Stud. Univ. Babeş-Bolyai Math. 56(2011), No. 2, 219–235

Iterates of multidimensional Kantorovich-type operators and their associated positive C_0-semigroups

Francesco Altomare, Mirella Cappelletti Montano and Vita Leonessa

Abstract. In this paper we deepen the study of a sequence of positive linear operators acting on $L^1([0,1]^N)$, $N \geq 1$, that have been introduced in [3] and that generalize the multidimensional Kantorovich operators (see [15]). We show that particular iterates of these operators converge on $\mathscr{C}([0,1]^N)$ to a Markov semigroup and on $L^p([0,1]^N)$, $1 \leq p < +\infty$, to a positive contractive C_0-semigroup (that is an extension of the previous one). The generators of these C_0-semigroups are the closures of some partial differential operators that belong to the class of Fleming-Viot operators arising in population genetics.

Mathematics Subject Classification (2010): 41A36, 47D06, 47F05.

Keywords: Multidimensional Kantorovich operator, positive approximation process, iterate of operators, positive C_0-semigroup, approximation of semigroups, Fleming-Viot operator.

1. Introduction

In the paper [3] we introduced and studied a sequence $(C_n)_{n\geq 1}$ of positive linear operators on $L^1([0,1]^N)$, $N \geq 1$, that are a generalization of the multidimensional Kantorovich operators, first introduced in [15], and that also extend to a multidimensional setting another sequence of positive linear operators on $L^1([0,1])$ studied in [5] and [6].

The operators C_n, $n \geq 1$, offer the advantage to reconstruct any Lebesgue-integrable function on $[0,1]^N$ by means of its mean values on a finite numbers of sub-cells of $[0,1]^N$ that do not constitute a subdivision of $[0,1]^N$.

Both in [6] and in [11] particular iterates of the (generalized) Kantorovich operators have been also investigated in connection with the existence of related C_0-semigroups of operators on $\mathscr{C}([0,1])$ and on $L^1([0,1])$.

Then, it seemed quite natural to tackle similar problems in a multidimensional setting and for the operators C_n, $n \geq 1$.

By using different methods from those employed in [6] and [11], in fact we first show that there exists a Markov semigroup $(T(t))_{t\geq 0}$ on $\mathscr{C}([0,1]^N)$ such that

$$T(t)(f) = \lim_{n\to\infty} C_n^{\rho_n}(f) \qquad \text{in } \mathscr{C}([0,1]^N) \tag{1.1}$$

for any $f \in \mathscr{C}([0,1]^N)$, $t \geq 0$ and for any sequence $(\rho_n)_{n\geq 1}$ of positive integers such that $\rho_n/n \to t$ as $n \to \infty$.

The generator $(A, D(A))$ of the Markov semigroup is determined on a core of $D(A)$, namely on $\mathscr{C}^2([0,1]^N)$, where it coincides with the second-order elliptic differential operator

$$V_l(u)(x) := \frac{1}{2}\sum_{i=1}^N x_i(1-x_i)\frac{\partial^2 u}{\partial x_i^2}(x) + \sum_{i=1}^N \left(\frac{l}{2} - x_i\right)\frac{\partial u}{\partial x_i}(x)$$

$(u \in \mathscr{C}^2([0,1]^N), x = (x_i)_{1\leq i\leq N} \in [0,1]^N)$, where $l \in [0,2]$.

Accordingly, formula (1.1) provides a constructive approximation of the solutions to the abstract Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = A(u(\cdot,t))(x) & x \in [0,1]^N, \quad t \geq 0, \\[2ex] u(x,0) = u_0(x) & u_0 \in D(A), \quad x \in [0,1]^N, \end{cases}$$

that, as it is well-known, are given by $u(x,t) = T(t)(u_0)(x)$ $(x \in [0,1]^N, t \geq 0)$.

The differential operator V_l falls in a class of Fleming-Viot operators arising in population genetics (see [2], [7], [10] for some additional references).

In addition, we also show that the subspace of all polynomials with a given degree and the subspace of all Hölder continuous functions on $[0,1]^N$ are invariant under $(T(t))_{t\geq 0}$. In some particular cases we finally show that the semigroup $(T(t))_{t\geq 0}$ can be extended to a positive contractive C_0-semigroup on $L^p([0,1]^N)$ for every $1 \leq p < +\infty$ and this semigroup can be equally approximated in the L^p-norm by iterates of the operators C_n, as in formula (1.1).

2. Preliminary results

Throughout this paper $[0,1]^N$ denotes the canonical hypercube in $\mathbf{R}^N$, $N \geq 1$, i.e.,

$$[0,1]^N := \{(x_i)_{1\leq i\leq N} \in \mathbf{R}^N \mid 0 \leq x_i \leq 1 \text{ for every } i = 1,\dots,N\}.$$

As usual we denote by $\mathscr{C}([0,1]^N)$ the space of all real valued continuous functions on $[0,1]^N$ and by $\mathscr{C}^2([0,1]^N)$ the space of all real valued continuous functions on $[0,1]^N$ which are twice continuously differentiable in the interior of $[0,1]^N$ and whose partial derivatives up to the order two can be continuously extended on $[0,1]^N$. The space $\mathscr{C}([0,1]^N)$, endowed with the natural (pointwise) ordering and the sup-norm $\|\cdot\|_\infty$, is a Banach lattice.

We also denote by $\mathbf{1}$ the constant function of constant value 1 on $[0,1]^N$. For a given $i \in \{1,\ldots,N\}$, the symbol pr_i stands for the i^{th} coordinate function on $[0,1]^N$, i.e., $pr_i(x) := x_i$ ($x = (x_i)_{1\le i\le N} \in [0,1]^N$). Moreover, fixed $x \in [0,1]^N$, we denote by Ψ_x the function defined as $\Psi_x(y) = y - x$ for every $y \in [0,1]^N$ (whenever $N = 1$ we use the symbol ψ_x) and by d_x the function defined by

$$d_x(y) := \|y - x\|_2 \quad (y \in [0,1]^N), \tag{2.1}$$

where $\|\cdot\|_2$ stands for the Euclidean norm on $\mathbf{R}^N$, i.e., $\|x\|_2 := \left(\sum_{i=1}^{N} x_i^2\right)^{1/2}$ ($x = (x_i)_{1\le i\le N} \in \mathbf{R}^N$).

We note that, given $x = (x_i)_{1\le i\le N} \in [0,1]^N$ and $i \in \{1,\ldots,N\}$,

$$pr_i \circ \Psi_x = pr_i - x_i\mathbf{1}, \tag{2.2}$$

and hence

$$(pr_i \circ \Psi_x)^2 = pr_i^2 - 2x_i pr_i + x_i^2\mathbf{1}. \tag{2.3}$$

Moreover,

$$d_x^2 = \sum_{i=1}^{N} (pr_i \circ \Psi_x)^2 \tag{2.4}$$

and

$$d_x^4 = \sum_{i=1}^{N} (pr_i \circ \Psi_x)^4 + 2 \sum_{1\le i<j\le N} (pr_i \circ \Psi_x)^2 (pr_j \circ \Psi_x)^2. \tag{2.5}$$

Given $1 \le p < +\infty$, the symbol $L^p([0,1]^N)$ stands for the spaces of all (equivalence classes of) Borel measurable functions f defined on $[0,1]^N$ such that

$$\|f\|_p := \left(\int_{[0,1]^N} |f|^p \, dx\right)^{1/p} < +\infty.$$

In [3] we introduced and studied a new sequence of positive linear operators acting on $L^1([0,1]^N)$, that will be also the object of interest of this paper.

