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Approximation of fuzzy numbers by trapezoidal fuzzy numbers preserving the core and the expected value

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Abstract. In this paper, we have suggested a new trapezoidal approximation of a fuzzy number, preserving the core and the expected value of fuzzy numbers. We have proved that the trapezoidal approximation of fuzzy numbers preserving the core and the expected value is always a fuzzy number. We have discussed the properties of this approximation.

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1. Introduction

Many recent papers process and transform imprecise information using the fuzzy theory. For a more efficient handling of information there is a natural need to approximate the fuzzy numbers by trapezoidal fuzzy numbers with or without additional conditions [2], [10], [8], [11], [14]. In [7] the trapezoidal approximation is a reasonable compromise between two opposite tendencies: to lose to much information and to introduce too sophisticated form of approximation from the point of view of computation.

In this paper we have proposed the trapezoidal approximation preserving the core and the expected value of a fuzzy number. Important properties (translation invariance, scale invariance, etc.) of this new trapezoidal approximation are studied in Section 4.

2. Preliminaries

We consider the following well-known description of a fuzzy number A:

$$A(x) = \begin{cases} l_A(x), & \text{if} \quad a_1 \leq x \leq a_2, \\ 1 & \text{if} \quad a_2 \leq x \leq a_3, \\ r_A(x), & \text{if} \quad a_3 \leq x \leq a_4, \\ 0, & \quad \text{otherwise}, \end{cases}$$

where $a_1,\ a_2,\ a_3,\ a_4,\ \in \mathbb{R}$, $l_A : [a_1, a_2] \longrightarrow [0,1]$ is a nondecreasing upper semicontinuous function, $l_A(a_1) = 0$, $l_A(a_2) = 1$, called the left side of the fuzzy number and $r_A : [a_3, a_4] \longrightarrow [0,1]$ is a nonincreasing upper semicontinuous function, $r_A(a_3) = 1$, $r_A(a_4) = 0$, called the right side of the fuzzy number. The $\alpha-$cut, $\alpha \in (0,1]$, of a fuzzy number A is a crisp set defined as $A_\alpha = \{x \in \mathbb{R} : A(x) \geq \alpha\}$.

Every $\alpha-$cut $\alpha \in [0,1]$, of a fuzzy number is a closed interval $A_\alpha = [A_L(\alpha), A_U(\alpha)]$ where

$$A_L(\alpha) = \inf\{x \in \mathbb{R} : A(x) \geq \alpha\},\ A_U(\alpha) = \sup\{x \in \mathbb{R} : A(x) \geq \alpha\}$$

for any $\alpha \in (0,1]$. If the sides of the fuzzy number A are strictly monotone then one can easily see that A_L and A_U are inverse functions of l_A and r_A, respectively.

The core or $1-$cut of a fuzzy number is defined as $core(A) = [a_2, a_3]$.

We denote by $F(\ \mathbb{R})$ the set of all fuzzy numbers.

Let $A, B \in F(\mathbb{R}), A_\alpha = [A_L(\alpha), A_U(\alpha)],\ B_\alpha = [B_L(\alpha), B_U(\alpha)], \alpha \in [0,1]$ and $\lambda \in \mathbb{R}$. We consider the sum $A + B$ and the scalar multiplication $\lambda \cdot A$ by (see [5]) $(A+B)_\alpha = A_\alpha + B_\alpha = [A_L(\alpha) + B_L(\alpha), A_U(\alpha) + B_U(\alpha)]$ and $(\lambda \cdot A)_\alpha = \lambda A_\alpha = \begin{cases} [\lambda A_L(\alpha), \lambda A_U(\alpha)], & \text{if } \lambda \geq 0, \\ [\lambda A_U(\alpha), \lambda A_L(\alpha)], & \text{if } \lambda < 0, \end{cases}$ respectively, for every $\alpha \in [0,1]$.

A metric on the set of fuzzy numbers, which is and extension of the Euclidean distance, is defined by (see [9]) $D^2(A,B) = \int_0^1 (A_L(\alpha) - B_L(\alpha))^2 d\alpha + \int_0^1 (A_U(\alpha) - B_U(\alpha))^2 d\alpha$.

