Stud. Univ. Babeş-Bolyai Math. 56(2011), No. 2, 199–206

Asymptotic expansions for Favard operators and their left quasi-interpolants

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Abstract. In 1944 Favard [5, pp. 229, 239] introduced a discretely defined operator which is a discrete analogue of the familiar Gauss-Weierstrass singular convolution integral. In the present paper we consider a slight generalization F_{n,σ_n} of the Favard operator and its Durrmeyer variant $\tilde{F}_{n,\sigma_n}$ and study the local rate of convergence when applied to locally smooth functions. The main result consists of the complete asymptotic expansions for the sequences $\left(F_{n,\sigma_n} f\right)(x)$ and $\left(\tilde{F}_{n,\sigma_n} f\right)(x)$ as n tends to infinity. Furthermore, these asymptotic expansions are valid also with respect to simultaneous approximation. Finally, we define left quasi-interpolants for the Favard operator and its Durrmeyer variant in the sense of Sablonniere.

Mathematics Subject Classification (2010): 41A36, 41A60, 41A28.

Keywords: Approximation by positive operators, asymptotic expansions, simultaneous approximation.

1. Introduction

In 1944 J. Favard [5, pp. 229, 239] introduced the operator

$$\left(F_n f\right)(x) = \frac{1}{\sqrt{\pi n}} \sum_{\nu=-\infty}^{\infty} f\left(\frac{\nu}{n}\right) \exp\left(-n\left(\frac{\nu}{n} - x\right)^2\right) \tag{1.1}$$

which is a discrete analogue of the familiar Gauss-Weierstrass singular convolution integral

$$\left(W_n f\right)(x) = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} f(t) \exp\left(-n(t-x)^2\right) dt.$$

Basic properties such as saturation in weighted spaces can be found in [3] and [2]. For a sequence of positive reals σ_n, the generalization

$$\left(F_{n,\sigma_n} f\right)(x) = \sum_{\nu=-\infty}^{\infty} p_{n,\nu,\sigma_n}(x)\, f\left(\frac{\nu}{n}\right), \tag{1.2}$$

where

$$p_{n,\nu,\sigma_n}(x) = \frac{1}{\sqrt{2\pi}\, n\sigma_n} \exp\left(-\frac{1}{2\sigma_n^2}\left(\frac{\nu}{n} - x\right)^2\right),$$

was introduced and studied by Gawronski and Stadtmüller [7]. The particular case $\sigma_n^2 = \gamma/(2n)$ with a constant $\gamma > 0$ reduces to Favard's classical operators (1.1). The operators can be applied to functions f defined on $\mathbb{R}$ satisfying the growth condition

$$f(t) = O\left(e^{Kt^2}\right) \qquad \text{as } |t| \to \infty, \tag{1.3}$$

for a constant $K > 0$.

In 2007 Nowak and Sikorska-Nowak [11] considered a Kantorovich variant [11, Eq. (1.5)]

$$\left(\hat{F}_{n,\sigma_n} f\right)(x) = n \sum_{\nu=-\infty}^{\infty} p_{n,\nu,\sigma_n}(x) \int_{\nu/n}^{(\nu+1)/n} p_{n,\nu,\sigma_n}(t)\, f(t)\, dt$$

and a Durrmeyer variant [11, Eq. (1.6)]

$$\left(\tilde{F}_{n,\sigma_n} f\right)(x) = n \sum_{\nu=-\infty}^{\infty} p_{n,\nu,\sigma_n}(x) \int_{-\infty}^{\infty} p_{n,\nu,\sigma_n}(t)\, f(t)\, dt \tag{1.4}$$

of Favard operators. Further related papers are [12] and [13].

The main result of this paper consists of the complete asymptotic expansions

$$F_{n,\sigma_n} f \sim f + \sum_{k=0}^{\infty} c_k(f)\, \sigma_n^k \quad \text{and} \quad \tilde{F}_{n,\sigma_n} f \sim f + \sum_{k=0}^{\infty} \tilde{c}_k(f)\, \sigma_n^k \qquad (n \to \infty),$$

for f sufficiently smooth. The coefficients c_k and $\tilde{c}_k$, which depend on f but are independent of n, are explicitly determined. It turns out that $c_k(f) = 0$, for all odd integers $k > 0$. Moreover, we deal with simultaneous approximation by the operators (1.2).