More precisely, let $(a_n)_{n\ge1}$ and $(b_n)_{n\ge1}$ be two sequences of real numbers such that $0 \le a_n < b_n \le 1$ for every $n \ge 1$.

If $n \ge 1$ and $h = (h_i)_{1\le i\le N} \in \{0,\ldots,n\}^N$, set

$$Q_{n,h}^{a_n,b_n} := \prod_{i=1}^{N} \left[\frac{h_i + a_n}{n+1}, \frac{h_i + b_n}{n+1}\right]$$

and consider the positive linear operator $C_n : L^1([0,1]^N) \longrightarrow \mathscr{C}([0,1]^N)$ defined by setting, for any $f \in L^1([0,1]^N)$ and $x = (x_i)_{1\leq i\leq N} \in [0,1]^N$,

$$\begin{aligned} C_n(f)(x) &= \sum_{h\in\{0,\ldots,n\}^N} P_{n,h}(x)\left(\frac{n+1}{b_n-a_n}\right)^N \int_{Q_{n,h}^{a_n,b_n}} f(t)\, dt \\ &= \sum_{\substack{h=(h_i)_{1\leq i\leq N} \\ h_i\in\{0,\ldots,n\}}} P_{n,h}(x)\left(\frac{n+1}{b_n-a_n}\right)^N \int_{\frac{h_1+a_n}{n+1}}^{\frac{h_1+b_n}{n+1}} \cdots \int_{\frac{h_N+a_n}{n+1}}^{\frac{h_N+b_n}{n+1}} f(t_1,\ldots,t_N)\, dt_1\cdots dt_N, \end{aligned} \tag{2.6}$$

where

$$P_{n,h}(x) := \prod_{i=1}^N p_{n,h_i}(x_i) = \prod_{i=1}^N \binom{n}{h_i} x_i^{h_i}(1-x_i)^{n-h_i} \tag{2.7}$$

for every $x = (x_i)_{1\leq i\leq N} \in [0,1]^N$ and $h = (h_i)_{1\leq i\leq N} \in \{0,\ldots,n\}^N$.

Note that C_n is positive and continuous and that, as an operator from $\mathscr{C}([0,1]^N)$ into itself, its norm is $||C_n|| = 1$, since $C_n(\mathbf{1}) = \mathbf{1}$ for any $n \geq 1$.

We point out that the sequence $(C_n)_{n\geq 1}$ represents a generalization of Kantorovich operators on $[0,1]^N$, that were introduced and studied by Zhou in [15] and that can be obtained from (2.6) by setting, for any $n \geq 1$, $a_n = 0$ and $b_n = 1$.

On the other hand, the C_n's generalize to the multidimensional case a class of operators first studied in [5, Examples 1.2, 1] and defined by

$$K_n(f)(x) = \sum_{h=0}^n p_{n,h}(x)\frac{n+1}{b_n-a_n}\int_{\frac{h+a_n}{n+1}}^{\frac{h+b_n}{n+1}} f(t)\, dt \tag{2.8}$$

for every $n \geq 1$, $f \in L^1([0,1])$ and $x \in [0,1]$, where, as above, $p_{n,h}(x) := \binom{n}{h}x^h(1-x)^{n-h}$.

A possible interest in the study of the sequence $(C_n)_{n\geq 1}$ lies in the fact that it allows to reconstruct a Lebesgue-integrable function by means of its mean values on the sets $Q_{n,h}^{a_n,b_n}$ which are smaller than the corresponding ones considered in [15]. In fact, the following result holds (see [3, Theorems 2.2 and 2.5]).

Proposition 2.1. *For every $f \in \mathscr{C}([0,1]^N)$,*

$$\lim_{n\to\infty} C_n(f) = f \quad \textit{uniformly on} \ \ [0,1]^N. \tag{2.9}$$

Moreover, for every $n \geq 1$ and $p \in [1,+\infty[$, the operator C_n is continuous from $L^p([0,1]^N)$ into itself and

$$\|C_n\|_{L^p,L^p} \leq \frac{1}{(b_n-a_n)^{N/p}}. \tag{2.10}$$

Finally, if $\sup_{n\geq 1} 1/(b_n-a_n) < +\infty$, then, for every $f \in L^p([0,1]^N)$,

$$\lim_{n\to\infty} C_n(f) = f \quad \textit{in} \ \ L^p([0,1]^N). \tag{2.11}$$

In [3, Propositions 2.4, 2.6 and 2.7] estimates of the rate of convergence in the previous approximation formulae are also given.

The main aim of this paper is to show that suitable iterates of the operators C_n converge to a positive C_0-semigroup of operators both in $\mathscr{C}([0,1]^N)$ and in $L^p([0,1]^N)$, $p \geq 1$.

To this end, first of all we recall some properties of the operators K_n defined in (2.8), that will be useful in the sequel (for a proof see [6, Section 2]).

Lemma 2.2. *For every $n \geq 1$, let K_n be the positive linear operator defined by (2.8) and, for every $0 \leq x \leq 1$, consider the functions $\psi_x(y) = y - x$ ($y \in [0,1]$). Then*

(i) $\lim_{n\to\infty} K_n(\psi_x^2)(x) = 0$ *uniformly on* $[0,1]$*;*

(ii) $\lim_{n\to\infty} nK_n(\psi_x^2)(x) = x(1-x)$ *uniformly on* $[0,1]$*;*

(iii) $\lim_{n\to\infty} nK_n(\psi_x^4)(x) = 0$ *uniformly on* $[0,1]$.

As regards the operators C_n, we have the following result (see [3, Lemma 2.1]).

Lemma 2.3. *Given $n \geq 1$ and $i \in \{1, \ldots, N\}$, then*

$$C_n(\mathbf{1}) = \mathbf{1}, \tag{2.12}$$

$$C_n(pr_i) = \frac{n}{n+1} pr_i + \frac{a_n + b_n}{2(n+1)} \mathbf{1} \tag{2.13}$$

and

$$C_n(pr_i^2) = \frac{1}{(n+1)^2} \left\{ n^2 pr_i^2 + n pr_i(1 - pr_i) + n(a_n + b_n) pr_i + \frac{1}{3}(a_n^2 + a_n b_n + b_n^2)\mathbf{1} \right\}. \tag{2.14}$$

Further, the following equalities will be useful (see [3, Lemma 2.1]).