Fuzzy numbers with simple membership functions are preferred in practice. The most used such fuzzy numbers are so-called trapezoidal fuzzy numbers, given by

$$T(x) = \begin{cases} \frac{x - t_1}{t_2 - t_1}, & \text{if} \quad t_1 \leq x \leq t_2, \\ 1, & \text{if} \quad t_2 \leq x \leq t_3, \\ \frac{t_4 - x}{t_4 - t_3}, & \text{if} \quad t_3 \leq x \leq t_4, \\ 0, & \quad \text{otherwise}. \end{cases}$$

We denote $T = (t_1, t_2, t_3, t_4)$ a trapezoidal fuzzy number as above. It is easy to prove that $T_L(\alpha) = t_1 + (t_2 - t_1)\alpha$ and $T_U(\alpha) = t_4 - (t_4 - t_3)\alpha$ for every $\alpha \in [0,1]$.

We denote by $F^T(\mathbb{R})$ the set of all trapezoidal fuzzy numbers.

The ambiguity Amb of $A \in F(\mathbb{R})$ is defined by (see [4])

$$Amb(A) = \int_0^1 \alpha \left(A_U(\alpha) - A_L(\alpha)\right) d\alpha$$

and the value Val of $A \in F(\mathbb{R})$ is defined by (see [4])

$$Val(A) = \int_0^1 \alpha \left(A_U(\alpha) + A_L(\alpha)\right) d\alpha.$$

The expected interval $EI(A)$ of a fuzzy number A,

$$A_\alpha = [A_L(\alpha), A_U(\alpha)]$$

is defined by (see [6], [12])

$$EI(A) = [E_*(A), E^*(A)] = \left[\int_0^1 A_L(\alpha)\, d\alpha, \int_0^1 A_U(\alpha)\, d\alpha\right],$$

expected value is given by (see [12]): $EV(A) = \frac{E_*(A)+E^*(A)}{2}$, core of A is given by (see [1]): $core(A) = [A_L(1), A_U(1)]$. The expected value for a trapezoidal fuzzy number $T = (t_1, t_2, t_3, t_4)$ is $EV(T) = \frac{t_1+t_2+t_3+t_4}{4}$.

Another kind of fuzzy number was introduced in [3] as follows:

$$A(x) = \begin{cases} \left(\frac{x-a}{b-a}\right)^n, x \in [a,b] \\ 1, x \in [b,c] \\ \left(\frac{d-x}{d-c}\right)^n, x \in [c,d] \\ 0, \text{ otherwise,} \end{cases}$$

where $n > 0$, $A = (a,b,c,d)_n$ with the parametric representation:

$$A_L(\alpha) = (b-a)\sqrt[n]{\alpha} + a,\ A_U(\alpha) = d - (d-c)\sqrt[n]{\alpha}, \alpha \in [0,1].$$

3. Trapezoidal approximation of fuzzy numbers

The below version of the well-known Karush-Kuhn-Tucker theorem is useful in the solving of the proposed problem.

Theorem 3.1. (*Rockafellar,* [13]) *Let $f, g_1, g_2, ..., g_m : \mathbb{R}^n \to \mathbb{R}$ be convex and differentiable function. Then $\bar{x}$ solves the convex programming problem*

$\min f(x)$

s.t. $g_i(x) \le b_i$ $i \in \{1,2,3,...,m\}$

if and only if exists μ_i, $i \in \{1,2,3,...,m\}$, *such that*

(*i*) $\nabla f\left(\bar{x}\right) + \sum_{i=1}^{m} \mu_i \nabla g_i\left(\bar{x}\right) = 0;$

(*ii*) $g_i\left(\bar{x}\right) - b_i \le 0;$

(*iii*) $\mu_i \ge 0;$

(*iv*) $\mu_i\left(b_i - g_i\left(\bar{x}\right)\right) = 0.$

Given a fuzzy number A, $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, $\alpha \in [0,1]$, the problem is to find a trapezoidal fuzzy number $T(A) = (t_1, t_2, t_3, t_4)$ which is the nearest to A with respect to metric D and preserves the expected value and the core of A, that is:

$$EV(A) = EV(T(A)), core(A) = core(T(A)).$$

The problem is reduced to minimize the distance between the fuzzy number A and the trapezoidal fuzzy number $T(A)$

$$\begin{aligned} F(t_1,t_2,t_3,t_4) &= \int_0^1 \left[A_L(\alpha)-(t_1+(t_2-t_1)\alpha)\right]^2 d\alpha + \\ &+\int_0^1 \left[A_U(\alpha)-(t_4+(t_3-t_4)\alpha)\right]^2 d\alpha \end{aligned}$$

s.t.

$$t_2 = A_L(1), \tag{3.1}$$

$$t_3 = A_U(1), \tag{3.2}$$

so

$$2\int_0^1 \left[A_L(\alpha)+A_U(\alpha)\right] d\alpha = t_1+t_2+t_3+t_4. \tag{3.3}$$

