Stud. Univ. Babeş-Bolyai Math. 56(2011), No. 2, 305–313

On the Szasz-Inverse Beta operators

Cristina S. Cismaşiu

Abstract. In this paper, we consider a probabilitistic representation of the Szasz-Inverse Beta operators, which are an mixed summation-integral type operators, and we study some approximation properties using probabilistic methods.

Mathematics Subject Classification (2010): 41A35, 41A36, 41A25,42A61.

Keywords: Szasz-Inverse Beta operators, Szasz-Mirakjan operators, Inverse Beta operators, Phillips operators, modified Szasz-Inverse Beta operators, estimates.

1. Probabilistic representation of the Szasz-Inverse Beta operators

In this paper we consider a probabilistic representation of the Szasz-Inverse Beta operators and study some approximation properties, using probabilistic methods. These operators were defined by (1.1)-(1.5) and were investigated by Gupta V., Noor M. A., [11] and some iterative constructions of these operators were studied recently by Finta Z., Govil N. K., Gupta V. [10]:

$$\begin{aligned} L_t(f;x) &= e^{-tx}f(0)+\sum_{k=1}^{\infty} s_{t,k}(x)\int_0^{\infty} b_{t,k}(u)f(u)du \qquad (1.1) \\ &= \int_0^{\infty} J_t(u;x)f(u)du,\ x\geq 0 \end{aligned}$$

with

$$s_{t,k}(x)=e^{-tx}\frac{(tx)^k}{k!},\ t>0,\ x\geq 0,\ k\in\mathbb{N}\cup\{0\} \qquad (1.2)$$

$$b_{t,k}(u)=\frac{1}{B(k,t+1)}\cdot\frac{u^{k-1}}{(1+u)^{t+k+1}},\ t>0,\ u>0, \qquad (1.3)$$

$$B(k,t+1)=\int\limits_0^\infty \frac{u^{k-1}}{(1+u)^{t+k+1}}du \tag{1.4}$$

being Inverse-Beta function

$$J_t(u;x)=e^{-tx}\delta(u)+\sum_{k=1}^{\infty} s_{t,k}(x)b_{t,k}(u), \tag{1.5}$$

$\delta(u)$ being the Dirac's delta function, for which $\int\limits_0^\infty \delta(u)f(u)du=f(0)$.

Using same ideea as Adell J. A., De la Cal J., [2], these operators can be represented as the mean value of the random variable $f\left(\frac{U_{N(tx)}}{V_{t+1}}\right)$ which has the probability density function $J_t(\cdot;x)$:

$$L_t(f;x)=E\left[f\left(Z_{tx}\right)\right]=E\left[f\left(\frac{U_{N(tx)}}{V_{t+1}}\right)\right],\ t>0,\ x\geq 0, \tag{1.6}$$

with $\{N(t) : t\geq 0\}$ a standard Poisson process and $\{U_t : t\geq 0\}$, $\{V_t : t\geq 0\}$ two mutually independent Gamma processes defined all on the same probability space.

Note that, the Poisson process is a stochastic process starting at the origin, having stationary independent increments with probability

$$P\left(N(t)=k\right)=\frac{e^{-t}t^k}{k!},\ t\geq 0,\ k\in\mathbb{N}\cup\{0\} \tag{1.7}$$

and the Gamma process is a stochastic process starting at the origin $(U_0=0, V_0=0)$, having stationary independent increments and such that for $t>0$, U_t, V_t have the Gamma probability density function

$$\rho_t(u)=\begin{cases}\dfrac{u^{t-1}e^{-u}}{\Gamma(t)} & ,\ t>0,\ u>0,\\ 0 & ,\ u=0\end{cases} \tag{1.8}$$

and without loss of generality [17] it can be assumed that $\{U_t : t\geq 0\}$, $\{V_t : t\geq 0\}$ for each $t>0$ has a.s. no decreasing right-continuous paths.