Proposition 2.4. *For every $x = (x_i)_{1 \leq i \leq N} \in [0,1]^N$ and $n \geq 1$,*

$$C_n(pr_i \circ \Psi_x)(x) = -\frac{1}{n+1} x_i + \frac{a_n + b_n}{2(n+1)}, \tag{2.15}$$

$$C_n((pr_i \circ \Psi_x)^2)(x) = \frac{1}{(n+1)^2} \left\{ x_i^2 + n x_i(1 - x_i) - (a_n + b_n) x_i + \frac{a_n^2 + a_n b_n + b_n^2}{3} \right\}, \tag{2.16}$$

$$C_n(d_x^2)(x) = \frac{1}{(n+1)^2} \left\{ (1-n)\|x\|_2^2 + (n - a_n - b_n) \sum_{i=1}^N x_i + N \frac{a_n^2 + a_n b_n + b_n^2}{3} \right\} \tag{2.17}$$

and

$$C_n(d_x^4)(x) = \sum_{i=1}^{N} K_n(\psi_{x_i}^4)(x_i) + 2 \sum_{1\le i<j\le N} K_n(\psi_{x_i}^2)(x_i)K_n(\psi_{x_j}^2)(x_j), \quad (2.18)$$

where, for any $n \ge 1$, the operator K_n is defined by (2.8) *and, for a given $i \in \{1,\dots,N\}$, $\psi_{x_i}(t_i) = t_i - x_i$ $(t = (t_i)_{1\le i\le N} \in [0,1]^N)$.*

Proof. Formulae (2.15)-(2.17) are a direct consequence of Lemma 2.3 and formulas (2.2)-(2.4). Taking both definition (2.6) of C_n's and formulae (2.2) and (2.5) into account, we obtain

$$\begin{aligned}
C_n(d_x^4)(x) &= \sum_{h\in\{0,\dots,n\}^N} P_{n,h}(x) \left(\frac{n+1}{b_n-a_n}\right)^N \int_{Q_{n,h}^{a_n,b_n}} d_x^4(t)\, dt \\
&= \sum_{h\in\{0,\dots,n\}^N} P_{n,h}(x) \left(\frac{n+1}{b_n-a_n}\right)^N \int_{Q_{n,h}^{a_n,b_n}} \sum_{i=1}^{N} (t_i-x_i)^4(t)\, dt \\
&+ \sum_{h\in\{0,\dots,n\}^N} P_{n,h}(x) \left(\frac{n+1}{b_n-a_n}\right)^N \int_{Q_{n,h}^{a_n,b_n}} 2 \sum_{1\le i<j\le N} (t_i-x_i)^2(t_j-x_j)^2\, dt \\
&= \sum_{\substack{h=(h_i)_{1\le i\le N}\\ h_i\in\{0,\dots,n\}}} P_{n,h}(x) \left(\frac{n+1}{b_n-a_n}\right)^N \int_{\frac{h_1+a_n}{n+1}}^{\frac{h_1+b_n}{n+1}} \cdots \int_{\frac{h_N+a_n}{n+1}}^{\frac{h_N+b_n}{n+1}} \sum_{i=1}^{N} \psi_{x_i}^4(t_i)\, dt_1 \cdots dt_N \\
&+2 \sum_{\substack{h=(h_i)_{1\le i\le N}\\ h_i\in\{0,\dots,n\}}} P_{n,h}(x) \left(\frac{n+1}{b_n-a_n}\right)^N \int_{\frac{h_1+a_n}{n+1}}^{\frac{h_1+b_n}{n+1}} \cdots \int_{\frac{h_N+a_n}{n+1}}^{\frac{h_N+b_n}{n+1}} \sum_{1\le i<j\le N} \psi_{x_i}^2(t_i)\psi_{x_j}^2(t_j) dt_1 \cdots dt_N \\
&= \sum_{\substack{h=(h_i)_{1\le i\le N}\\ h_i\in\{0,\dots,n\}}} P_{n,h}(x) \left(\frac{n+1}{b_n-a_n}\right) \sum_{i=1}^{N} \int_{\frac{h_i+a_n}{n+1}}^{\frac{h_i+b_n}{n+1}} \psi_{x_i}^4(t_i)\, dt_i \\
&+ 2 \sum_{\substack{h=(h_i)_{1\le i\le N}\\ h_i\in\{0,\dots,n\}}} P_{n,h}(x) \left(\frac{n+1}{b_n-a_n}\right)^2 \sum_{1\le i<j\le N} \int_{\frac{h_i+a_n}{n+1}}^{\frac{h_i+b_n}{n+1}} \int_{\frac{h_j+a_n}{n+1}}^{\frac{h_j+b_n}{n+1}} \psi_{x_i}^2(t_i)\psi_{x_j}^2(t_j)\, dt_i\, dt_j.
\end{aligned}$$

Now keeping (2.7) in mind and using the identities

$$\sum_{h_k=0}^{n} p_{n,h_k}(x_k) = 1 \quad \text{for every } k \in \{1,\dots,N\},$$

we have

$$C_n(d_x^4)(x) = \sum_{i=1}^{N} \sum_{h_i=0}^{n} p_{n,h_i}(x_i) \left(\frac{n+1}{b_n - a_n}\right) \int_{\frac{h_i+a_n}{n+1}}^{\frac{h_i+b_n}{n+1}} \psi_{x_i}^4(t_i)\, dt_i$$
$$+ 2 \sum_{1\le i<j\le N} \sum_{h_i=0}^{n} p_{n,h_i}(x_i) \left(\frac{n+1}{b_n - a_n}\right) \int_{\frac{h_i+a_n}{n+1}}^{\frac{h_i+b_n}{n+1}} \psi_{x_i}^2(t_i)\, dt_i$$
$$\times \sum_{h_j=0}^{n} p_{n,h_j}(x_j) \left(\frac{n+1}{b_n - a_n}\right) \int_{\frac{h_j+a_n}{n+1}}^{\frac{h_j+b_n}{n+1}} \psi_{x_j}^2(t_j)\, dt_j,$$

and hence formula (2.18) follows. □

Remark 2.5. A more explicit expression of (2.18) can be obtained using some computations contained in the proof of [6, Theorem 2.2].

Another useful result is shown below.

Proposition 2.6. *Under each of the following sets of conditions:*

(a) $a_n = 0$ *and* $b_n = 1$ *for every* $n \geq 1$,

or

(b) (i) $0 < b_n - a_n < 1$ *for every* $n \geq 1$;
(ii) *there exist* $\lim_{n\to\infty} a_n = 0$ *and* $\lim_{n\to\infty} b_n = 1$;
(iii) $M_1 := \sup_{n\geq 1} n(1 - (b_n - a_n)) < +\infty$,

for every $p \geq 1$ *there exists* $\omega_p \geq 0$ *such that, for every* $k \geq 1$ *and* $n \geq 1$,

$$\|C_n^k\|_{L^p,L^p} \leq e^{\frac{k}{n}\omega_p}, \tag{2.19}$$

where C_n^k *denotes the iterate of* C_n *of order* k.