The conditions for $T(A)=(t_1,t_2,t_3,t_4)$ to be a trapezoidal fuzzy number are

$$t_1 \leq t_2 \tag{3.4}$$

$$t_2 \leq t_3 \tag{3.5}$$

and

$$t_3 \leq t_4. \tag{3.6}$$

Taking into account the relations $(3.1)-(3.3)$, t_4 becomes:

$$t_4 = 2\int_0^1 \left(A_L(\alpha)+A_U(\alpha)\right) d\alpha - t_1 - A_L(1) - A_U(1),$$

so $F(t_1,t_2,t_3,t_4)$ becomes $g(t_1)$

$$\begin{aligned} g(t_1) = \tfrac{2t_1^2}{3} + 2t_1\textstyle\int_0^1 \left(A_L(\alpha)-E(\alpha)\right)(\alpha-1)\,d\alpha + \tfrac{t_1}{3}A_L(1) + \\ +\textstyle\int_0^1 \left(E(\alpha)\right)^2 d\alpha + \int_0^1 \left(A_L(\alpha)-A_L(1)\alpha\right)^2 d\alpha, \end{aligned} \tag{3.7}$$

where

$$E(\alpha) = A_U(\alpha)$$

$$+2(\alpha-1)\int_0^1 \left(A_L(\alpha)+A_U(\alpha)\right) d\alpha - A_L(1)(\alpha-1) - A_U(1)(2\alpha-1),$$

so conditions $(3.4)-(3.6)$ are:

$$t_1 \leq A_L(1) \tag{3.8}$$

$$A_L(1) \leq A_U(1) \tag{3.9}$$

and

$$t_1 \leq 2\int_0^1 A_L(\alpha)\,d\alpha + 2\int_0^1 A_U(\alpha)\,d\alpha - A_L(1) - 2A_U(1). \tag{3.10}$$

For any fuzzy number the relation (3.9) is always true.

Theorem 3.2. *If A, $A_\alpha = [A_L(\alpha), A_U(\alpha)]$ is a fuzzy number and $T(A) = (t_1, t_2, t_3, t_4)$ denotes the nearest (with respect to metric D) trapezoidal fuzzy number A preserving the expected value and the core, then*

(i) If

$$\int_0^1 [(2-6\alpha) A_U(\alpha) + (6\alpha - 10) A_L(\alpha)] \, d\alpha + 7A_L(1) + A_U(1) < 0 \quad (3.11)$$

and

$$\int_0^1 [A_L(\alpha) + A_U(\alpha)] \, d\alpha \geq A_L(1) + A_U(1), \quad (3.12)$$

then

$$t_1 = t_2 = A_L(1); \; t_3 = A_U(1);$$

$$t_4 = 2\int_0^1 A_L(\alpha) \, d\alpha + 2\int_0^1 A_U(\alpha) \, d\alpha - 2A_L(1) - A_U(1).$$

(ii) If

$$\int_0^1 [(2-6\alpha) A_L(\alpha) + (6\alpha - 10) A_U(\alpha)] \, d\alpha + A_L(1) + 7A_U(1) > 0 \quad (3.13)$$

and

$$\int_0^1 [A_L(\alpha) + A_U(\alpha)] \, d\alpha \leq A_L(1) + A_U(1), \quad (3.14)$$

then

$$t_1 = 2\int_0^1 A_L(\alpha) \, d\alpha + 2\int_0^1 A_U(\alpha) \, d\alpha - A_L(1) - 2A_U(1);$$

$$t_2 = A_L(1); \; t_3 = t_4 = A_U(1).$$

(iii) If

$$\int_0^1 [(2-6\alpha) A_U(\alpha) + (6\alpha - 10) A_L(\alpha)] \, d\alpha + 7A_L(1) + A_U(1) \geq 0 \quad (3.15)$$

and

$$\int_0^1 [(2-6\alpha) A_L(\alpha) + (6\alpha - 10) A_U(\alpha)] \, d\alpha + A_L(1) + 7A_U(1) \leq 0 \quad (3.16)$$

then

$$t_1 = -\frac{3}{2}\int_0^1 \alpha (A_L(\alpha) - A_U(\alpha)) \, d\alpha - \frac{3}{4}A_L(1)$$
$$+\frac{1}{2}\int_0^1 (5A_L(\alpha) - A_U(\alpha)) \, d\alpha - \frac{1}{4}A_U(1);$$
$$t_2 = A_L(1); \; t_3 = A_U(1);$$
$$t_4 = \frac{3}{2}\int_0^1 \alpha (A_L(\alpha) - A_U(\alpha)) \, d\alpha - \frac{1}{4}A_L(1)$$
$$-\frac{1}{2}\int_0^1 (A_L(\alpha) - 5A_U(\alpha)) \, d\alpha - \frac{3}{4}A_U(1).$$