Indeed, in our paper [4] we showed that

$$E\left[f\left(\frac{U_{N(tx)}}{V_{t+1}}\right)\right]=\int\limits_0^\infty f(u)\left(\int\limits_0^\infty y\rho_{U_{N(tx)}}(yu)\rho_{V_{t+1}}(y)dy\right)du$$

$$=\int\limits_0^\infty f(u)\left(\int\limits_0^\infty y\sum_{k=0}^{\infty}\frac{e^{-tx}(tx)^k}{k!}\rho_{U_k}(yu)\rho_{V_{t+1}}(y)dy\right)du$$

$$=e^{-tx}f(0)+$$

$$+\sum_{k=1}^{\infty}s_{t,k}(x)\int\limits_0^\infty f(u)\left(\int\limits_0^\infty \frac{y^{k+t}}{\Gamma(k)}\cdot\frac{u^{k-1}}{\Gamma(t+1)}e^{-y(u+1)}dy\right)du$$

$$= e^{-tx} f(0) +$$

$$+ \sum_{k=1}^{\infty} s_{t,k}(x) \int_0^{\infty} f(u) \left(\int_0^{\infty} \frac{\left(\frac{v}{u+1}\right)^{k+t}}{\Gamma(k)} \cdot \frac{u^{k-1}}{\Gamma(t+1)} e^{-v} \frac{dv}{u+1} \right) du$$

$$= e^{-tx} f(0) +$$

$$+ \sum_{k=1}^{\infty} s_{t,k}(x) \int_0^{\infty} f(u) \frac{b_{t,k}(u)}{\Gamma(k+t+1)} \left(\int_0^{\infty} v^{k+t} e^{-v} dv \right) du$$

$$= e^{-tx} f(0) + \sum_{k=1}^{\infty} s_{t,k}(x) \int_0^{\infty} f(u) b_{t,k}(u) du = L_t(f; x).$$

On the other hand, the Szasz-Inverse Beta operators (1.1)-(1.5) can be represented as the composition between Szasz-Mirakjan operators and Inverse-Beta operators:

$$L_t(f; x) = (S_t \circ T_t)(f; x) = S_t(T_t)(f; x),\ t > 0,\ x \geq 0 \tag{1.9}$$

with the Szasz-Mirakjian operators

$$S_t(f; x) = E\left[f\left(\frac{N_{tx}}{t} \right) \right] = \sum_{k=0}^{\infty} s_{t,k}(x) f\left(\frac{k}{t} \right) \text{ with (1.2)} \tag{1.10}$$

and the Inverse-Beta operators or the Stancu operators of second kind [19]:

$$\begin{cases} T_t(f; x) = E\left[f\left(W_{tx,t+1} \right) \right] \\ \qquad = \dfrac{1}{B(tx, t+1)} \int_0^{\infty} \dfrac{u^{tx-1}}{(1+u)^{tx+t+1}} f(u) du \\ \qquad = \int_0^{\infty} f(u) b_{tx,t+1}(u) du,\ t > 0,\ x > 0, \\ T_t(f; 0) = f(0), \end{cases} \tag{1.11}$$

with $W_{tx,t+1}$ a random variable having the Inverse-Beta distribution with probability density function as

$$b_{tx,t+1}(u) = \frac{1}{B(tx, t+1)} \cdot \frac{u^{tx-1}}{(1+u)^{tx+t+1}},\ t > 0,\ x > 0,\ u > 0 \tag{1.12}$$

and $B(tx, t+1) = \int_0^{\infty} \frac{u^{tx-1}}{(1+u)^{tx+t+1}} du,\ t > 0,\ x > 0.$

It is known [16. IV.10.(3)] that, if we consider two independent random variables U_{tx}, V_{t+1} having Gamma distribution with probability density function (1.8) for $t := tx$ respectively $t := t+1$, then the probability density function of the ratio $\frac{U_{tx}}{V_{t+1}}$ is $b_{tx,t+1}(u) = \int_0^{\infty} y \rho_{U_{tx}}(uy) \rho_{V_{t+1}}(y) dy$ a Inverse-Beta probability density function as (1.12).

