n,j})$, for all $j \in \{0, 1, ..., n\}$, in all calculations and estimations we may suppose that $x \neq x_{n,j}$, for all $j \in \{0, 1, ..., n\}$.

Denote $\Omega_n(x) = \Pi_{i=0}^n (x - x_{n,i})$. It is easy to see that for any $x \in [a,b]$, with $x \neq x_{n,j}$, $j \in \{0, 1, ..., n\}$, we can write

$$E_n(x) = \frac{\bigvee_{k \in I_n^+(x)} l_{n,k}(x)|x_{n,k} - x|}{\bigvee_{k \in I_n^+(x)} l_{n,k}(x)} = \frac{|\Omega_n(x)|}{\bigvee_{k \in I_n^+(x)} l_{n,k}(x)} = \frac{1}{\bigvee_{k \in I_n^+(x)} \frac{1}{|x - x_{n,k}|}}$$

$$= \min\{|x - x_{n,k}|; k \in I_n^+(x)\}.$$

Denote $x_{n,-1} := a$ and $x_{n,n+1} := b$ and fix $j \in \{-1, 0, ..., n, n+1\}$. We have three possibilities : 1) $j = -1$; 2) $0 \le j \le n-1$; 3) $j = n$. Let $x \in (x_{n,j}, x_{n,j+1})$.

Case 1). We may suppose that $a < x_{n,0}$. We have $l_{n,0}(x) > 0$ for all $x \in [a, x_{n,0})$. Indeed, by using (1.1) we easily get that for $x \in [a, x_{n,0})$, we have $sign[l_{n,0}(x)] = (-1)^n \cdot (-1)^n = +1$. Therefore $0 \in I_n^+(x)$, for all $x \in [a, x_{n,0})$. We also get $|x - x_{n,0}| \le |x - x_{n,k}|$, for all $k \in I_n^+(x)$ and $x \in [a, x_{n,0})$, which implies $E_n(x) = |x - x_{n,0}| = x_{n,0} - x \le x_{n,0} - a \le d_n$, for all $x \in [a, x_{n,0})$.

Case 2). We have $l_{n,j}(x) > 0$ and $l_{n,j+1}(x) > 0$ for all $x \in (x_{n,j}, x_{n,j+1})$. Indeed, by using (1.1) we easily get that for $x \in (x_{n,j}, x_{n,j+1})$, we have $sign[l_{n,j}(x)] = (-1)^{n-j} \cdot (-1)^{n-j} = +1$ and $sign[l_{n,j+1}(x)] = (-1)^{n-j-1} \cdot (-1)^{n-j-1} = +1$. Therefore $j, j+1 \in I_n^+(x)$, for all $x \in (x_{n,j}, x_{n,j+1})$.

We also get $|x - x_{n,j}| \le |x - x_{n,k}|$ for all $k \in \{0.1, ..., j\}$ and $|x - x_{n,j+1}| \le |x - x_{n,k}|$ for all $k \in \{j+1, j+2, ..., n\}$, which implies $E_n(x) = \min\{|x - x_{n,j}|, |x - x_{n,j+1}|\} \le \frac{d_n}{2}$, for all $x \in (x_{n,j}, x_{n,j+1})$.

Case 3). We may suppose that $x_{n,n} < b$. We have $l_{n,n}(x) > 0$ for all $x \in (x_{n,n}, b]$. Indeed, by using (1.1) we easily get that for $x \in (x_{n,n}, b]$, we have $sign[l_{n,n}(x)] = (-1)^0 \cdot (-1)^0 = +1$. Therefore $n \in I_n^+(x)$, for all $x \in (x_{n,0}, b]$. We also get $|x - x_{n,n}| \le |x - x_{n,k}|$, for all $k \in I_n^+(x)$ and $x \in (x_{n,n}, b]$, which implies $E_n(x) = |x - x_{n,n}| = x - x_{n,n} \le b - x_{n,n} \le d_n$, for all $x \in (x_{n,n}, b]$.

Collecting all the above estimates and applying Corollary 2.2, the theorem is proved. □

Remark. The order of approximation in terms of $\omega_1(f; d_n)_{[a,b]}$ in Theorem 3.1 cannot be improved, in the sense that it easily follows from the proof of Theorem 3.1, that the estimate $E_n(x) \le \mathcal{O}(d_n)$ cannot be improved.