Proof. Let us remark that the hypothesis of convexity and differentiability in the Karush-Kuhn-Tucker theorem are satisfied for the function g given by (3.7) under conditions (3.8) – (3.10). After some calculations we can write the conditions of Karush-Kuhn-Tucker theorem to minimize the function g, in the following way:

$$\frac{4t_1}{3}+2\int_0^1 \alpha\left[A_L(\alpha)-A_U(\alpha)\right]d\alpha+\frac{2}{3}\int_0^1 \left(A_U(\alpha)-5A_L(\alpha)\right)d\alpha+$$
$$+A_L(1)+\frac{A_U(1)}{3}+\mu_1+\mu_2=0 \tag{3.17}$$
$$\mu_1\left(t_1-A_L(1)\right)=0 \tag{3.18}$$
$$\mu_2\left(t_1-2\int_0^1 A_L(\alpha)\,d\alpha-2\int_0^1 A_U(\alpha)\,d\alpha+A_L(1)+2A_U(1)\right)=0 \tag{3.19}$$
$$\mu_1\geq 0 \tag{3.20}$$
$$\mu_2\geq 0 \tag{3.21}$$
$$t_1-A_L(1)\leq 0 \tag{3.22}$$
$$t_1-2\int_0^1 A_L(\alpha)\,d\alpha-2\int_0^1 A_U(\alpha)\,d\alpha+A_L(1)+2A_U(1)\leq 0 \tag{3.23}$$

If $\mu_1\neq 0$ and $\mu_2\neq 0$, then the solution is: $t_1=A_L(1)$ and

$$t_1=2\int_0^1 A_L(\alpha)\,d\alpha+2\int_0^1 A_U(\alpha)\,d\alpha-A_L(1)-2A_U(1),$$

so

$$t_4=2\int_0^1 A_L(\alpha)\,d\alpha+2\int_0^1 A_U(\alpha)\,d\alpha-2A_L(1)-A_U(1)$$

or $t_4=A_U(1)$, from (3.17) we obtain that

$$\mu_1+\mu_2=-2\int_0^1 \alpha\left[A_L(\alpha)-A_U(\alpha)\right]d\alpha$$
$$-\frac{2}{3}\int_0^1 \left(A_U(\alpha)-5A_L(\alpha)\right)d\alpha-A_L(1)-\frac{A_U(1)}{3} \tag{3.24}$$

but $\mu_1\neq 0$ and $\mu_2\neq 0$, so

$$\int_0^1 A_L(\alpha)\,d\alpha+\int_0^1 A_U(\alpha)\,d\alpha=A_L(1)+A_U(1),$$

by (3.24), we have

$$\mu_1+\mu_2=\int_0^1 (1-2\alpha)\,A_L(\alpha)\,d\alpha+\int_0^1 (2\alpha-3)\,A_U(\alpha)\,d\alpha+2A_U(1)$$

so

$$\mu_1+\mu_2=\int_0^1 (1-2\alpha)\,A_L(\alpha)\,d\alpha+\int_0^1 (2\alpha-1)\,A_U(\alpha)\,d\alpha$$
$$+2\left(A_U(1)-\int_0^1 A_U(\alpha)\,d\alpha\right)$$

taking into account Lemma 1 from [2] results that $\mu_1 + \mu_2 \leq 0$. In fact, because $\mu_1 \neq 0$ and $\mu_2 \neq 0$ we obtain that $\mu_1 + \mu_2 < 0$, so Karush-Kuhn-Tucker conditions can not be verified, which means that we have no solution in this case.