Proof. Fix $p \geq 1$. Under assumption (a), on account of (2.10), the result obviously follows with $\omega_p = 0$.

Assume that conditions (i), (ii) and (iii) of (b) hold true; since

$$\lim_{n\to\infty} \frac{\log(b_n - a_n)}{1 - (b_n - a_n)} = -1,$$

there exists

$$M_2 := \sup_{n\geq 1} \frac{-\log(b_n - a_n)}{1 - (b_n - a_n)} > 0. \tag{2.20}$$

By means of (2.10), we then get

$$\|C_n^k\|_{L^p,L^p} \leq \frac{1}{(b_n - a_n)^{kN/p}} = e^{-\frac{kN}{p}\log(b_n - a_n)}$$
$$= e^{\frac{k}{n}\left(-\frac{N}{p} n(1-(b_n-a_n))\frac{\log(b_n-a_n)}{1-(b_n-a_n)}\right)} \leq e^{\frac{k}{n}\omega_p},$$

where $\omega_p := NM_1M_2/p$, and this completes the proof of (2.19). □

We also point out that, as in the one-dimensional case (see [5, formula (4.2)]), the operators C_n are closely related to the Bernstein operators on $[0,1]^N$ that are defined by

$$B_n(f)(x) := \sum_{\substack{h=(h_i)_{1\le i\le N}\\ h_i\in\{0,\dots,N\}}} P_{n,h}(x) f\left(\frac{h_1}{n},\dots,\frac{h_N}{n}\right) \tag{2.21}$$

$(f \in \mathscr{C}([0,1]^N),\ x = (x_i)_{1\le i\le N} \in [0,1]^N, n \ge 1)$, $P_{n,h}(x)$ being defined by (2.7).

More precisely, for every $f \in L^1([0,1]^N)$, considering the function

$$F_n(f)(x) := \left(\frac{n+1}{b_n - a_n}\right)^N \int_{\frac{nx_1+a_n}{n+1}}^{\frac{nx_1+b_n}{n+1}} dt_1 \cdots \int_{\frac{nx_N+a_n}{n+1}}^{\frac{nx_N+b_n}{n+1}} f(t_1,\dots,t_N)\, dt_N$$
$$= \int_0^1 dt_1 \cdots \int_0^1 f\left(\frac{(b_n-a_n)t_1 + a_n + nx_1}{n+1},\dots,\frac{(b_n-a_n)t_N + a_n + nx_N}{n+1}\right) dt_N \tag{2.22}$$

$(x = (x_i)_{1\le i\le N} \in [0,1]^N), n \ge 1)$, it turns out that

$$C_n(f)(x) = B_n(F_n(f))(x) \tag{2.23}$$

$(f \in L^1([0,1]^N),\ x = (x_i)_{1\le i\le N} \in [0,1]^N, n \ge 1)$.

Formula (2.23) allows us to easily determine some subsets of $\mathscr{C}([0,1]^N)$ that are invariant under the operators C_n, $n \ge 1$.

Given any $m \in \mathbf{N}$, we shall denote by $\mathbb{P}_m$ the linear subspace of the (restrictions to $[0,1]^N$ of the) polynomials of degree no greater than m.

Finally, given $M \ge 0$ and $0 < \alpha \le 1$, the symbol $Lip^1_M\alpha$ stands for the subset of all functions $f \in \mathscr{C}([0,1]^N)$ such that, for every $x, y \in [0,1]^N$,

$$|f(x) - f(y)| \le M\|x-y\|_1^\alpha,$$

where $\|\cdot\|_1$ denotes the l_1-norm on $\mathbf{R}^N$, i.e., $\|z\|_1 := \sum_{i=1}^N |z_i|$ for every $z = (z_i)_{1\le i\le N} \in \mathbf{R}^N$.

Proposition 2.7. *The subsets $\mathbb{P}_m$, $m \ge 1$, and $Lip^1_M\alpha$ are invariant under the operators C_n, $n \ge 1$, i.e.,*

$$C_n(\mathbb{P}_m) \subset \mathbb{P}_m \tag{2.24}$$

and

$$C_n(Lip^1_M\alpha) \subset Lip^1_M\alpha. \tag{2.25}$$

Proof. Both the subsets $\mathbb{P}_m$ and $Lip^1_M\alpha$ are invariant under the operators B_n, $n \ge 1$ (see, respectively, [1, Section 6.3.12, condition (6.2.18) and the proof of Theorem 6.2.6, p. 441] and [1, Corollary 6.1.22 and Section 6.3.12, p. 476]).

Therefore, on account of (2.23), it suffices to show that $F_n(f) \in \mathbb{P}_m$ (resp., $F_n(f) \in Lip^1_M\alpha$) provided that $f \in \mathbb{P}_m$ or $f \in Lip^1_M\alpha$, respectively, and this can be easily verified by virtue of (2.22). □

3. The C_0-semigroups associated with the operators C_n

In this section we shall prove that suitable iterates of the operators C_n converge on $\mathscr{C}([0,1]^N)$ to a Markov semigroup and on $L^p([0,1]^N)$, $1 \le p < +\infty$, to a positive contractive C_0-semigroup (that is an extension of the previous one).

From now on we assume that there exists

$$l := \lim_{n\to\infty}(a_n + b_n) \in \mathbf{R}. \tag{3.1}$$

Clearly, $0 \le l \le 2$.

Under this assumption we shall prove that the sequence $(C_n)_{n\ge 1}$ satisfies an asymptotic formula with respect to the elliptic second order differential operator $V_l : \mathscr{C}^2([0,1]^N) \longrightarrow \mathscr{C}([0,1]^N)$ defined by setting

$$V_l(u)(x) := \frac{1}{2}\sum_{i=1}^{N} x_i(1-x_i)\frac{\partial^2 u}{\partial x_i^2}(x) + \sum_{i=1}^{N}\left(\frac{l}{2} - x_i\right)\frac{\partial u}{\partial x_i}(x), \tag{3.2}$$

for every $u \in \mathscr{C}^2([0,1]^N)$ and $x = (x_i)_{1\le i\le N} \in [0,1]^N$.

Theorem 3.1. *Under assumption* (3.1), *for every* $u \in \mathscr{C}^2([0,1]^N)$,

$$\lim_{n\to\infty} n(C_n(u) - u) = V_l(u) \tag{3.3}$$

uniformly on $[0,1]^N$ *and hence in* $L^p([0,1]^N)$.

Proof. According to [4, Theorem 3.5], the claim will be proved after showing that, for every $i \in \{1,\dots,N\}$,

(a) $\lim_{n\to\infty}[nC_n(pr_i \circ \Psi_x)(x) - (l/2 - x_i)] = 0$ uniformly on $[0,1]^N$,

(b) $\lim_{n\to\infty}[nC_n((pr_i \circ \Psi_x)^2)(x) - x_i(1-x_i)] = 0$ uniformly on $[0,1]^N$,

(c) $\sup_{\substack{n\ge 1\\ x\in[0,1]^N}} nC_n(d_x^2)(x) < +\infty$

and

(d) $\lim_{n\to\infty} nC_n(d_x^4)(x) = 0$ uniformly on $[0,1]^N$,

where d_x is defined by (2.1).