Finally, we define left quasi-interpolants for the Favard operator and its Durrmeyer variant in the sense of Sablonniere.

2. Complete asymptotic expansions

Throughout the paper, we assume that

$$\sigma_n > 0, \quad \sigma_n \to 0, \quad \sigma_n^{-1} = O\left(n^{1-\eta}\right) \qquad (n \to \infty) \tag{2.1}$$

with (an arbitrarily small) constant $\eta > 0$. Note that the latter condition implies that $n\sigma_n \to \infty$ as $n \to \infty$.

Under these conditions, the operators possess the basic property that $(F_n f)(x)$ converges to $f(x)$ in each continuity point x of f. Among other results, Gawronski and Stadtmüller [7, Eq. (0.6)] established the Voronovskaja-type theorem

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2} \left[(F_{n,\sigma_n} f)(x) - f(x) \right] = \frac{1}{2} f''(x) \tag{2.2}$$

uniformly on proper compact subsets of $[a,b]$, for $f \in C^2[a,b]$ $(a,b \in \mathbb{R})$ and $\sigma_n \to 0$ as $n \to \infty$, provided that certain conditions on the first three moments of F_{n,σ_n} are satisfied. Actually, Eq. (2.2) was proved for a truncated variant of (1.2) which possesses the same asymptotic properties as (1.2) [7, cf. Theorem 1 (iii) and Remark (i), p. 393]. For a Voronovskaja-type theorem in the particular case $\sigma_n^2 = \gamma/(2n)$ cf. [3, Theorem 4.3]. Abel and Butzer extended Formula (2.2) by deriving a complete asymptotic expansion of the form

$$F_{n,\sigma_n} f \sim f + \sum_{k=0}^{\infty} c_k(f)\, \sigma_n^k \qquad (n \to \infty),$$

for f sufficiently smooth. The latter formula means that, for all positive integers q, there holds pointwise on $\mathbb{R}$

$$F_{n,\sigma_n} f = f + \sum_{k=1}^{q} c_k(f)\, \sigma_n^k + o(\sigma_n^q) \qquad (n \to \infty).$$

The following theorem presents the main result of this paper, the complete asymptotic expansion for the sequence $\left(\tilde{F}_{n,\sigma_n}\right)(x)$ as $n \to \infty$. For $r \in \mathbb{N}$ and $x \in \mathbb{R}$ let $W[r;x]$ be the class of functions on $\mathbb{R}$ satisfying growth condition (1.3), which admit a derivative of order r at the point x.

Theorem 2.1. *Let $q \in \mathbb{N}$ and $x \in \mathbb{R}$. Suppose that the real sequence (σ_n) satisfies the conditions* (2.1). *For each function $f \in W[2q;x]$, the Favard-Durrmeyer operators* (1.4) *possess the complete asymptotic expansions*

$$(F_{n,\sigma_n} f)(x) = f(x) + \sum_{k=1}^{q} \frac{f^{(2k)}(x)}{(2k)!!} \sigma_n^{2k} + o\left(\sigma_n^{2q}\right) \tag{2.3}$$

and

$$\left(\tilde{F}_{n,\sigma_n} f\right)(x) = f(x) + \sum_{k=1}^{q} \frac{f^{(2k)}(x)}{k!} \sigma_n^{2k} + o\left(\sigma_n^{2q}\right) \tag{2.4}$$

as $n \to \infty$.

Here $m!!$ denote the double factorial numbers defined by $0!! = 1!! = 1$ and $m!! = m \times (m-2)!!$ for integers $m \geq 2$. It turns out that the asymptotic expansions contain only terms with even order derivatives of the function f.

As an immediate consequence we obtain the following Voronosvkaja-type theorems.