(i) If $\mu_1 \neq 0$ and $\mu_2 = 0$, then from (3.17) and (3.18) we obtain that:

$$t_1 = A_L(1)$$

and

$$\mu_1 = 2\int_0^1 \alpha \left[A_U(\alpha) - A_L(\alpha)\right] d\alpha$$

$$-\frac{2}{3}\int_0^1 \left(A_U(\alpha) - 5A_L(\alpha)\right) d\alpha - \frac{A_U(1)}{3} - \frac{7A_L(1)}{3}, \mu_2 = 0$$

and from (3.1), (3.2) and (3.3) we obtain that

$$t_4 = 2\int_0^1 A_L(\alpha)\, d\alpha + 2\int_0^1 A_U(\alpha)\, d\alpha - 2A_L(1) - A_U(1).$$

(ii) If $\mu_1 = 0$ and $\mu_2 \neq 0$, then from (3.17) and (3.19) we obtain that

$$t_1 = 2\int_0^1 A_L(\alpha)\, d\alpha + 2\int_0^1 A_U(\alpha)\, d\alpha - A_L(1) - 2A_U(1) \text{ and } \mu_1 = 0,$$

$$\mu_2 = \frac{1}{3}\int_0^1 \left[(2-6\alpha) A_L(\alpha) + (6\alpha - 10) A_U(\alpha)\right] d\alpha + \frac{1}{3}\left(A_L(1) + 7A_U(1)\right)$$

and $t_4 = A_U(1)$.

(iii) If $\mu_1 = 0$ and $\mu_2 = 0$, then from (3.17), (3.22) and (3.23) we obtain that

$$t_1 = -\frac{3}{2}\int_0^1 \alpha \left(A_L(\alpha) - A_U(\alpha)\right) d\alpha + \frac{1}{2}\int_0^1 \left(5A_L(\alpha) - A_U(\alpha)\right) d\alpha$$

$$-\frac{3}{4}A_L(1) - \frac{1}{4}A_U(1)$$

and

$$t_4 = \frac{3}{2}\int_0^1 \alpha \left(A_L(\alpha) - A_U(\alpha)\right) d\alpha - \frac{1}{2}\int_0^1 \left(A_L(\alpha) - 5A_U(\alpha)\right) d\alpha$$

$$-\frac{1}{4}A_L(1) - \frac{3}{4}A_U(1).$$

□

Remark 3.3. Any fuzzy number can apply one and only case of the Theorem 3.2.

Proof. Let us denote

$$\Gamma_1 = \{A : A \in F(\mathbb{R}) \text{ and the case } (i) \text{ is applicable to } A\},$$

$$\Gamma_2 = \{A : A \in F(\mathbb{R}) \text{ and the case } (ii) \text{ is applicable to } A\}$$

and

$$\Gamma_3 = \{A : A \in F(\mathbb{R}) \text{ and the case } (iii) \text{ is applicable to } A\}.$$

It is obvious that $\Gamma_3 = (\Gamma_1 \cup \Gamma_2)^c$, so, the three cases of Theorem 3.2 cover the set of all fuzzy numbers. On the other hand $\Gamma_1 \cap \Gamma_3 = \varnothing$ because the relation (3.11) is complementary with the relation (3.15). $\Gamma_2 \cap \Gamma_3 = \varnothing$ because the relation (3.13) is complementary with the relation (3.16). □

Example 3.4. Let A be a fuzzy number $A_\alpha = [1 + 99\sqrt{\alpha}, 200 - 95\sqrt{\alpha}]$ then the trapezoidal approximation preserving the core and the expected value is $T(A) = (t_1, t_2, t_3, t_4)$ and can be calculated with case (iii) of Theorem 3.2 as follows: $t_1 = \frac{923}{30}$; $t_2 = 100$; $t_3 = 105$; $t_4 = \frac{5147}{30}$.

Theorem 3.5. *Let $A = (a, b, c, d)_n$ be a fuzzy number and $T(A) = (t_1, t_2, t_3, t_4)$ the nearest (with respect to metric D) trapezoidal fuzzy number A preserving the expected value and the core*

(i) If $(n-1)(d-c) + (17n+7)(b-a) < 0$ and $a - b - c + d \geq 0$, then

$$t_1 = t_2 = b;\ t_3 = c;\ t_4 = \frac{2a - 2b - c + 2d + cn}{n+1}.$$

(ii) If $(b-a)(1-n) - (17n+7)(d-c) > 0$ and $a - b - c + d \leq 0$, then

$$t_1 = \frac{2a - b - 2c + 2d + bn}{n+1};\ t_2 = b;\ t_3 = t_4 = c.$$

(iii) If $(b-a)(1-n) - (17n+7)(d-c) \leq 0$ and

$$(n-1)(d-c) + (17n+7)(b-a) \geq 0$$

then

$$t_1 = \frac{7a - 3b - c + d + 8bn^2 + 17an - 5bn + cn - dn}{4(n+1)(2n+1)};\ t_2 = b;\ t_3 = c;$$