As applications we obtain the following two results.

Corollary 3.2. *(i) Let $I = [a, b]$, $f \in C_+([a, b])$ and the equidistant knots in $I = [a, b]$, $x_{n,k} = a + kh$, $k \in \{0, ..., n\}$, with $h = (b - a)/n$. Then we have*

$$|L_n^{(M)}(f)(x) - f(x)| \le 2\omega_1\left(f; \frac{b-a}{n}\right)_{[a,b]}, \text{ for all } x \in [a, b].$$

(ii) Let $w(x)$ be a weight function on the finite interval $I = [a, b]$, satisfying $w(x) \ge \nu > 0$, for all $x \in [a, b]$. If $a < x_{n,0} < x_{n,1} < ... < x_{n,n} < b$ are the the zeros of the associated orthonormal polynomial $p_{n+1}(x)$ of degree $\le n+1$, then for any $f \in C_+([a, b])$ we have

$$|L_n^{(M)}(f)(x) - f(x)| \le C\omega_1\left(f; \frac{ln(n+1)}{n+1}\right)_{[a,b]}, \text{ for all } x \in [a, b],$$

where $C > 0$ is a constant depending only on ν, a and b.

(iii) Let $w(x)$ be a weight function on the interval $I = [-1, 1]$, satisfying $A \le \sqrt{1 - x^2}w(x) \le B$, for all $x \in [-1, 1]$, where $A, B > 0$ are constants. If $-1 < x_{n,0} < x_{n,1} < ... < x_{n,n} < 1$ are the the zeros of the associated orthonormal polynomial $p_{n+1}(x)$ of degree $\le n+1$, then for any $f \in C_+([-1, 1])$

we have

$$|L_n^{(M)}(f)(x) - f(x)| \le C\omega_1\left(f; \frac{1}{n+1}\right)_{[-1,1]}, \text{ for all } x \in [-1,1],$$

where $C > 0$ is a constant depending only on A and B.

(iv) If $-\frac{1}{2} \le \alpha \le +\frac{1}{2}$, $-\frac{1}{2} \le \beta \le +\frac{1}{2}$ and $-1 < x_{n,0} < x_{n,1} < ... < x_{n,n} < 1$ are the the zeros of the associated orthonormal Jacobi polynomial $J_{n+1}(x)$ of degree $\le n+1$, associated to the weight $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$, then for any $f \in C_+([-1,1])$ we have

$$|L_n^{(M)}(f)(x) - f(x)| \le C\omega_1\left(f; \frac{1}{n+1}\right)_{[-1,1]}, \text{ for all } x \in [-1,1],$$

where $C > 0$ is a constant depending only on α and β.

Proof. (i) It is immediate from Theorem 3.1 for $d_n = \frac{b-a}{n}$.

(ii) It follows from Theorem 3.1, taking into account that by Theorem 6.11.1, pp. 112-113 in [18], we have $d_n \le c\frac{ln(n+1)}{n+1}$, with $c > 0$ depending on ν, a and b only.

(iii) It follows from Theorem 3.1, taking into account that by Theorem 6.11.2, p. 114 in [18], we have $d_n \le c\frac{1}{n+1}$, with $c > 0$ depending on A and B only.

(iv) It follows from Theorem 3.1, taking into account that by Theorem 6.3.1, p. 125 in [18], we have $d_n \le c\frac{1}{n+1}$, with $c > 0$ depending on α and β only. □

It is of interest to have a more explicit form for the operator $L_n(f)(x)$ in Theorem 3.1. In this sense we present the following.