Remark 1.1. The Inverse-Beta probability density function can be represented with a negative binomial probability for $t>0$ and with convention $\binom{t}{k}=\frac{t(t-1)(t-2)\cdots(t-k+1)}{k!}$, $t>0$, $k\in\mathbb{N}$, we have

$$\begin{aligned} p_{t,k-1}(u) &= \binom{t+k}{k-1}\left(\frac{u}{1+u}\right)^{k-1}\left(\frac{1}{1+u}\right)^{t+2} \\ &= \binom{t+k}{k-1}\cdot\frac{u^{k-1}}{(1+u)^{t+k+1}}, \end{aligned} \tag{1.13}$$

$t>0$, $u>0$, $k\in\mathbb{N}$, for which $\int\limits_0^\infty p_{t,k-1}(u)du=\dfrac{1}{t+1}$ and so

$$\begin{aligned} b_{t,k}(u) &= \frac{1}{B(k,t+1)}\cdot\frac{u^{k-1}}{(1+u)^{t+k+1}} \\ &= (t+1)p_{t,k-1}(u)=\frac{p_{t,k-1}(u)}{\int\limits_0^\infty p_{t,k-1}(u)du}. \end{aligned} \tag{1.14}$$

The probability density function (1.5) becomes the kernel:

$$\begin{aligned} J_t(u;x) &= e^{-tx}\delta(u)+\sum_{k=1}^\infty s_{t,k}(x)b_{t,k}(u) \\ &= e^{-tx}\delta(u)+(t+1)\sum_{k=1}^\infty s_{t,k}(x)p_{t,k-1}(u) \end{aligned}$$

and the operators (1.1) have a Durrmeyer-type construction

$$\begin{aligned} L_t(f;x) &= e^{-tx}f(0)+\sum_{k=1}^\infty s_{t,k}(x)\frac{\int\limits_0^\infty p_{t,k-1}(u)f(u)du}{\int\limits_0^\infty p_{t,k-1}(u)du} \\ &= e^{-tx}f(0)+(t+1)\sum_{k=1}^\infty s_{t,k}(x)\int_0^\infty p_{t,k-1}(u)f(u)du. \end{aligned} \tag{1.15}$$

Using the representation (1.9) and the images of the test functions $e_i(x)=x^i$, $i=0,1,2$, $x\geq 0$ with these operators (1.10) and (1.11)-(1.12), it is easy to prove that

$$\begin{aligned} L_t\left(e_i;x\right) &= e_i(x),\ i=\overline{0,1},\ x\geq 0; \\ L_t\left(e_2;x\right) &= \frac{t}{t-1}x^2+\frac{2}{t-1}x,\ t>1,\ x\geq 0; \\ L_t\left(e_2-x^2;x\right) &= L_t\left((e_1-x)^2;x\right)=D^2\left[\frac{U_{N(tx)}}{V_{t+1}}\right] \\ &= E\left[\left(\frac{U_{N(tx)}}{V_{t+1}}-x\right)^2\right]=\frac{x(2+x)}{t-1},\ t>1,\ x\geq 0. \end{aligned} \tag{1.16}$$

2. Approximation properties of Szasz-Inverse Beta operators

In view of (1.9) because a part of the properties of Szasz-Inverse Beta operators depends on the same properties of Szasz-Mirakjan operators (1.10) and of the Inverse-Beta operators (1.11)-(1.12), next time, using a probabilistic method which was presented in [1], we studied [4] the monotonic convergence under convexity for the Szasz-Inverse Beta operators (1.1)-(1.5):

Theorem 2.1. *Let $t > 1$ be fixed. For the Szasz-Inverse Beta operators (1.1)-(1.5) following:*