Corollary 2.2. *Let $x \in \mathbb{R}$. Suppose that the real sequence (σ_n) satisfies the conditions* (2.1)*. For each function $f \in W[2;x]$, there hold the asymptotic relations*

$$\lim_{n\to\infty} \sigma_n^{-2} \left(\left(F_{n,\sigma_n} f \right)(x) - f(x) \right) = \frac{1}{2} f''(x)$$

and

$$\lim_{n\to\infty} \sigma_n^{-2} \left(\left(\tilde{F}_{n,\sigma_n} f \right)(x) - f(x) \right) = f''(x).$$

Concerning simultaneous approximation, it turns out that the complete asymptotic expansion (2.3) can be differentiated term-by-term. Indeed, there holds

Theorem 2.3. *Let $\ell \in \mathbb{N}_0$, $q \in \mathbb{N}$ and $x \in \mathbb{R}$. Suppose that the real sequence (σ_n) satisfies condition* (2.1)*. For each function $f \in W[2(\ell+q);x]$, the following complete asymptotic expansions are valid as $n \to \infty$:*

$$\left(F_{n,\sigma_n} f \right)^{(\ell)}(x) = f(x) + \sum_{k=1}^{q} \frac{f^{(2k+\ell)}(x)}{(2k)!!} \sigma_n^{2k} + o\left(\sigma_n^{2q}\right) \tag{2.5}$$

and

$$\left(\tilde{F}_{n,\sigma_n} f \right)^{(\ell)}(x) = f^{(\ell)}(x) + \sum_{k=1}^{q} \frac{f^{(2k+\ell)}(x)}{k!} \sigma_n^{2k} + o\left(\sigma_n^{2q}\right). \tag{2.6}$$

Remark 2.4. The latter formulas can be written in the equivalent form

$$\lim_{n\to\infty} \sigma_n^{-2q} \left(\left(F_{n,\sigma_n} f \right)^{(\ell)}(x) - f^{(\ell)}(x) - \sum_{k=1}^{q} \frac{f^{(2k+\ell)}(x)}{(2k)!!} \sigma_n^{2k} \right) = 0,$$

$$\lim_{n\to\infty} \sigma_n^{-2q} \left(\left(\tilde{F}_{n,\sigma_n} f \right)^{(\ell)}(x) - f^{(\ell)}(x) - \sum_{k=1}^{q} \frac{f^{(2k+\ell)}(x)}{k!} \sigma_n^{2k} \right) = 0.$$

Assuming smoothness of f on intervals $I = (a,b)$, $a, b \in \mathbb{R}$, it can be shown that the above expansions hold uniformly on compact subsets of I.

The proofs are based on localization theorems which are interesting in themselves. We quote only the result for the ordinary Favard operator (1.2).

Proposition 2.5. *Fix $x \in \mathbb{R}$ and let $\delta > 0$. Assume that the function $f : \mathbb{R} \to \mathbb{R}$ vanishes in $(x-\delta, x+\delta)$ and satisfies, for positive constants M_x, K_x, the growth condition*

$$|f(t)| \le M_x e^{K_x (t-x)^2} \qquad (t \in \mathbb{R}). \tag{2.7}$$

Then, for positive $\sigma < 1/\sqrt{2K_x}$, there holds the estimate

$$\left| \left(F_{n,\sigma} f \right)(x) \right| \le \sqrt{\frac{2}{\pi}} \frac{M_x \sigma / \delta}{1 - 2K_x \sigma^2} \exp\left(-\frac{1 - 2K_x \sigma^2}{2} \left(\frac{\delta}{\sigma} \right)^2 \right).$$

Consequently, under the general assumption (2.1) a positive constant A (independent of δ) exists such that the sequence $((F_{n,\sigma_n} f)(x))$ can be estimated by

$$(F_{n,\sigma_n} f)(x) = o\left(\exp\left(-A\frac{\delta^2}{\sigma_n^2}\right)\right) \qquad (n \to \infty).$$

Remark 2.6. Note that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies condition (2.1) if and only if condition (2.7) is valid. The elementary inequality $(t-x)^2 \leq 2\left(t^2+x^2\right)$ implies that

$$M_x e^{K_x (t-x)^2} \leq M e^{Kt^2} \qquad (t, x \in \mathbb{R})$$

with constants $M = M_x e^{2Kx^2}$ and $K = 2K_x$.