We proceed to verify (a). According to formula (2.15) we get that, for every $i = 1,\dots,N$,

$$\left|nC_n(pr_i \circ \Psi_x)(x) - \left(\frac{l}{2} - x_i\right)\right| \le \frac{1}{n+1}|x_i| + \left|\frac{n}{n+1}\frac{a_n+b_n}{2} - \frac{l}{2}\right|$$
$$\le \frac{1}{n+1} + \left|\frac{n}{n+1}\frac{a_n+b_n}{2} - \frac{l}{2}\right|;$$

hence the required assertion follows from (3.1).

To prove statement (b) we preliminary notice that, by virtue of formula (2.16), for every $i = 1, \ldots, N$,

$$nC_n((pr_i \circ \Psi_x)^2)(x) - x_i(1-x_i)$$
$$= \left[\frac{n^2}{(n+1)^2} - 1\right] x_i(1-x_i) + \frac{n}{(n+1)^2}\left\{x_i^2 - (a_n+b_n)x_i + \frac{a_n^2+a_nb_n+b_n^2}{3}\right\};$$

therefore

$$|nC_n((pr_i \circ \Psi_x)^2)(x) - x_i(1-x_i)|$$
$$\leq \left|\frac{n^2}{(n+1)^2} - 1\right| x_i(1-x_i) + \frac{n}{(n+1)^2}\left(x_i^2 + (a_n+b_n)x_i + \frac{a_n^2+a_nb_n+b_n^2}{3}\right)$$
$$\leq \frac{2n+1}{4}\frac{1}{(n+1)^2} + \frac{4n}{(n+1)^2}$$

and this completes the proof of (b).

As regards conditions (c) and (d), from (2.17) we achieve that, for every $x \in [0,1]^N$,

$$C_n(d_x^2)(x) \leq \frac{N}{n+1},$$

and hence condition (c) follows. Finally, condition (d) is a consequence of (2.18) and Lemma 2.2. □

We recall that a Markov semigroup on $\mathscr{C}([0,1]^N)$ is a C_0-semigroup $(T(t))_{t\geq 0}$ of positive linear operators on $\mathscr{C}([0,1]^N)$ such that $T(t)(\mathbf{1}) = \mathbf{1}$ for every $t \geq 0$ (for more details on the theory of C_0-semigroups of operators we refer, e.g., to [8], [9] and [12]). In particular, we refer to [8, Section 13.6] for some remarkable aspects concerning Markov semigroups (see also [1, Section 1.6]).

We also recall that, given a Banach space $(E, \|\cdot\|)$, a *core* for a linear operator $A : D(A) \longrightarrow E$, defined on a linear subspace $D(A)$ of E, is a linear subspace D_0 of E that is dense in $D(A)$ with respect to the graph norm $\|u\|_A := \|u\| + \|A(u)\|$ $(u \in D(A))$.

If $(A, D(A))$ is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ of operators on E, then a dense (in E) linear subspace D_0 of $D(A)$ that is invariant under $(T(t))_{t\geq 0}$, i.e., $T(t)(D_0) \subset D_0$ for every $t \geq 0$, is a core for $(A, D(A))$ (see, e.g., [9, Chapter II, Proposition 1.7]). Moreover, if D_0 is a core for $(A, D(A))$, then $(A, D(A))$ is the closure of (A, D_0) as well.

As in Section 2, given any $m \in \mathbf{N}$, we denote by $\mathbb{P}_m$ the linear subspace of the (restrictions to $[0,1]^N$ of the) polynomials on $\mathbf{R}^N$ of degree no greater than m. Thus $\mathbb{P} := \bigcup_{m=0}^{+\infty} \mathbb{P}_m$ is the subalgebra of all the (restrictions to $[0,1]^N$ of the) polynomials on $\mathbf{R}^N$ and it is dense in $\mathscr{C}([0,1]^N)$ by the Stone-Weierstrass theorem.

Fix $0 \leq l \leq 2$ and consider the differential operator $V_l : \mathscr{C}^2([0,1]^N) \longrightarrow \mathscr{C}([0,1]^N)$ defined by (3.2). This operator falls in the class of Fleming-Viot operators arising in population genetics, that are usually studied in the setting of the multidimensional simplex. However, in the framework of hypercubes they have been investigated in [2], [7], [10].

Theorem 3.2. *There exists a Markov semigroup* $(T_l(t))_{t\geq 0}$ *on* $\mathscr{C}([0,1]^N)$ *satisfying the following properties:*

(1) *If* $(a_n)_{n\geq 1}$ *and* $(b_n)_{n\geq 1}$ *are two sequences of real numbers satisfying* $0 \leq a_n < b_n \leq 1$ *for every* $n \geq 1$ *and* $\lim_{n\to\infty}(a_n + b_n) = l$*, then for every* $t \geq 0$ *and for every sequence* $(\rho_n)_{n\geq 1}$ *of positive integers such that* $\lim_{n\to\infty} \rho_n/n = t$

$$\lim_{n\to\infty} C_n^{\rho_n}(f) = T_l(t)(f) \qquad \textit{uniformly on } [0,1]^N \tag{3.4}$$

for every $f \in \mathscr{C}([0,1]^N)$*, where each* $C_n^{\rho_n}$ *denotes the iterate of* C_n *of order* ρ_n*. In particular,*

$$\lim_{n\to\infty} C_n^{[nt]}(f) = T_l(t)(f) \qquad \textit{uniformly on } [0,1]^N \tag{3.5}$$

for every $f \in \mathscr{C}([0,1]^N)$*, where* $[nt]$ *stands for the integer part of* nt*.*

(2) *Denoted by* $(A_l, D(A_l))$ *the generator of the semigroup* $(T_l(t))_{t\geq 0}$*, then* $\mathscr{C}^2([0,1]^N)$ *is a core for* $(A_l, D(A_l))$*, so that* $(A_l, D(A_l))$ *is the closure of* $(V_l, \mathscr{C}^2([0,1]^N))$*.*

(3) *The subalgebra* $\mathbb{P}$ *is a core for* $(A_l, D(A_l))$ *and* $T_l(t)(\mathbb{P}_m) \subset \mathbb{P}_m$ *for every* $t \geq 0$ *and* $m \geq 0$*.*

(4) $T_l(t)(Lip_M^1\alpha) \subset Lip_M^1\alpha$ *for every* $t \geq 0$*,* $M \geq 0$ *and* $0 < \alpha \leq 1$*.*

Proof. The proof is similar in spirit to the one of Theorem 4.1 of [2]. Consider two sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ of real numbers satisfying $0 \leq a_n < b_n \leq 1$ for every $n \geq 1$ and $\lim_{n\to\infty}(a_n + b_n) = l$, and denote by $(C_n)_{n\geq 1}$ the relevant operators defined by (2.6).