1. $L_t(e_i; x) = e_i(x)$, $i = \overline{0,1}$;
2. $L_t(e_2; x) = \dfrac{t}{t-1}x^2 + \dfrac{2}{t-1}x$;
3. *If f is a convex function on $(0, +\infty)$ then $L_t f$ is convex too and in addition, f is nondecreasing then for $1 < r < s$, $L_r f \geq L_s f \geq f$;*
4. *If $f \in Lip_{(0,+\infty)}(C, \alpha)$, $\alpha \in (0, 1]$ then $L_t f \in Lip_{(0,+\infty)}(C, \alpha)$, $\alpha \in (0, 1]$.*

The proof is immediately [4] using the following two lemmas:

Lemma 2.2. *If $(U_{tx})_{t>0,\, x\geq 0}$, $(V_{t+1})_{t>0}$ are two independent Gamma processes defined on the same probability space, then for all $1 < r \leq s$ and $x > 0$ we have*

$$E\left(\frac{U_{rx}}{V_{r+1}} \mid \frac{U_{sx}}{V_{s+1}}\right) = \frac{U_{sx}}{V_{s+1}}\ a.\,s.$$

Lemma 2.3. *Let $t > 1$ be fixed. For the Inverse-Beta operators (1.11)-(1.12) following:*

1. *If f is a real convex function on $(0, +\infty)$ then $T_t f$ is convex too.*
2. *If f is a nondecreasing and convex function on $(0, +\infty)$and $1 < r < s$ then $T_r f \geq T_s f \geq f$.*
3. *If $f \in Lip_{(0,+\infty)}(C, \alpha)$, $\alpha \in (0, 1]$ then $T_t f \in Lip_{(0,+\infty)}(C, \alpha)$, $\alpha \in (0, 1]$*

Theorem 2.4. *For any function $f \in \mathbf{C}_B[0, +\infty)$ and for any compact set $K \subset [0, +\infty)$ we have $\lim_{t\to\infty} L_t(f) = f$ uniform on K.*

Proof. It follows from the Bohmann-Korovkin's theorem and from Theorem 2.1. □

In the next theorem we give in 1 and 2 an approximation using the modulus of continuity of f and of derivative f' and in 3 an asymptotic approximation of Voronovskaja type.

Theorem 2.5. **1.** *If $f \in \mathbf{C}_B[0, +\infty)$, then for every $x \in [0, +\infty)$*

$$|L_t(f; x) - f(x)| \leq \left(1 + \sqrt{x(2 + x)}\right)\omega\left(f; \frac{1}{\sqrt{t-1}}\right),\ t > 1.$$

2. *If* $f' \in \mathbf{C}_B[0,+\infty)$*, then for every* $x \in [0,+\infty)$

$$|L_t(f;x) - f(x)| \le$$
$$\le \sqrt{\frac{x(2+x)}{t-1}}\left(1+\sqrt{x(2+x)}\right)\omega\left(f'; \frac{1}{\sqrt{t-1}}\right), \; t>1.$$

3. *If* f *is bounded on* $[0,+\infty)$*, differentiable in some neighborhood of* x *and has second derivative* f*" for some* $x \in [0,+\infty)$*, then for* $t>1$

$$\lim_{t\to\infty}(t-1)\left[L_t(f;x) - f(x)\right] = \frac{x(2+x)}{2}f''(x).$$

If $f \in \mathbf{C}^2[0,+\infty)$*, then the convergence is uniform on any compact* $K \subset [0,+\infty)$*.*

Proof. For 1 and 2 see (1.16) and a result of Shisha O., Mond B., [18] and for 3 see Cismaşiu C. [3]. □

Remark 2.6. An interesting result which was obtained by De la Cal J., Carcamo J., [7] for the operators of Bernstein-type which preserves the affine functions, namely centered Bernstein-type operators, can be used for Szasz-Inverse Beta operators (1.1)-(1.5) :