Theorem 3.3. *Given $f \in C_+([a,b])$ and the nodes $-\infty < a \le x_{n,0} < x_{n,1} < ... < x_{n,n} \le b < \infty$, the max-product operator $L_n^{(M)}(f)(x)$ is continuous on $[a,b]$, $L_n^{(M)}(f)(x_{n,j}) = f(x_{n,j})$ for all $j \in \{0,1,...,n\}$ and we can write :*

$$L_n^{(M)}(f)(x) = \bigvee_{k=0}^{n} (-1)^k \frac{x - x_{n,0}}{x - x_{n,k}} f(x_{n,k}), \text{ for } x \in [a, x_{n,0}),$$

$$L_n^{(M)}(f)(x)$$
$$= \bigvee_{k=0}^{n} (-1)^{j-k} \frac{x - x_{n,j}}{x - x_{n,k}} f(x_{n,k}), x \in (x_{n,j}, (x_{n,j} + x_{n,j+1})/2], j = \overline{0, n-1},$$

$$L_n^{(M)}(f)(x)$$
$$= \bigvee_{k=0}^{n} (-1)^{j+1-k} \frac{x - x_{n,j+1}}{x - x_{n,k}} f(x_{n,k}), x \in [(x_{n,j} + x_{n,j+1})/2, x_{n,j+1}),$$

$$j = \overline{0, n-1},$$

$$L_n^{(M)}(f)(x) = \bigvee_{k=0}^{n} (-1)^{n-k} \frac{x - x_{n,n}}{x - x_{n,k}} f(x_{n,k}), \text{ for } x \in (x_{n,n}, b].$$

Proof. The continuity and the interpolation properties were already established by the Remark from the beginning of Section 2. In order to get the rest of the statement in the theorem, it suffices to prove the following formulas :

$$\bigvee_{k\in I_n^+(x)} l_{n,k}(x) = l_{n,0}(x), \text{ for } x \in [a, x_{n,0}),$$

$$\bigvee_{k\in I_n^+(x)} l_{n,k}(x) = l_{n,j}(x), \text{ for } x \in (x_{n,j}, (x_{n,j} + x_{n,j+1})/2], j = \overline{0, n-1},$$

$$\bigvee_{k\in I_n^+(x)} l_{n,k}(x) = l_{n,j+1}(x), \text{ for } x \in [(x_{n,j} + x_{n,j+1})/2, x_{n,j+1}), j = \overline{0, n-1},$$

$$\bigvee_{k\in I_n^+(x)} l_{n,k}(x) = l_{n,n}(x), \text{ for } x \in (x_{n,n}, b].$$

We have three cases : 1) $x \in [a, x_{n,0})$; 2) $x \in (x_{n,j}, x_{n,j+1})$, $j \in \{0, 1, ..., n-1\}$; 3) $x \in (x_{n,n}, b]$.

Case 1). By the proof of Theorem 3.1, Case 1), we have $l_{n,0}(x) > 0$, for $x \in [a, x_{n,0})$. Also, for any $k \in I_n^+(x)$, we have

$$\frac{l_{n,0}(x)}{l_{n,k}(x)} = \frac{x_{n,k} - x}{x_{n,0} - x} \geq 1.$$

Case 2). Let $j \in \{0, 1, ..., n-1\}$ be fixed. By the proof of Theorem 3.1, Case 2), we have $l_{n,j}(x) > 0$ and $l_{n,j+1}(x) > 0$, for $x \in (x_{n,j}, x_{n,j+1})$. We have

$$\frac{l_{n,j}(x)}{l_{n,j+1}(x)} = \frac{x_{n,j+1} - x}{x - x_{n,j}}.$$

Therefore, for any $x \in (x_{n,j}, (x_{n,j} + x_{n,j+1})/2]$ we have $l_{n,j}(x) \geq l_{n,j+1}(x)$ and for any $x \in [(x_{n,j} + x_{n,j+1})/2, x_{n,j+1})$ we have $l_{n,j+1}(x) \geq l_{n,j}(x)$.

Let $k \in I_n^+(x)$. If $k \leq j$ then

$$\frac{l_{n,j}(x)}{l_{n,k}(x)} = \frac{x - x_{n,k}}{x - x_{n,j}} \geq 1$$

and if $k \geq j + 1$ then

$$\frac{l_{n,j+1}(x)}{l_{n,k}(x)} = \frac{x_{n,k} - x}{x_{n,j+1} - x} \geq 1.$$

Case 3). By the proof of Theorem 3.1, Case 3), we have $l_{n,n}(x) > 0$, for $x \in (x_{n,n}, b]$. Also, in this case, for any $k \in I_n^+(x)$, we have

$$\frac{l_{n,n}(x)}{l_{n,k}(x)} = \frac{x - x_{n,k}}{x - x_{n,n}} \geq 1$$

and the theorem is proved. □

In what follows, would be of interest to compare the approximation results for the max-product Lagrange interpolation operators, with their linear counterparts. Thus, in the case of Lagrange interpolatory polynomials, it is well-known the fact that the divergence phenomenon is very pronounced.

