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Discrete operators associated with certain integral operators

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Abstract. We associate to a given sequence of positive linear integral operators a sequence of discrete operators and investigate the relationship between the two sequences. Several examples illustrate the general results.

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1. Introduction

Let $I_n : C[a,b] \longrightarrow C[a,b],\ n \geq 1$, be a sequence of positive linear operators of the form

$$I_n(f;x) = \sum_{k=0}^{n} h_{n,k}(x) A_{n,k}(f),\ f \in C[a,b],\ x \in [a,b],$$

where $h_{n,k} \in C[a,b],\ h_{n,k} \geq 0$ and

$$A_{n,k}(f) = \int_a^b f(t) d\mu_{n,k}(t)$$

with $\mu_{n,k}$ probability Borel measures on $[a,b], n \geq 1, k = 0, 1, \ldots, n$.

Let $x_{n,k} \in [a,b]$ be the barycenter of $\mu_{n,k}$, i.e.,

$$x_{n,k} = \int_a^b t d\mu_{n,k}(t).$$

We associate with the sequence (I_n) the sequence of operators

$$D_n(f;x) = \sum_{k=0}^{n} h_{n,k}(x) f(x_{n,k}).$$

Generally speaking, the operators D_n are simpler than I_n. We investigate the properties of D_n in relation with those of I_n.

2. Some examples

For $n \geq 1$ and $k = 0, 1, \ldots, n$ let

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \; x \in [0,1].$$

Example 2.1. Let $U_n : C[0,1] \longrightarrow C[0,1]$ be the genuine Bernstein-Durrmeyer operators (see [3] and the references therein) defined by

$$U_n(f;x) := f(0)p_{n,0}(x) + f(1)p_{n,n}(x) +$$
$$+(n-1)\sum_{k=1}^{n-1} p_{n,k}(x) \int_0^1 p_{n-2,k-1}(t)f(t)dt.$$

It is easy to see that the associated operators are the classical Bernstein operators

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\Big(\frac{k}{n}\Big).$$

Example 2.2. Consider the sequences of real numbers a_n and b_n such that $0 \leq a_n < b_n \leq 1, n \geq 1$. In [1] the authors introduced and investigated the operators

$$C_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) \Big(\frac{n+1}{b_n - a_n} \int_{\frac{k+a_n}{n+1}}^{\frac{k+b_n}{n+1}} f(t)dt \Big),$$

where $f \in C[0,1]$ and $x \in [0,1]$.

The associated operators are the Stancu type operators (see [15])

$$S_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\Big(\frac{2k + a_n + b_n}{2(n+1)}\Big).$$

In particular, for $a_n = 0$ and $b_n = 1, (C_n)$ becomes the sequence of classical Kantorovich operators.

Example 2.3. Let $a, b > -1$ and $\alpha \geq 0$. Consider the positive linear functionals $T_{n,k} : C[0,1] \longrightarrow \mathbb{R}$,

$$T_{n,k}(f) := \frac{\int_0^1 f(t) t^{ck+a}(1-t)^{c(n-k)+b} dt}{B(ck+a+1, c(n-k)+b+1)},$$

where $c := c_n := [n^\alpha]$ and B is the Beta function.

For $f \in C[0,1]$ and $x \in [0,1]$ let

$$P_n(f;x) := \sum_{k=0}^{n} p_{n,k}(x) T_{n,k}(f), ; n \geq 1.$$

The sequence of positive linear operators (P_n) was introduced by D. Mache (see [5], [6]); it represents a link between the Durrmeyer operators with Jacobi weights (obtained for $\alpha = 0$) and the Bernstein operators (obtained as a limiting case when $\alpha \longrightarrow \infty$). Concerning the properties of the operators P_n and their relationship with Durrmeyer, Bernstein, and other operators, see [5], [6], [8], [9], [10], [11]. The semigroup of operators, represented in terms of iterates of P_n, is investigated in [2], [9], [10], [11], [12].