3. Quasi-interpolants

The results of the preceding section show that the optimal degree of approximation cannot be improved in general by higher smoothness properties of the function f. In order to obtain much faster convergence quasi-interpolants were considered. Let us shortly recall the definition of the quasi-interpolants in the sense of Sablonniere [14]. For another method to construct quasi-interpolants see [8] and [9].

If the operators $\mathcal{B}_n$ let invariant the space of algebraic polynomials Π_j of each order $j = 0, 1, 2, \ldots$ (the most approximation operators possess this property), i.e.,

$$\mathcal{B}_n(\Pi_j) \subseteq \Pi_j \qquad (0 \leq j \leq n),$$

$B_n : \Pi_n \to \Pi_n$ is an isomorphism which can be represented by linear differential operators

$$\mathcal{B}_n = \sum_{k=0}^{n} \beta_{n,k} D^k$$

with polynomial coefficients $\beta_{n,k}$ and $Df = f'$, D^0 =id. The inverse operator $\mathcal{B}_n^{-1} \equiv \mathcal{A} : \Pi_n \to \Pi_n$ satisfies

$$\mathcal{A} = \sum_{k=0}^{n} \alpha_{n,k} D^k$$

with polynomial coefficients $\alpha_{n,k}$. Sablonniere defined new families of intermediate operators obtained by composition of B_n and its truncated inverses

$$\mathcal{A}_n^{(r)} = \sum_{k=0}^{r} \alpha_{n,k} D^k.$$

In this way he obtained a family of left quasi-interpolants (LQI) defined by

$$\mathcal{B}_n^{(r)} = \mathcal{A}_n^{(r)} \circ \mathcal{B}_n, \quad 0 \leq r \leq n,$$

and a family of right quasi-interpolants (RQI) defined by

$$\mathcal{B}_n^{[r]} = \mathcal{B}_n \circ \mathcal{A}_n^{(r)}, \quad 0 \leq r \leq n.$$

Obviously, there holds $\mathcal{B}_n^{(0)} = \mathcal{B}_n^{[0]} = \mathcal{B}_n$, and $\mathcal{B}_n^{(n)} = \mathcal{B}_n^{[n]} = I$ when acting on Π_n. In the following we consider only the family of LQI. The definition reveals that $\mathcal{B}_n^{(r)} f$ is a linear combination of derivatives of $\mathcal{B}_n f$. Furthermore, $\mathcal{B}_n^{(r)}$ $(0 \le r \le n)$ has the nice property to preserve polynomials of degree up to r, because, for $p \in \Pi_r$, we have

$$\begin{aligned}\mathcal{B}_n^{(r)} p &= \left(\mathcal{A}_n^{(r)} \circ \mathcal{B}_n\right) p = \sum_{k=0}^{r} \alpha_{n,k} D^k \underbrace{(\mathcal{B}_n p)}_{\in \Pi_r} = \sum_{k=0}^{n} \alpha_{n,k} D^k (\mathcal{B}_n p) \\ &= \left(\mathcal{A}_n^{-1} \circ \mathcal{B}_n\right) p = p.\end{aligned}$$

In many instances there holds $L_n^{(r)} f - f = O\left(n^{-\lfloor r/2+1 \rfloor}\right)$ as $n \to \infty$.