Theorem 2.7 (De la Cal J., Carcamo J., [7]**).** *If* $L_1 = L_2 \circ L_3$*, where* L_1, L_2, L_3 *are centered Bernstein-type operators (*$Lf(x) = E[f(Y_x)]$*,* $x \in I \subset \mathbb{R}$*,* $L_1(x) = E[Y_x] = x$*) over the same interval* I *and if* $\mathbf{L}_{cx}$ *is the set of all convex functions in the domain of the three operators, then* $L_1 f \ge L_2 f$*,* $f \in \mathbf{L}_{cx}$*.*

If, in addition L_3 *preserves convexity, then* $L_1 f \ge L_2 f \vee L_3 f$*,* $f \in \mathbf{L}_{cx}$ *where* $f \vee g$ *denotes the maximum of* f *and* g*.*

In view of this result and using the representation (1.9) for Szasz-Inverse-Beta operators, we have $L_t f \ge S_t f$, $f \in \mathbf{L}_{cx}[0,+\infty)$ and $L_t f \ge S_t f \vee T_t f$, $f \in \mathbf{L}_{cx}[0,+\infty)$, where S_t are the Szasz-Mirakjan operators (1.10), T_t are the Inverse-Beta operators (1.11)-(1.12) and L_t are the Szasz-Inverse Beta operators (1.1)-(1.5).

An estimate of the difference $|L_t(f;x) - S_t(f,x)|$ was given by us in [6]:

Theorem 2.8. *If* $f \in \mathbf{C}_B[0,+\infty) \cap \mathbf{L}_{cx}[0,+\infty)$ *then for every* $x \in [0,+\infty)$ *and* $t>1$

$$|L_t(f;x) - S_t(f,x)| \le \left(1+\delta^{-2}\left(\frac{x(x+1)}{t-1} + \frac{x}{t(t-1)}\right)\right)\omega(f,\delta)$$

with $\omega(f,\delta) = \sup\{|f(x)-f(y)| \,:\, x,y \ge 0,\, |x-y| \le \delta\}$ *the modulus of continuity of* f*.*

Using the probabilistic representation of these operators, result for $t>1$, $\delta>0$

$$\left|E\left[f\left(\frac{U_{N(tx)}}{V_{t+1}}\right)\right] - E\left[f\left(\frac{N(tx)}{t}\right)\right]\right| \le$$
$$\le \left(1+\delta^{-2}\left(D^2\left(\frac{U_{tx}}{V_{t+1}}\right) + \frac{1}{t-1}D^2\left(\frac{N(tx)}{t}\right)\right)\right)\omega(f,\delta)$$

3. Approximating Phillips operators by modified Szasz-Inverse Beta operators

Using the same ideea as De la Cal J. , Luquin F. [8] or as Adell J. A., De la Cal J. [2] , we consider a new operator defined as the aid of Szasz-Inverse Beta operator (1.1)-(1.5) for $r > 0$, $t > 0$, $x \geq 0$:

$$\begin{aligned} \Theta_{r,t}(f;x) &= L_{rt}\left(f(tu);\frac{x}{t}\right) = \int\limits_0^\infty \frac{1}{t} J_{rt}\left(\frac{u}{t};\frac{x}{t}\right) f(u)du \qquad (3.1) \\ &= \int\limits_0^\infty \frac{1}{t}\left[e^{-rx}\delta\left(\frac{u}{t}\right) + \sum_{k=1}^\infty s_{rt,k}\left(\frac{x}{t}\right) b_{rt,k}\left(\frac{u}{t}\right)\right] f(u)du \\ &= e^{-rx}f(0) + \sum_{k=1}^\infty s_{r,k}(x)\int\limits_0^\infty \frac{1}{t} b_{rt,k}\left(\frac{u}{t}\right) f(u)du \end{aligned}$$

where f is any real function defined on $[0,\infty)$ such that $\Theta_{r,t}(|f|);x) < \infty$.