$$t_4 = \tfrac{a - b - 3c + 7d + 8cn^2 - an + bn - 5cn + 17dn}{4(n+1)(2n+1)}.$$

Proof. Let $A = (a, b, c, d)_n$ be a fuzzy number, then $A_L(1) = b$, $A_U(1) = c$ and

$$\int_0^1 A_L(\alpha)\, d\alpha = \frac{a + bn}{n+1};\ \int_0^1 A_U(\alpha)\, d\alpha = \frac{cn + d}{n+1};$$

$$\int_0^1 \alpha A_L(\alpha)\, d\alpha = \frac{a + 2bn}{4n+2};\ \int_0^1 \alpha A_U(\alpha)\, d\alpha = \frac{2cn + d}{4n+2}.$$

Applying Theorem 3.2 the result is immediately. □

Example 3.6. For a fuzzy number $A = (2, 3, 4, 40)_{\frac{1}{2}}$ applying the case (i) of Theorem 3.5 we obtain the trapezoidal approximation which preserves the expected value and the core of A: $T(A) = \left(3, 3, 4, \frac{152}{3}\right)$.

Example 3.7. For a fuzzy number $A = (-200, 0, 10, 20)_{\frac{1}{5}}$ applying the case (ii) of Theorem 3.5 we obtain the trapezoidal approximation which preserves the expected value and the core of A: $T(A) = \left(-\frac{950}{3}, 0, 10, 10\right)$.

Example 3.8. For a fuzzy number $A = (1, 2, 3, 100)_2$ applying the case (iii) of Theorem 3.5 we obtain the trapezoidal approximation which preserves the expected value and the core of A: $T(A) = \left(-\frac{3}{10}, 2, 3, \frac{693}{10}\right)$.

4. Properties

In fuzzy theory are many approximate methods of fuzzy numbers and an infinite number of approximation operators. For the present approximation operator we study some properties proposed by Grzgorzewski and Mrowka in paper [7].

Theorem 4.1. *The trapezoidal approximation preserving the core and the expected value given in Theorem* 3.2*:*

(i) is invariant to translation;
(ii) is scale invariant;
(iii) fulfills the nearness criterion;
(iv) fulfills the identity criterion.

Proof. (i) If $A \in \Gamma_1$ then the conditions $(3.11 - 3.12)$ are verified, so

$$\int_0^1 \left[(2-6\alpha)(A+z)_U(\alpha) + (6\alpha-10)(A+z)_L(\alpha)\right] d\alpha$$
$$+7(A+z)_L(1) + (A+z)_U(1)$$
$$= \int_0^1 \left[(2-6\alpha) A_U(\alpha) + 2z + (6\alpha-10) A_L(\alpha) - 10z\right] d\alpha$$
$$+7A_L(1) + A_U(1) + 8z$$
$$= \int_0^1 \left[(2-6\alpha) A_U(\alpha) + (6\alpha-10) A_L(\alpha)\right] d\alpha + 7A_L(1) + A_U(1) < 0,$$

and

$$\int_0^1 \left[(A+z)_L(\alpha) + (A+z)_U(\alpha)\right] d\alpha - (A+z)_L(1) - (A+z)_U(1)$$
$$= \int_0^1 \left[A_L(\alpha) + A_U(\alpha)\right] d\alpha - A_L(1) - A_U(1) \geq 0,$$

so $A + z \in \Gamma_1$.

We obtain that

$$t_1(A+z) = (A+z)_L(1) = A_L(1) + z = t_1(A) + z,$$
$$t_2(A+z) = (A+z)_L(1) = A_L(1) + z = t_2(A) + z,$$
$$t_3(A+z) = (A+z)_U(1) = A_U(1) + z = t_3(A) + z,$$

and

$$t_4(A+z) = 2\int_0^1 \left[(A+z)_L(\alpha) + (A+z)_U(\alpha)\right] d\alpha$$
$$-2(A+z)_L(1) - (A+z)_U(1)$$
$$= 2\int_0^1 A_L(\alpha)\, d\alpha + 2\int_0^1 A_U(\alpha)\, d\alpha - 2A_L(1) - A_U(1) + 4z - 2z - z = t_4(A) + z,$$

so $T(A+z) = T(A) + z$, $A \in \Gamma_1$.

It is obviously that for $A \in \Gamma_2$ we obtain that $A + z \in \Gamma_2$ and for $A \in \Gamma_3$ we obtain that $A + z \in \Gamma_3$ so $T(A+z) = T(A) + z$, for every $A \in \Gamma_i$, $i \in \{1, 2, 3\}$.