Unfortunately, the Favard operator as well as its Durrmeyer variant doesn't let invariant the spaces Π_j, for $0 \le j \le n$. However, under appropriate assumptions on the sequence (σ_n) they do it asymptotically up to a remainder which decays exponentially fast as n tends to infinity. Writing $\simeq$ for this "asymptotic equality" we obtain, for fixed $n \in \mathbb{N}$,

$$\begin{aligned} F_{n,\sigma_n} p_k &\simeq e_k \\ \text{with } p_k &= k! \sum_{j=0}^{\lfloor k/2 \rfloor} (-1)^j \frac{\sigma_n^{2j}}{2^j j!\,(k-2j)!} e_{k-2j}, \end{aligned}$$

where e_m denote the monomials $e_m(t) = t^m$ $(m = 0, 1, 2, \ldots)$. Hence, for the inverse,

$$(F_{n,\sigma_n})^{-1} e_k \simeq p_k = \sum_{j=0}^{\lfloor k/2 \rfloor} \underbrace{(-1)^j \frac{\sigma_n^{2j}}{2^j j!}}_{=\alpha_{n,2j}} D^{2j} e_k$$

Note that $\beta_{n,2k+1} = \alpha_{n,2k+1} = 0$ $(k = 0, 1, 2, \ldots)$ and that neither $\beta_{n,k}$ nor $\alpha_{n,k}$ depend on the variable x. The analogous results for the Favard-Durrmeyer operators are similar. Proceeding in this way we define the following operators:

Definition 3.1 (Favard quasi-interpolants). *The left quasi-interpolants* $F_{n,\sigma_n}^{(r)}$ *and* $\tilde{F}_{n,\sigma_n}^{(r)}$ $(r = 0, 1, 2, \ldots)$ *of the Favard and Favard-Durrmeyer operators, respectively, are given by*

$$F_{n,\sigma_n}^{(r)} = \sum_{k=0}^{r} \alpha_{n,k} D^k F_{n,\sigma_n} := \sum_{k=0}^{\lfloor r/2 \rfloor} (-1)^k \frac{\sigma_n^{2k}}{2^k k!} D^{2k} F_{n,\sigma_n}$$

and

$$\tilde{F}_{n,\sigma_n}^{(r)} = \sum_{k=0}^{r} \tilde{\alpha}_{n,k} D_{n,\sigma_n}^k \tilde{F}_{n,\sigma_n} := \sum_{k=0}^{\lfloor r/2 \rfloor} (-1)^k \frac{\sigma_n^{2k}}{k!} D^{2k} \tilde{F}_{n,\sigma_n}.$$

Remark 3.2. Note that $F_{n,\sigma_n}^{(2r)} = F_{n,\sigma_n}^{(2r+1)}$ and $\tilde{F}_{n,\sigma_n}^{(2r)} = \tilde{F}_{n,\sigma_n}^{(2r+1)}$ $(r = 0, 1, 2, \ldots)$.

The local rate of convergence is given by the next theorem.

Theorem 3.3. *Let* $\ell \in \mathbb{N}_0$, $q \in \mathbb{N}$ *and* $x \in \mathbb{R}$. *Suppose that the real sequence* (σ_n) *satisfies condition* (2.1). *For each function* $f \in W[2(\ell+q);x]$, *the following complete asymptotic expansions are valid as* $n \to \infty$*:*

$$\left(F_{n,\sigma_n}^{(2r)} f\right)^{(\ell)}(x) \sim f^{(\ell)}(x) + (-1)^r \sum_{k=r+1}^{\infty} \binom{k-1}{r} \frac{f^{(2k+\ell)}(x)}{(2k)!!} \sigma_n^{2k}$$

and

$$\left(\tilde{F}_{n,\sigma_n}^{(2r)} f\right)^{(\ell)}(x) = f^{(\ell)}(x) + (-1)^r \sum_{k=1}^{q} \binom{k-1}{r} \frac{f^{(2k+\ell)}(x)}{k!} \sigma_n^{2k} + o\left(\sigma_n^{2q}\right).$$

Remark 3.4. An immediate consequence are the asymptotic relations

$$\left(F_{n,\sigma_n}^{(2r)} f\right)(x) - f(x) = O\left(\sigma_n^{2(r+1)}\right)$$

and

$$\left(\tilde{F}_{n,\sigma_n}^{(2r)} f\right)(x) - f(x) = O\left(\sigma_n^{2(r+1)}\right)$$

as $n \to \infty$.

Acknowledgment. The author would like to thank the anonymous referee for the very careful reading of the manuscript. The valuable comments and suggestions led to several improvements in the paper.

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