Moreover, consider the linear operator $B : D(B) \longrightarrow \mathscr{C}([0,1]^N)$ defined by

$$B(u) := \lim_{n\to\infty} n(C_n(u) - u) \qquad (u \in D(B)),$$

where

$$D(B) := \left\{ u \in \mathscr{C}([0,1]^N) \;\middle|\; \text{there exists } \lim_{n\to\infty} n(C_n(u) - u) \text{ in } \mathscr{C}([0,1]^N) \right\}.$$

By Theorem 3.1, $\mathscr{C}^2([0,1]^N) \subset D(B)$ and $B = V_l$ on $\mathscr{C}^2([0,1]^N)$. In particular, each $\mathbb{P}_m$ is contained in $D(B)$, it is finite dimensional and invariant under the operators C_n by virtue of Proposition 2.7. By a result of Schnabl ([14]; see also [13] or [1, Theorem 1.6.8]) the operator $(B, D(B))$ is then closable in $\mathscr{C}([0,1]^N)$ and its closure, that we denote by $(A_l, D(A_l))$, is the generator of a positive C_0-semigroup $(T_l(t))_{t\geq 0}$ of linear contractions of $\mathscr{C}([0,1]^N)$, satisfying (3.4) and (3.5).

Since $C_n(\mathbf{1}) = \mathbf{1}$ for any $n \geq 1$, from (3.5) it follows that $T_l(t)(\mathbf{1}) = \mathbf{1}$ for every $t \geq 0$. Moreover, each $\mathbb{P}_m$ is closed in $\mathscr{C}([0,1]^N)$ and it is invariant under the C_n's. Therefore, iterating and passing to the limit, we obtain that $T_l(t)(\mathbb{P}_m) \subset \mathbb{P}_m$ for every $t \geq 0$.

Accordingly, we get that $T_l(t)(\mathbb{P}) \subset \mathbb{P}$ for any $t \geq 0$ and hence $\mathbb{P}$ is a core for $(A_l, D(A_l))$. In particular, $\mathscr{C}^2([0,1]^N)$ is a core for $(A_l, D(A_l))$ as well and $A_l = B = V_l$ on $\mathscr{C}^2([0,1]^N)$, which implies that $(A_l, D(A_l))$ is the closure of $(V_l, \mathscr{C}^2([0,1]^N))$, too.

This last statement shows, indeed, that the generator $(A_l, D(A_l))$ is independent on the sequence $(C_n)_{n\geq 1}$ and hence on the sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$. On the other hand, the generator $(A_l, D(A_l))$ determines the generated semigroup uniquely (see [9, Chapter II, Theorem 1.4]) and so the semigroup $(T_l(t))_{t\geq 0}$ does not depend on the particular sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$, as well.

Finally, statement (4) follows from formula (2.25) of Proposition 2.7 and from the fact that $Lip_M^1\alpha$ is closed under the pointwise (and hence under the uniform) convergence on $[0,1]^N$. □

Remarks 3.3.

1. Let us now consider the abstract Cauchy problem associated with $(A_l, D(A_l))$, i.e.,

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = A_l(u(\cdot,t))(x) & x \in [0,1]^N, \quad t \geq 0, \\ \\ u(x,0) = u_0(x) & u_0 \in D(A_l), \quad x \in [0,1]^N. \end{cases}$$

Since $(A_l, D(A_l))$ generates a C_0-semigroup, the above Cauchy problem admits a unique solution $u : [0,1]^N \times [0,+\infty[\to \mathbf{R}$ given by $u(x,t) = T_l(t)(u_0)(x)$ for every $x \in [0,1]^N$ and $t \geq 0$ (see, e.g., [12, Chapter A-II]). Hence, by Theorem 3.2, it is possible to approximate such solutions by means of iterates of the C_n's, i.e.,

$$u(x,t) = T_l(t)(u_0)(x) = \lim_{n\to\infty} C_n^{[nt]}(u_0)(x),$$

the limit being uniform with respect to $x \in [0,1]^N$.

Moreover, since A_l coincides with the elliptic second-order differential operator V_l defined by (3.2) on $\mathbb{P}_m$, $m \geq 1$, if $u_0 \in \mathbb{P}_m$, then $u(x,t)$ is the unique solution to the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \dfrac{1}{2}\sum\limits_{i=1}^{N} x_i(1-x_i)\dfrac{\partial^2 u(x,t)}{\partial x_i^2} + \sum\limits_{i=1}^{N}\left(\dfrac{l}{2} - x_i\right)\dfrac{\partial u(x,t)}{\partial x_i} & x \in [0,1]^N, \\ & t \geq 0, \\ \\ u(x,0) = u_0(x) & x \in [0,1]^N \end{cases}$$

and $u(\cdot,t) \in \mathbb{P}_m$ for every $t \geq 0$ (see statement (3) of Theorem 3.2).

Finally, according to statement (4) of Theorem 3.2, if $u_0 \in D(A_l) \cap Lip_M^1\alpha$ $(M \geq 0, 0 < \alpha \leq 1)$, then $u(\cdot,t) \in Lip_M^1\alpha$ for every $t \geq 0$.

2. Theorem 3.2 extends Theorem 3.3 of [6] from the one-dimensional case to a multidimensional context. However, there an explicit description of

the generator $(A_l, D(A_l))$ is given, namely

$$D(A_l) := \left\{ u \in \mathscr{C}([0,1]) \ \mid u \in \mathscr{C}^2(]0,1[) \text{ and } \lim_{\substack{x\to 0^+ \\ x\to 1^-}} A_l(u)(x) \in \mathbf{R} \right\} \tag{3.6}$$

and

$$A_l(u)(x) := \begin{cases} \dfrac{x(1-x)}{2} u''(x) + \left(\dfrac{l}{2} - x\right) u'(x) & \text{if } 0 < x < 1, \\ \\ \lim\limits_{t\to x} A_l(u)(t) & \text{if } x = 0, 1 \end{cases} \tag{3.7}$$

$(u \in D(A_l), 0 \le x \le 1)$.

An analogous description of $(A_l, D(A_l))$ in multidimensional setting seems to be a difficult but very interesting problem.

3. Statement (2) of Theorem 3.2 has been also obtained in [7, Theorem 2.1] with a different approach.

Next, we shall show that, in some particular cases, the Markov semigroup considered in Theorem 3.2 extends to a positive contractive C_0-semigroup on $L^p([0,1]^N)$, $1 \le p < +\infty$.

In fact, in these cases the limit (3.1) is $l = 1$, that leads to consider the differential operator

$$\begin{aligned} V(u)(x) := V_1(u)(x) &= \frac{1}{2}\sum_{i=1}^{N} x_i(1-x_i)\frac{\partial^2 u}{\partial x_i^2}(x) + \sum_{i=1}^{N}\left(\frac{1}{2} - x_i\right)\frac{\partial u}{\partial x_i}(x) \\ &= \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\frac{x_i(1-x_i)}{2}\frac{\partial u}{\partial x_i}\right)(x) \end{aligned} \tag{3.8}$$

$(u \in \mathscr{C}^2([0,1]^N)$ and $x = (x_i)_{1\le i\le N} \in [0,1]^N)$.

