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# Estimates for general positive linear operators on non-compact interval using weighted moduli of continuity

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**Abstract.** We give estimates with explicit constants of the degree of approximation by general positive linear operators on the interval $[0, \infty)$, using a weighted modulus of continuity. In particular we obtain a quantitative version of a result of Totik concerning Szász-Mirakjan operators.

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## 1. Introduction

The moduli of continuity or smoothness of different kinds play a crucial role in estimating the degree of approximation by using linear methods. In approximation on non-compact intervals more convenient are the weighted moduli. There are several types of constructions of weighted moduli of first order. A very short list of contributions in this directions are given in References.

In this paper we introduce a class of first order weighted moduli of continuity constructed starting from a family of "admissible" functions and we deduce estimates for general positive operators. These estimates are with explicit constants. Such type of estimates are already obtained for weighted moduli on a compact interval, for the Ditzian-Totik modulus of second order, (see [9], [8], [12]).

Finally we remark that, in the case of a certain admissible function, our modulus is equivalent to the usual modulus applied to a certain modification of the function. This last modulus was used by Totik [14] for Szász-Mirakjan operators.

## 2. A general estimate with the modulus $\omega^{\varphi}$

Denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $k \in \mathbb{N}$ denote by $\Pi_k$, the space of polynomials of degree at most $k$ and for $j \in \mathbb{N}_0$ consider the monomial functions $e_j(t) = t^j$, $t \in [0,\infty)$. Denote by $[a]$, the integer part of a number $a \in \mathbb{R}$. Denote also by $\mathcal{F}(I)$, the space of real functions defined on an interval $I$.

We adopt the following

**Definition 2.1.** *A function $\varphi \in C([0,\infty))$ is named admissible if it satisfies the following conditions:*

i) $\varphi(t) > 0$, *for* $t \in (0,\infty)$*;*
ii) $\frac{1}{\varphi}$ *is convex on interval* $(0,\infty)$*;*
iii) *we have*

$$\lim_{a\to+0} \int_a^x \frac{\mathrm{d}t}{\varphi(t)} < \infty \quad \textit{for all } x > 0; \tag{2.1}$$

iv) *we have*

$$\int_0^\infty \frac{\mathrm{d}t}{\varphi(t)} = +\infty. \tag{2.2}$$

In this definition we use the Riemann improper integral. Using an admissible function $\varphi$ we introduce the following first order weighted modulus.

**Definition 2.2.** *For $f \in \mathcal{F}([0,\infty))$, and $h > 0$ set:*

$$\omega^{\varphi}(f,h) = \sup\left\{|f(v) - f(u)| \ : \ u,v \in [0,\infty),\ |v-u| \le h\varphi\left(\frac{u+v}{2}\right)\right\}. \tag{2.3}$$

We admit in this definition that the supremum could be equal to $+\infty$.

**Remark 2.3.** Function $e_0$ is admissible and for $\varphi = e_0$ we obtain $\omega^{\varphi} = \omega$, where $\omega$ denotes the usual first order modulus.

Property iii) allows to take $\varphi$ with condition $\frac{1}{\varphi(x)} = \mathrm{O}(x^{\alpha})$ $(x \to 0)$, with $\alpha > -1$. Very suitable for applications is the case $\varphi(x) \sim \sqrt{x}$ $(x \to 0)$, when the dependence of modulus $\omega^{\varphi}(f,\cdot)$ on the values taken by a function $f$ in a neighbourhood of the point $x = 0$ is similar with the dependence of the first order Ditzian-Totik modulus on the values taken by a function near the end points of the interval $[0,1]$. However if we take $\varphi(x) = \sqrt{x}$, for $x \ge 0$, then $\omega^{\varphi}(f,h)$ is finite for any $h > 0$ only if $f$ satisfies the restrictive condition $f(x) = \mathrm{O}(\sqrt{x})$ $(x \to \infty)$. This fact can be deduced, for instance, from Remark 2.6 in Section 2.

In order to enlarge the class of functions for which $\omega^{\varphi}(f,h) < \infty$, for any $h > 0$, by condition iv), we have the possibility to take $\varphi$ rapidly decreasing to 0 when $x \to \infty$. For instance an admissible function is $\varphi(x) = \frac{\sqrt{x}}{1+x^m}$, $x \ge 0$, for $m \in \mathbb{N}$, $m \ge 2$. Then we have $\omega^{\varphi}(f,h) < \infty$, for any differentiable function $f$ such that $|f'(x)| \le Mx^{m-\frac{1}{2}}$.