Let $e_i(x) = x^i,\ x \in [0,1],\ i = 0, 1, \ldots$. Then $T_{n,k}(e_0) = 1$ and the barycenter of the probability Radon measure $T_{n,k}$ is

$$T_{n,k}(e_1) = \frac{ck + a + 1}{cn + a + b + 2}.$$

As in Section 1, we associate with the sequence (P_n) the simpler sequence of positive linear operators (V_n) defined, for $f \in C[0,1]$ and $x \in [0,1]$, by

$$V_n(f;x) := \sum_{k=0}^{n} p_{n,k}(x) f\Big(\frac{ck + a + 1}{cn + a + b + 2}\Big).$$

When $a = b = -1$, or when $\alpha \longrightarrow \infty$, we get the classical Bernstein operators; when $\alpha = 0$, the operators V_n reduce to the operators considered by D.D. Stancu in [15].

In the next sections we investigate the properties of the operators (V_n) in connection with the properties of (P_n); see also [7].

3. Approximation properties

By direct computation we get

$$V_n e_0 = e_0,$$

$$V_n e_1 = \frac{cne_1 + (a+1)e_0}{cn + a + b + 2},$$

$$V_n e_2 = \frac{c^2 n(n-1)e_2 + cn(c + 2a + 2)e_1 + (a+1)^2 e_0}{(cn + a + b + 2)^2}.$$

Let us remark that

$$\lim_{n \to \infty} V_n e_i = e_i,\ i = 0, 1, 2,$$

uniformly on $[0,1]$.

From the classical Korovkin Theorem we infer:

Proposition 3.1. *For all* $f \in C[0,1]$,

$$\lim_{n \to \infty} V_n f = f,\ \textit{uniformly on } [0,1].$$

In the sequel we shall use the inequality

$$|L(f) - f(b)| \le (L(e_2) - b^2)\frac{\|f''\|}{2}, \quad f \in C^2[0,1],$$

where L is a probability Radon measure on $[0,1],\ b = L(e_1)$ is the barycenter of L, and $\|\cdot\|$ is the uniform norm. To prove this inequality, it suffices to apply the *barycenter inequality*

$$L(h) \ge h(b),\ h \in C[0,1] \text{ convex},$$

to the convex functions $\dfrac{\|f''\|}{2} e_2 \pm f$.

Theorem 3.2. *For $n \geq 1,\ x \in [0,1],$ and $f \in C^2[0,1]$ we have*

$$|P_n(f;x) - V_n(f;x)| \leq \frac{c^2 n(n-1)x(1-x) + cn(b-a)x + cn(a+1) + (a+1)(b+1)}{2(cn+a+b+2)^2(cn+a+b+3)} \|f''\|.$$

Proof. Since the barycenter of $T_{n,k}$ is

$$\frac{ck+a+1}{cn+a+b+2},$$

we have

$$|T_{n,k}(f) - f\Big(\frac{ck+a+1}{cn+a+b+2}\Big)| \leq \Big(T_{n,k}(e_2) - \Big(\frac{ck+a+1}{cn+a+b+2}\Big)^2\Big)\frac{\|f''\|}{2}$$
$$= \frac{(ck+a+1)(c(n-k)+b+1)}{(cn+a+b+2)^2(cn+a+b+3)}\frac{\|f''\|}{2}.$$

Consequently,

$$|P_n(f;x) - V_n(f;x)| \leq \frac{\|f''\|}{2}\sum_{k=0}^{n} p_{n,k}(x)\frac{(ck+a+1)(c(n-k)+b+1)}{(cn+a+b+2)^2(cn+a+b+3)}$$
$$= \frac{\|f''\|}{2}\frac{c^2n(n-1)x(1-x) + cn(b-a)x + cn(a+1) + (a+1)(b+1)}{(cn+a+b+2)^2(cn+a+b+3)}.$$

□

Let us remark that for $\alpha = a = b = 0$ the operators P_n reduce to the classical Durrmeyer operators M_n. Consequently, the previous theorem yields

Corollary 3.3. *For $n \geq 1, x \in [0,1]$ and $f \in C^2[0,1]$ we have*

$$|M_n(f;x) - \sum_{k=0}^{n} p_{n,k}(x) f\Big(\frac{k+1}{n+2}\Big)| \leq$$
$$\leq \frac{n(n-1)x(1-x)+n+1}{2(n+2)^2(n+3)}||f''||.$$