Similarly, we shall simply denote by $(T(t))_{t\ge 0}$ and by $(A, D(A))$ the semigroup $(T_1(t))_{t\ge 0}$ and its generator $(A_1, D(A_1))$.

Theorem 3.4. *The Markov semigroup $(T(t))_{t\ge 0}$ extends to a positive contractive C_0-semigroup $(\widetilde{T}(t))_{t\ge 0}$ on $L^p([0,1]^N)$ for each $p \in [1, +\infty[$.*

Moreover, $\mathscr{C}^2([0,1]^N)$ is a core for the generator $(\widetilde{A}, D(\widetilde{A}))$ of $(\widetilde{T}(t))_{t\ge 0}$, so that $(\widetilde{A}, D(\widetilde{A}))$ is the closure of $(V, \mathscr{C}^2([0,1]^N))$ in $L^p([0,1]^N)$.

Finally, if $(a_n)_{n\ge 1}$ and $(b_n)_{n\ge 1}$ are two sequences of real numbers such that $0 \le a_n < b_n \le 1$ and if, in addition, they satisfy one of the following sets of conditions:

(a) $a_n = 0$ *and* $b_n = 1$ *for every* $n \ge 1$,

or

(b) (i) $0 < b_n - a_n < 1$ *for every* $n \ge 1$;
(ii) *there exist* $\lim\limits_{n\to\infty} a_n = 0$ *and* $\lim\limits_{n\to\infty} b_n = 1$;
(iii) $M_1 := \sup\limits_{n\ge 1} n(1 - (b_n - a_n)) < +\infty$,

then for every $t \geq 0$*, for every sequence* $(\rho_n)_{n\geq 1}$ *of positive integers such that* $\lim_{n\to\infty} \rho_n/n = t$ *and for every* $f \in L^p([0,1]^N)$*,*

$$\lim_{n\to\infty} C_n^{\rho_n}(f) = \widetilde{T}(t)(f) \qquad \text{in } L^p([0,1]^N). \tag{3.9}$$

In particular, for every $f \in L^p([0,1]^N)$*,*

$$\lim_{n\to\infty} C_n^{[nt]}(f) = \widetilde{T}(t)(f) \qquad \text{in } L^p([0,1]^N). \tag{3.10}$$

Here, again, the operators C_n*,* $n \geq 1$*, are defined by* (2.6)*.*

Proof. Fix $t \geq 0$ and consider an arbitrary sequence $(\rho_n)_{n\geq 1}$ of positive integers such that $\rho_n/n \to t$. Furthermore, consider the sequence $(C_n)_{n\geq 1}$ associated with $a_n = 0$ and $b_n = 1$, $n \geq 1$. From (2.10) it follows that $\|C_n\|_{L^p,L^p} \leq 1$ and hence, on account of (3.4)

$$\|T(t)f\|_p = \lim_{n\to\infty} \|C_n^{\rho_n}(f)\|_p \leq \|f\|_p$$

for every $f \in \mathscr{C}([0,1]^N)$.

Therefore, there exists a unique linear continuous extension $\widetilde{T}(t) : L^p([0,1]^N) \longrightarrow L^p([0,1]^N)$ of $T(t)$. Moreover, $\|\widetilde{T}(t)\|_{L^p,L^p} \leq 1$ for every $t \geq 0$.

It is not difficult to show that $\widetilde{T}(t)$ is positive because if $f \in L^p([0,1]^N)$, $f \geq 0$, then there exists a sequence $(f_n)_{n\geq 1}$ in $\mathscr{C}([0,1]^N)$ such that $\lim_{n\to\infty} f_n = f$ in $L^p([0,1]^N)$. We may assume that $f_n \geq 0$ for every $n \geq 1$ (if not, we replace f_n with its positive part f_n^+). Therefore,

$$\widetilde{T}(t)(f) = \lim_{n\to\infty} \widetilde{T}(t)(f_n) = \lim_{n\to\infty} T(t)(f_n) \geq 0.$$

The family $(\widetilde{T}(t))_{t\geq 0}$ is obviously a semigroup and, in addition, it is strongly continuous; this easily follows, for instance, from ([9, Chapter I, Proposition 5.3]) thanks to the fact that, for every $t \in [0,1]$ and for every $f \in \mathscr{C}([0,1]^N)$,

$$\lim_{t\to 0^+} \widetilde{T}(t)(f) = \lim_{t\to 0^+} T(t)(f) = f$$

in $\mathscr{C}([0,1]^N)$ and hence in $L^p([0,1]^N)$, because $(T(t))_{t\geq 0}$ is a C_0-semigroup on $\mathscr{C}([0,1]^N)$.

Let $(\widetilde{A}, D(\widetilde{A}))$ be the generator of $(\widetilde{T}(t))_{t\geq 0}$. Then, from the definition of domain of generators, it follows that $D(A) \subset D(\widetilde{A})$ and $\widetilde{A} = A$ on $D(A)$. Moreover, $D(A)$ is a core for $(\widetilde{A}, D(\widetilde{A}))$, since $\widetilde{T}(t)(D(A)) = T(t)(D(A)) \subset D(A)$ for every $t \geq 0$.

In order to show that $\mathscr{C}^2([0,1]^N)$ is a core for $(\widetilde{A}, D(\widetilde{A}))$, fix $u \in D(\widetilde{A})$ and $\varepsilon > 0$; then there exists $v \in D(A)$ such that

$$\|u - v\|_p \leq \frac{\varepsilon}{2} \quad \text{and} \quad \|\widetilde{A}(u) - A(v)\|_p \leq \frac{\varepsilon}{2}. \tag{3.11}$$

On the other hand, by Theorem 3.2, $\mathscr{C}^2([0,1]^N)$ is a core for $(A, D(A))$ and hence there exists $w \in \mathscr{C}^2([0,1]^N)$ such that

$$\|v - w\|_\infty \le \frac{\varepsilon}{2} \quad \text{and} \quad \|A(v) - A(w)\|_\infty \le \frac{\varepsilon}{2}. \tag{3.12}$$

From (3.11) and (3.12) it follows that

$$\|u - w\|_p \le \|u - v\|_p + \|v - w\|_p \le \|u - v\|_p + \|v - w\|_\infty \le \varepsilon$$

and, analogously,

$$\|\widetilde{A}(u) - A(w)\|_p \le \varepsilon.$$

In order to prove (3.9), fix $t \ge 0$ and consider a sequence $(\rho_n)_{n\ge1}$ of positive integers such that $\lim_{n\to\infty} \rho_n/n = t$; formula (3.4) implies that, for every $f \in \mathscr{C}([0,1]^N)$,

$$\lim_{n\to\infty} C_n^{\rho_n}(f) = \widetilde{T}(t)(f)$$

in $L^p([0,1]^N)$. Since $\|C_n^{\rho_n}\|_{L^p,L^p} \le 1$ for every $n \ge 1$, then (3.9) and (3.10) follow.