(ii) For $\lambda > 0$ the proof is immediate because $(\lambda A)_L(\alpha) = \lambda A_L(\alpha)$ and $(\lambda A)_U(\alpha) = \lambda A_U(\alpha)$, $\alpha \in [0,1]$ so $T(\lambda \cdot A) = \lambda \cdot T(A)$, $A \in \Gamma_i$, $i \in \{1,2,3\}$ and from $A \in \Gamma_i$ results that $\lambda \cdot A \in \Gamma_i$, $i \in \{1,2,3\}$.

In case $\lambda < 0$ we have $(\lambda A)_L(\alpha) = \lambda A_U(\alpha)$ and $(\lambda A)_U(\alpha) = \lambda A_L(\alpha)$, for every $\alpha \in [0,1]$.

If $A \in \Gamma_1$ then the conditions $(3.11 - 3.12)$ are verified

$$\lambda \int_0^1 [(2-6\alpha) A_U(\alpha) + (6\alpha - 10) A_L(\alpha)]\, d\alpha + 7\lambda A_L(1) + \lambda A_U(1) > 0,$$

so

$$\int_0^1 [(2-6\alpha) \lambda A_U(\alpha) + (6\alpha - 10) \lambda A_L(\alpha)]\, d\alpha + 7\lambda A_L(1) + \lambda A_U(1) > 0$$

and is equivalent to

$$\int_0^1 [(2-6\alpha)(\lambda A)_L(\alpha) + (6\alpha - 10)(\lambda A)_U(\alpha)]\, d\alpha$$
$$+7(\lambda A)_U(1) + (\lambda A)_L(1) > 0$$

and

$$\lambda \int_0^1 [A_L(\alpha) + A_U(\alpha)]\, d\alpha \le \lambda A_L(1) + \lambda A_U(1)$$

so

$$\int_0^1 [\lambda A_L(\alpha) + \lambda A_U(\alpha)]\, d\alpha \le \lambda A_L(1) + \lambda A_U(1)$$

and is equivalent to

$$\int_0^1 [(\lambda A)_U(\alpha) + (\lambda A)_L(\alpha)]\, d\alpha \le (\lambda A)_U(1) + (\lambda A)_L(1),$$

we obtain that $\lambda \cdot A \in \Gamma_2$, so

$$T(\lambda \cdot A) = \Big(2\int_0^1 (\lambda \cdot A)_L(\alpha)\, d\alpha + 2\int_0^1 (\lambda \cdot A)_U(\alpha)\, d\alpha - 2(\lambda \cdot A)_L(1) -$$
$$-(\lambda \cdot A)_U(1), (\lambda \cdot A)_L(1), (\lambda \cdot A)_U(1), (\lambda \cdot A)_U(1))$$
$$= (\lambda \cdot A_L(1), \lambda \cdot A_L(1), \lambda \cdot A_U(1),$$
$$2\int_0^1 \lambda \cdot A_L(\alpha)\, d\alpha + 2\int_0^1 \lambda \cdot A_U(\alpha)\, d\alpha - 2\lambda \cdot A_U(1) - \lambda \cdot A_L(1)\Big)$$
$$= \lambda \cdot T(A)$$

then $T(\lambda \cdot A) \in \Gamma_1$. $\lambda \cdot A$ is in case (ii) of Theorem 3.2 if and only if A is in case (i) of Theorem 3.2.

For $A \in \Gamma_2$ we obtain that $\lambda \cdot A \in \Gamma_2$, so $\lambda \cdot A$ is in case (i) of Theorem 3.2 if and only if A is in case (ii) of Theorem 3.2.

Similarly it can be shown that $\lambda \cdot A$ is in case (iii) of Theorem 3.2 if and only if A is in case (iii) of Theorem 3.2.

(iii) By the construction of the operator under study.