4. Asymptotic formulae

The moments of the operator V_n are defined by

$$M_{n,m}(x) := V_n((e_1 - xe_0)^m; x) = \sum_{k=0}^{n}\Big(\frac{ck+a+1}{cn+a+b+2} - x\Big)^m p_{n,k}(x).$$

Let us remark that

$$M'_{n,m}(x) = \sum_{k=0}^{n}\Big(\frac{ck+a+1}{cn+a+b+2} - x\Big)^m p'_{n,k}(x) - mM_{n,m-1}(x).$$

Since

$$x(1-x)p'_{n,k}(x) = (k-nx)p_{n,k}(x),$$

we get

$$
\begin{aligned}
x(1-x)M'_{n,m}(x) &= \sum_{k=0}^{n}\Big(\frac{ck+a+1}{cn+a+b+2}-x\Big)^{m}(k-nx)p_{n,k}(x)\\
&- mx(1-x)M_{n,m-1}(x) =\\
&= \frac{cn+a+b+2}{c}\sum_{k=0}^{n}\Big(\frac{ck+a+1}{cn+a+b+2}-x\Big)^{m+1}p_{n,k}(x)\\
&- \frac{a+1-(a+b+2)x}{c}\sum_{k=0}^{n}\Big(\frac{ck+a+1}{cn+a+b+2}-x\Big)^{m}p_{n,k}(x)\\
&- mx(1-x)M_{n,m-1}(x).
\end{aligned}
$$

Consequently, the following recurrence formula for the moments of V_n is valid:

Theorem 4.1. *For all $n \geq 1$ and $m \geq 1$,*

$$
\begin{aligned}
&(cn+a+b+2)M_{n,m+1}(x) = cx(1-x)M'_{n,m}(x)+\\
&+(a+1-(a+b+2)x)M_{n,m}(x) + cmx(1-x)M_{n,m-1}(x).
\end{aligned}
$$

It is easy to verify that

$$M_{n,0}(x) = 1,\ M_{n,1}(x) = \frac{a+1-(a+b+2)x}{cn+a+b+2}.$$

By using the recurrence formula we get

$$M_{n,2}(x) = \frac{c^2nx(1-x)+(a+1-(a+b+2)x)^2}{(cn+a+b+2)^2}.$$

The same recurrence formula can be used in order to verify that

$$M_{n,m}(x) = O(n^{-[\frac{m+1}{2}]}),\ m \geq 0,$$

uniformly for $x \in [0,1]$.

Now the assumptions of Sikkema's theorem [14] are fulfilled; consequently, we have the following Voronovskaja type formula:

Theorem 4.2.

$$\lim_{n\to\infty} n(V_n(f;x)-f(x)) = \begin{cases} \frac{x(1-x)}{2}f''(x)+(a+1-(a+b+2)x)f'(x),\ \alpha=0\\ \frac{x(1-x)}{2}f''(x),\ \alpha>0, \end{cases}$$

for all $f \in C[0,1]$ such that $f''(x)$ exists and is finite.Moreover, if $f \in C^2[0,1]$, the convergence is uniform on $[0,1]$.

Concerning the (similar) Voronovskaja formula for the operators P_n, see [10] and the references given there.

5. Iterates of V_n

Let r be a non-negative integer, $r \le n$. It is well-known (see, e.g., [4] and the references given there) that

$$B_n e_r = \frac{n(n-1)\dots(n-r+1)}{n^r} e_r \ + \textit{terms of lower degree},$$

where B_n are the classical Bernstein operators.