Given an admissible function $\varphi$, we consider the following corresponding function

$$\Phi(x) = \int_0^x \frac{\mathrm{d}t}{\varphi(t)},\ x \in (0,\infty). \tag{2.4}$$

**Lemma 2.4.** *Let $f \in \mathcal{F}([0,\infty))$, $h > 0$ and $0 \le a < b$, such that $\Phi(b) - \Phi(a) = h$. Then for all points $c, d$ such that $a \le c \le d \le b$, we have*

$$|f(d) - f(c)| \le \omega^{\varphi}(f,h). \tag{2.5}$$

*Proof.* We have to show that $d - c \le h\varphi\left(\frac{c+d}{2}\right)$.

From condition iii) of Definition 2.1 we deduce, using Jensen inequality:

$$\frac{d-c}{\varphi\left(\frac{c+d}{2}\right)} \le \int_c^d \frac{\mathrm{d}t}{\varphi(t)}.$$

But

$$\int_c^d \frac{\mathrm{d}t}{\varphi(t)} \le \int_a^b \frac{\mathrm{d}t}{\varphi(t)} = \Phi(b) - \Phi(a) = h.$$

□

**Lemma 2.5.** *Let $f \in \mathcal{F}([0,\infty))$, $x > 0$ and $h > 0$. We have*

$$|f(t) - f(x)| \le \left(1 + \frac{1}{h^2}\left(\Phi(t) - \Phi(x)\right)^2\right)\omega^{\varphi}(f,h). \tag{2.6}$$

*Proof.* We may consider only the case $\omega^{\varphi}(f,h) < \infty$. Note that function $\Phi : (0,\infty) \to (0,\infty)$ is a strictly increasing bijection. Therefore it admits an inverse $\Phi^{-1} : (0,\infty) \to (0,\infty)$.

Put $p = \left[\frac{\Phi(x)}{h}\right]$. Define the sequence $(u_j)_{j \ge -p}$ by

$$u_j = \Phi^{-1}(jh + \Phi(x)),\ j \ge -p.$$

From this it immediately follows that

$$\Phi(u_{j+1}) - \Phi(u_j) = h,\ j \ge -p.$$

Consider the decomposition

$$[0,\infty) = [0,u_{-p}) \cup \bigcup_{j=-p}^{\infty} [u_j, u_{j+1}),$$

where $[0,u_{-p}) = \emptyset$, if $u_{-p} = 0$. Let $t \in [0,\infty)$. We have to consider several cases.

Case 1: $t \in [x,\infty)$. Then there is an index $n \in \mathbb{N}_0$, such that $t \in [u_n, u_{n+1})$. We have

$$|f(t) - f(x)| \le |f(t) - f(u_n)| + \sum_{j=0}^{n-1} |f(u_{j+1}) - f(u_j)|,$$

where the last sum is 0 if $n = 0$. Using Lemma 2.4 we have $|f(t) - f(u_n)| \le \omega^{\varphi}(f,h)$ and $|f(u_{j+1}) - f(u_j)| \le \omega^{\varphi}(f,h)$, for $0 \le j \le n-1$. Hence

$$|f(t) - f(x)| \le (n+1)\omega^{\varphi}(f,h).$$

If $n = 0$, from this we obtain directly relation (2.6). If $n \geq 1$ we have successively:

$$\begin{aligned} 1 + n &= 1 + \frac{1}{h} \sum_{j=0}^{n-1} (\Phi(u_{j+1}) - \Phi(u_j) = 1 + \frac{1}{h}(\Phi(u_n) - \Phi(x)) \\ &\leq 1 + \frac{1}{h}|\Phi(t) - \Phi(x)| \leq 1 + \frac{1}{h^2} \cdot (\Phi(t) - \Phi(x))^2 \end{aligned}$$

It follows relation (2.6).