(iv) If $A=(a_1,a_2,a_3,a_4)$, $A_\alpha=[a_1+\alpha(a_2-a_1),a_4-\alpha(a_4-a_3)]$, then

$$\int_0^1 [(2-6\alpha)A_U(\alpha)+(6\alpha-10)A_L(\alpha)]\,d\alpha+7A_L(1)+A_U(1)$$
$$=4(a_2-a_1)\geq 0$$

and

$$\int_0^1 [(2-6\alpha)A_L(\alpha)+(6\alpha-10)A_U(\alpha)]\,d\alpha+A_L(1)+7A_U(1)$$
$$=4(a_3-a_4)\leq 0$$

so (3.15) and (3.16) are verified and the case (iii) of Theorem 3.2 is applicable to A. We obtain that

$$t_1=-\frac{3}{2}\int_0^1 \alpha\left((a_1+\alpha(a_2-a_1))-(a_4-\alpha(a_4-a_3))\right)d\alpha$$
$$-\frac{1}{4}(a_4-(a_4-a_3))+\frac{1}{2}\int_0^1 \left(5(a_1+\alpha(a_2-a_1))-(a_4-\alpha(a_4-a_3))\right)d\alpha$$
$$-\frac{3}{4}(a_1+(a_2-a_1))=a_1,$$
$$t_2=A_L(1)=a_2;t_3=A_U(1)=a_3$$

and

$$t_4=\frac{3}{2}\int_0^1 \alpha\left((a_1+\alpha(a_2-a_1))-(a_4-\alpha(a_4-a_3))\right)d\alpha$$
$$-\frac{3}{4}(a_4-(a_4-a_3))-\frac{1}{2}\int_0^1 \left((a_1+\alpha(a_2-a_1))-5(a_4-\alpha(a_4-a_3))\right)d\alpha$$
$$-\frac{1}{4}(a_1+(a_2-a_1))=a_4,$$

so $T(A)=A,\forall A\in F^T(\mathbb{R})$. □

Remark 4.2. Let A be a trapezoidal fuzzy number, the trapezoidal approximation preserving the core and the expected value does not preserve the ambiguity and the value of a fuzzy number: $Amb(A)\neq Amb(T(A))$, $Val(A)\neq Val(T(A))$, for any $A\in F(\mathbb{R})$.

Example 4.3. Let A be a fuzzy number $A=(0,1,2,3)_2$, $A_L(\alpha)=\sqrt{\alpha}$, $A_U(\alpha)=3-\sqrt{\alpha}$. The trapezoidal approximation which preserves the expected value and core is $T(A)=\left(\frac{3}{10},1,2,\frac{27}{10}\right)$.The ambiguity of the fuzzy number A is: $Amb(A)=\frac{7}{10}$ and the ambiguity of the trapezoidal approximation is: $Amb(T(A))=\frac{11}{15}$, so $Amb(A)\neq Amb(T(A))$.

Example 4.4. For a fuzzy number $A=[\sqrt{\alpha}+1,30-27\sqrt{\alpha}]$ the value of the fuzzy number is $Val(A)=\frac{51}{10}$, the trapezoidal approximation which preserves the expected value and the core is $T(A)=\left(\frac{13}{15},2,3,\frac{322}{15}\right)$ so it's value is: $Val(T(A))=\frac{97}{18}$, then $Val(A)\neq Val(T(A))$.

The following example studies the continuity of the trapezoidal operator which preserves the core and the expected value.

Example 4.5. The trapezoidal operator introduced by Theorem 3.2 is not continuous, with respect to the distance D. Indeed, if $A \in F(\mathbb{R}), A_n \in F(\mathbb{R}), n \in \mathbb{N}$ such that $A_L(\alpha) = \sqrt{\alpha}$, $A_U(\alpha) = 3 - \sqrt{\alpha}$ and $(A_n)_L = \sqrt{\alpha} + \alpha^n$, $(A_n)_U = 3 - \sqrt{\alpha}, \alpha \in [0,1]$, then the trapezoidal fuzzy numbers which preserve the expected value and the core of A and A_n are:

$$T(A) = \left(\frac{3}{10}, 1, 2, \frac{27}{10}\right),$$

$$T(A_n) = \left(-\frac{7n + 9n^2 - 52}{20(n+2)(n+1)}, 2, 2, \frac{27}{10}\right), n \in \mathbb{N}$$

and

$$\lim_{n \to \infty} D(T(A_n), T(A)) = \frac{\sqrt{39}}{12}$$

and

$$\lim_{n \to \infty} D(A_n, A) = \lim_{n \to \infty} \sqrt{\frac{1}{2n+1}} = 0,$$

so $\lim\limits_{n \to \infty} D(T(A_n), T(A)) = \frac{\sqrt{39}}{12} \neq 0 = \lim\limits_{n \to \infty} D(A_n, A)$.

5. Conclusion

In the present paper a new trapezoidal approximation of a fuzzy number was added to trapezoidal approximations already introduced in [1], [2], [7], [8], [14]. It has multiple advantages: can be easy calculated, has some important properties: scale invariance, identity, translation invariance, nearness criterion.

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