We obtain for the operators (3.1) a Durrmeyer-type construction in a similar way as for representation (1.15) with (1.14) for the Szasz-Inverse Beta operators (1.1)-(1.5):

$$\begin{aligned} \Theta_{r,t}(f;x) &= L_{rt}\left(f(tu);\frac{x}{t}\right) = \int\limits_0^\infty \frac{1}{t} J_{rt}\left(\frac{u}{t};\frac{x}{t}\right) f(u)du \qquad (3.2) \\ &= \int\limits_0^\infty \frac{1}{t}\left[e^{-rx}\delta\left(\frac{u}{t}\right) + \sum_{k=1}^\infty s_{rt,k}\left(\frac{x}{t}\right) b_{rt,k}\left(\frac{u}{t}\right)\right] f(u)du \\ &= e^{-rx}f(0) + \\ &\quad + \left(r+\frac{1}{t}\right)\sum_{k=1}^\infty s_{r,k}(x)\int\limits_0^\infty p_{rt,k-1}\left(\frac{u}{t}\right) f(u)du. \end{aligned}$$

and a probabilistic representation

$$\Theta_{r,t}(f;x) = L_{rt}\left(f(tu);\frac{x}{t}\right) = E\left[f\left(t\frac{U_{N(rx)}}{V_{rt+1}}\right)\right] \qquad (3.3)$$

These operators $\Theta_{r,t}(f;\cdot)$ approximate the Phillips' operators [14] defined as

$$\begin{aligned} P_r(f;x) &= E\left[f\left(\frac{U_{N(rx)}}{r}\right)\right] \qquad (3.4) \\ &= e^{-rx}f(0) + r\sum_{k=1}^\infty s_{r,k}(x)\int\limits_0^\infty s_{r,k-1}(u)f(u)du \\ &= \int\limits_0^\infty H_r(u;x)f(u)du,\ r>0,\ x\geq 0, \end{aligned}$$

with $s_{r,k}(x)$ as (1.2) ,

$$H_r(u;x) = e^{-rx}\delta(u) + r\sum_{k=1}^{\infty} s_{r,k}(x)s_{r,k-1}(u) \tag{3.5}$$

$x \geq 0$, $k \in \mathbb{N}\cup\{0\}$, $r > 0$, δ the Dirac's Delta function and for $f : [0,\infty) \longrightarrow \mathbb{R}$ any integrabile function, such that $P_r(|f|;x) < \infty$.

The Phillips operators (3.4)-(3.5) were studied by several authors (see [9],[12], [13], [14]) and are considered "the genuine Durrmeyer-Szasz-Mirakjan operators". A generalization of these operators, using two continuous parameters was obtained by Păltănea R. [15].

Theorem 3.1. *Let $x \geq 0$, $r,t,u > 0$ be. If, f is a real bounded function on $[0,\infty)$ then*

$$\begin{aligned} |\Theta_{r,t}(f;x) - P_r(f;x)| &= \left|L_{rt}\left(f(tu);\frac{x}{t}\right) - P_r(f;x)\right| \\ &\leq \|f\| \cdot \frac{r^2x^2+4rx+2}{rt+1} \end{aligned}$$

and we have uniform convergence as $t \to \infty$ on every bounded interval $[0,a]$, $a > 0$.

Proof. We presented in detail the proof in [5] and we gave a bound for the total variation distance between the probability distributions of the random variables $t\dfrac{U_{N(rx)}}{V_{rt+1}}$ and $\dfrac{U_{N(rx)}}{r}$, respectively between $\left|\dfrac{1}{t}b_{rt,k}\left(\dfrac{u}{t}\right) - rs_{r,k-1}(u)\right|$. □

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Cristina S. Cismaşiu
"Transilvania" University
Department of Mathematics
Eroilor 29, 500 036, Braşov, Romania
e-mail: ccismasiu@yahoo.com