Let

$$\varphi_r := \Big(\frac{cne_1 + (a+1)e_0}{cn+a+b+2}\Big)^r.$$

Then, for $k = 0, 1, \dots, n$,

$$\varphi_r(\frac{k}{n}) = \Big(\frac{ck+a+1}{cn+a+b+2}\Big)^r,$$

so that

$$\begin{aligned} V_n e_r = B_n \varphi_r &= \Big(\frac{cn}{cn+a+b+2}\Big)^r B_n e_r + \text{terms of lower degree} \\ &= \frac{n(n-1)\dots(n-r+1)}{(cn+a+b+2)^r} c^r e_r + \text{terms of lower degree.} \end{aligned}$$

It follows that:

Theorem 5.1. *The numbers*

$$\lambda_r := \frac{n(n-1)\dots(n-r+1)}{(cn+a+b+2)^r} c^r, \quad r = 0, 1, \dots, n,$$

are eigenvalues of V_n, and the eigenfunction corresponding to λ_r can be chosen as a monic polynomial of degree r.

Now let us describe V_n as

$$V_n(f;x) = \sum_{k=0}^n p_{n,k}(x) f\Big(\frac{k + \frac{a+1}{c}}{n + \frac{a+b+2}{c}}\Big)$$

Under this form we see that V_n coincides with the operator $S_n^{<0,\beta,\gamma>}$ defined in [4;(1)], if

$$\beta := \frac{a+1}{c}\ ,\ \gamma := \frac{a+b+2}{c}.$$

Now the above Theorem 5.1 can be considered also as a consequence of Theorem 1 in [4].

The over-iterates of V_n can be studied by using the results of [4] or [13]. Indeed, let

$$a_j := \frac{j+\beta}{n+\gamma} = \frac{cj+a+1}{cn+a+b+2}\ ,\ j = 0, 1, \dots, n.$$

From [4;(9), (11), (12)] or from [13; Th. 5.3] we deduce for $f \in C[0,1]$:

$$\lim_{m\to\infty} V_n^m f = e_0 \sum_{j=0}^n d_j f\Big(\frac{cj+a+1}{cn+a+b+2}\Big),$$

uniformly on $[0,1]$, where $(d_0, d_1, \dots, d_n)$ is the unique solution of the system

$$\begin{pmatrix} p_{n,0}(a_0) & \dots & p_{n,0}(a_n) \\ \dots & \dots & \dots \\ p_{n,n}(a_0) & \dots & p_{n,n}(a_n) \end{pmatrix} \begin{pmatrix} d_0 \\ \dots \\ d_n \end{pmatrix} = \begin{pmatrix} d_0 \\ \dots \\ d_n \end{pmatrix}$$

satisfying $d_0 \geq 0, \dots, d_n \geq 0, d_0 + \dots + d_n = 1$.

6. Shape preserving properties

For each $m \geq 0$ consider the function

$$\varphi_m(t) = \Big(\frac{cnt + a + 1}{cn + a + b + 2}\Big)^m, \ t \in [0,1].$$

Let B_n be the classical Bernstein operators on $C[0,1]$. Then we have

$$V_n e_m = B_n \varphi_m, \ n \geq 1.$$

Consequently, the technique used in [16, Section 25.2] can be applied; as in [16, Cor.25.2] we get

Theorem 6.1. *If $0 \leq m \leq n$ and $f \in C[0,1]$ is convex of order m, then $V_n f$ is convex of order m.*

For convex functions of order 1, i.e., usual convex functions, we have also

Theorem 6.2. *If $f \in C[0,1]$ is convex, then*

$$P_n(f;x) \geq V_n(f;x) \geq f\Big(\frac{cnx + a + 1}{cn + a + b + 2}\Big), \ x \in [0,1].$$

Proof. Let $f \in C[0,1]$ be convex, and $x \in [0,1]$. From the barycenter inequality we know that

$$T_{n,k}(f) \geq f\Big(\frac{ck + a + 1}{cn + a + b + 2}\Big), \ k = 0, 1, \dots, n,$$

which immediately yields

$$P_n(f;x) \geq V_n(f;x).$$

On the other hand, consider the probability Radon measure

$$g \longrightarrow V_n(g;x), \ g \in C[0,1].$$

The corresponding barycenter is

$$V_n(e_1;x) = \frac{cnx + a + 1}{cn + a + b + 2}.$$

Again by the barycenter inequality we get

$$V_n(f;x) \geq f\Big(\frac{cnx + a + 1}{cn + a + b + 2}\Big). \qquad \square$$

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