Finally, consider two sequences $(a_n)_{n\ge1}$ and $(b_n)_{n\ge1}$ satisfying assumption (b) and denote by $(C_n)_{n\ge1}$ the relevant operators. Given $t \ge 0$ and a sequence $(\rho_n)_{n\ge1}$ of positive integers such that $\rho_n/n \to t$, from (3.4) it follows that

$$\widetilde{T}(t)(f) = \lim_{n\to\infty} C_n^{\rho_n}(f) \qquad \text{in } L^p([0,1]^N)$$

for every $f \in \mathscr{C}([0,1]^N)$. Moreover, (2.19) implies that

$$\|C_n^{\rho_n}\|_{L^p,L^p} \le \exp\left(\omega_p \frac{\rho_n}{n}\right) \le \exp(\rho\, \omega_p),$$

where $\rho := \sup_{n\ge1} \rho_n/n$ and $\omega_p = NM_1M_2/p$, M_2 being defined by formula (2.20) in the proof of Proposition 2.6. Consequently, $(C_n^{\rho_n})_{n\ge1}$ is equibounded in $L^p([0,1]^N)$ and hence the above limit relationship extends from $\mathscr{C}([0,1]^N)$ to $L^p([0,1]^N)$. □

Remarks 3.5.

1. Examples of sequences satisfying assumptions (b) in Theorem 3.4 can be easily furnished. For instance, fix $\alpha \ge 1$ and, for every $n \ge 1$, set $a_n := \frac{1}{2}\left(1 + \frac{1}{2n^\alpha} - \frac{n^\alpha}{n^\alpha + 1}\right)$ and $b_n := \frac{1}{2}\left(1 + \frac{1}{2n^\alpha} + \frac{n^\alpha}{n^\alpha + 1}\right)$.
2. Theorem 3.4 seems to be new even in the one-dimensional case where, according to Remark 3.3, 2, the generator $(A, D(A))$ is described by (3.6) and (3.7). However, for $N = 1$ and for $a_n = 0$ and $b_n = 1$, $n \ge 1$, a similar result has been already proved in [11, Theorem 1] with a completely different method. Moreover, in the same paper a representation of the semigroup in terms of the Legendre polynomials is also given.
3. The differential operator $(V_l, \mathscr{C}^2([0,1]^N))$ falls within a more general class of second order differential operators that have been investigated in [2] (see, in particular, Section 4, formula (4.1) and Examples 2.2, 2). From Theorem 4.1 of that paper it already follows that $(V_l, \mathscr{C}^2([0,1]^N))$ is closable

and its closure is the generator of a Markov semigroup on $\mathscr{C}([0,1]^N)$ that can be approximated, as in (3.4), by iterates of modified Bernstein-Schnabl operators. However, in general, these approximating operators are not defined on $L^p([0,1]^N)$, so that formulae (3.9) and (3.10) cannot be available for them.
4. The generation property of the operator $(V, \mathscr{C}^2([0,1]^N))$ in the space $L^p([0,1]^N)$ has been also investigated in [10, Theorem 2.5]. Moreover, in this paper it is shown that the semigroup $(\widetilde{T}(t))_{t\geq 0}$ is analytic and a description of the domain $D(\widetilde{A})$ in terms of weighted Sobolev spaces is given.

References

[1] Altomare, F., Campiti, M., *Korovkin-type approximation theory and its applications*, de Gruyter Studies in Mathematics, **17**, Walter de Gruyter & Co., Berlin, 1994.

[2] Altomare, F., Cappelletti Montano, M., Diomede, S., *Degenerate elliptic operators, Feller semigroups and modified Bernstein-Schnabl operators*, Math. Nachr., 2011, DOI:10.1002/mana.200810196.

[3] Altomare, F., Cappelletti Montano, M., Leonessa, V., *On a generalization of Kantorovich operators on simplices and hypercubes*, Adv. Pure Appl. Math., **1**(2010), no. 3, 359-385.

[4] Altomare, F., Diomede, S., *Asymptotic formulae for positive linear operators on convex subsets of Banach spaces*, J. Concr. Appl. Math., **3**(2005), no. 4, 413-435.

[5] Altomare, F., Leonessa, V., *On a sequence of positive linear operators associated with a continuous selection of Borel measures*, Mediterr. J. Math., **3**(2006), no. 3-4, 363-382.

[6] Altomare, F., Leonessa, V., *Continuous selections of Borel measures, positive operators and degenerate evolution problems*, Rev. Anal. Numér. Théor. Approx., **36**(2007), no. 1, 9-23.

[7] Cerrai, S., Clément, Ph., *Schauder estimates for a degenerate second order elliptic operator on a cube*, J. Differential Equations, **242**(2007), 287-321.

[8] Davies, E.B., *Linear operators and their spectra*, Cambridge Studies in Advanced Mathematics **106**, Cambridge University Press, Cambridge, 2007.

[9] Engel, K.J., Nagel, R., *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics, **194**, Springer, New York-Berlin, 2000.

[10] Mugnolo, D., Rhandi, A., *On the domain of a Fleming-Viot type operator on an L^p-space with invariant measure*, preprint n. 63, 2009, Dipartimento di Ingegneria dell'Informazione e Matematica Applicata, Università degli Studi di Salerno.

[11] Nagel, J., *Asymptotic properties of powers of Kantorovič operators*, J. Approx. Theory, **36**(1982), no. 3, 268-275.

[12] Nagel, R., (ed.), *One-parameter semigroups of positive operators*, Lecture Notes in Math., **1184**, Springer-Verlag, Berlin, 1986.

[13] Nishishiraho, T., *Saturation of bounded linear operators*, Tôhoku Math. J., **30**(1978), 69-81.

[14] Schnabl, R., *Über gleichmäßige Approximation durch positive lineare Operatoren*, in Constructive Theory of Functions (Proc. Internat. Conf. Varna, 1970) 287-296; Izdat. Bolgar. Akad. Nauk, Sofia, 1972.

[15] Zhou, D.X., *Converse theorems for multidimensional Kantorovich operators*, Anal. Math., **19**(1993), 85-100.

Francesco Altomare
Dipartimento di Matematica
Università degli Studi di Bari "A. Moro"
Campus Universitario, Via E. Orabona, 4
70125 Bari-Italia
e-mail: `altomare@dm.uniba.it`

Mirella Cappelletti Montano
Dipartimento di Matematica
Università degli Studi di Bari "A. Moro"
Campus Universitario, Via E. Orabona, 4
70125 Bari-Italia
e-mail: `montano@dm.uniba.it`

Vita Leonessa
Dipartimento di Matematica e Informatica
Università degli Studi della Basilicata
Viale dell'Ateneo Lucano no. 10, Macchia Romana
85100 Potenza-Italia
e-mail: `vita.leonessa@unibas.it`