In this sense, let us briefly recall some results (for details, see e.g. Chapter 4 in the book Szabados-Vértesi [17]). Thus, Bernstein [6] proved that for $f(x) = |x|$, the Lagrange interpolatory polynomials attached to the system of equidistant nodes in $[-1, 1]$ does not converge to $f(x)$, for any $x \in (-1, 1) \setminus \{0\}$. Grümwald [13] and independently Marcinkiewicz [15], proved that when the system of interpolation nodes consists in the Chebyshev nodes of the first kind, there exists a function $f \in C([-1, 1])$ such that for the attached Lagrange interpolatory polynomials $L_n(f)(x)$, we have $\limsup_{n\to\infty} |L_n(f)(x)| = +\infty$, for all $x \in [-1, 1]$. More general, a similar result holds for the system of Jacobi nodes in $[-1, 1]$ (see the book Szabados-Vértesi [17], relationship (4.1), p. 126). For an arbitrary system of interpolation nodes in $[-1, 1]$, in Erdös-Vértesi [11] it is proved that there exists a function $f \in C([-1, 1])$, such that for the attached Lagrange interpolatory polynomials we have $\limsup_{n\to\infty} |L_n(f)(x)| = +\infty$, almost everywhere $x \in [-1, 1]$. By using the condensation singularities principle in Functional Analysis, Muntean [16], Cobzas-Muntean [8] proved that for any system of nodes in $[0, 1]$, there exists a superdense subset $X_0 \subset C([0, 1])$, such that for any $f \in X_0$, the subset of divergence points in $[0, 1]$ for the attached Lagrange interpolatory polynomials $L_n(f)(x)$, is superdense in $[0, 1]$ (a countable intersection of open subsets which, in addition, is infinite, uncountable and dense subset, is called superdense).

In contrast with these results, the results in Theorem 3.1 and Corollary 3.2 show that for the max-product interpolatory operator $L_n^{(M)}(f)(x)$, the situation is essentially better, having uniform convergence with good rates of convergence for some of the most important systems of interpolation nodes.

Let us note that on the other hand, in Hermann-Vértesi [14], starting from a Lagrange interpolatory process (convergent or not)

$$P_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f(x_{n,k}),$$

with

$$p_{n,k}(x) = \frac{(x - x_{n,0})...(x - x_{n,k-1})(x - x_{n,k+1})...(x - x_{n,n})}{(x_{n,k} - x_{n,0})...(x_{n,k} - x_{n,k-1})(x_{n,k} - x_{n,k+1})...(x_{n,k} - x_{n,n})},$$

new linear interpolatory rational operators are constructed, of the form

$$R_n(f)(x) = \frac{\sum_{k=0}^{n} f(x_{n,k})|p_{n,k}(x)|^r}{\sum_{k=0}^{n} f(x_{n,k})|p_{n,k}(x)|^r},$$

are constructed, for which in the case when $r > 2$ and $x_{n,k}$ are some Jacobi knots, the Jackson-type order of approximation

$$\|R_n(f) - f\| \le C\omega_1(f; 1/n),$$

is obtained (see Theorem 3.2 in Hermann-Vértesi [14]).

In other words, for the linear rational construction $R_n(f)(x)$, we get the same order of approximation as for the interpolatory rational max-product operator in Theorem 3.1 of the form

$$L_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} p_{n,k}(x) f\left(x_{n,k}\right)}{\bigvee_{k=0}^{n} p_{n,k}(x)}.$$

Clearly that with respect to $R_n(f)(x)$, the max-product rational operator $L_n^{(M)}(f)(x)$ present the advantage that it provides an estimate in terms of $\omega_1(f; 1/n)$ for any kind of interpolatory systems of points, with the properties that the distance between two consecutive nodes converges to zero as $n \to \infty$.

But still it is an interesting open problem, a comparison from computational point of view, between a rational max-product type product like that given by Theorem 3.1 (that is of the form $L_n^{(M)}(f)(x)$) and the linear rational one like $R_n(f)(x)$ mentioned above.

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