Case 2: $t \in [u_{-p}, x)$. This implies that $p \geq 1$. Then there is $n \in \mathbb{N}$, such that $t \in [u_{-n-1}, u_{-n})$. We have

$$|f(t) - f(x)| \leq |f(t) - f(u_{-n})| + \sum_{j=0}^{n-1} |f(u_{-j}) - f(u_{-j-1})|,$$

where the last sum is 0 if $n = 0$. Using Lemma 2.4 we have $|f(t) - f(u_{-n})| \leq \omega^{\varphi}(f, h)$ and $|f(u_{-j}) - f(u_{-j-1})| \leq \omega^{\varphi}(f, h)$, for $0 \leq j \leq n - 1$. Hence

$$|f(t) - f(x)| \leq (n + 1)\omega^{\varphi}(f, h).$$

If $n = 0$, from this we obtain directly relation (2.6). If $n \geq 1$ we have successively, similarly as in Case 1:

$$\begin{aligned} 1 + n &= 1 + \frac{1}{h} \sum_{j=0}^{n-1} (\Phi(u_{-j}) - \Phi(u_{-j-1})) = 1 + \frac{1}{h}(\Phi(x) - \Phi(u_{-n})) \\ &\leq 1 + \frac{1}{h}|\Phi(x) - \Phi(t)| \leq 1 + \frac{1}{h^2} \cdot (\Phi(t) - \Phi(x))^2 \end{aligned}$$

Case 3: $t \in [0, u_{-p})$. We have

$$|f(t) - f(x)| \leq |f(t) - f(u_{-p})| + \sum_{j=0}^{p-1} |f(u_{-j}) - f(u_{-j-1})|,$$

where the last sum is 0 if $p = 0$. Let show that $|f(t) - f(u_{-p})| \leq \omega^{\varphi}(f, h)$. We must to prove $u_{-p} - t \leq h\varphi\left(\frac{u_{-p}+t}{2}\right)$. But from the convexity of function $\frac{1}{\varphi}$ we obtain

$$\frac{u_{-p} - t}{\varphi\left(\frac{u_{-p}+t}{2}\right)} \leq \int_t^{u_{-p}} \frac{\mathrm{d}s}{\varphi(s)} = \Phi(u_{-p}) - \Phi(t) \leq \Phi(u_{-p}).$$

Since function $\Phi^{-1}$ is strictly increasing and $\Phi(x) - ph < h$ it follows that $u_{-p} \leq \Phi^{-1}(h)$. Hence $\Phi(u_{-p}) \leq h$. Then we continue like in Case 2, for $n = p$. □

**Remark 2.6.** From the proof of Lemma 2.5 it follows that for $f \in \mathcal{F}([0, \infty))$, $x > 0$ and $h > 0$, we have also

$$|f(t) - f(x)| \leq \left(1 + \frac{1}{h}|\Phi(t) - \Phi(x)|\right) \omega^{\varphi}(f, h). \tag{2.7}$$

The main result of this section is the following

**Theorem 2.7.** *Let $W$ be a linear subspace of $\mathcal{F}([0,\infty))$ and let $F : W \to \mathbb{R}$ be a positive linear functional. Let $x \in [0,\infty)$ and let $\varphi$ be an admissible function. Suppose that $(\Phi - \Phi(x)e_0)^2 \in W$ and $e_0 \in W$. Then, for all $f \in W$ and all $h > 0$ we have*

$$\begin{aligned} |F(f) - f(x)| &\leq |f(x)| \cdot |F(e_0) - 1| \\ &\quad + \Big(F(e_0) + h^{-2} F((\Phi - \Phi(x)e_0)^2)\Big)\omega^{\varphi}(f,h). \quad (2.8) \end{aligned}$$

*Proof.* The theorem follows from Lemma 2.5 and the inequality:

$$|F(f) - f(x)| \leq |f(x)| \cdot |F(e_0) - 1| + F(|f - f(x)e_0|).$$

□

**Corollary 2.8.** *Let $W$ be a linear subspace of $\mathcal{F}([0,\infty))$ and let $L : W \to \mathcal{F}([0,\infty))$ be a positive linear operator. Let $\varphi$ an admissible function. Suppose that $(\Phi - \Phi(x)e_0)^2 \in W$ for each $x \in [0,\infty)$ and also $e_0 \in W$. Then for all $f \in W$, all $x \in [0,\infty)$ and $h > 0$ we have*

$$\begin{aligned} &|L(f,x) - f(x)| \leq |f(x)| \cdot |L(e_0,x) - 1| \\ &+ \Big(L(e_0,x) + h^{-2} L((\Phi - \Phi(x)e_0)^2, x)\Big)\omega^{\varphi}(f,h). \quad (2.9) \end{aligned}$$

**Remark 2.9.** In the case $\varphi = e_0$, we have $\Phi = e_1$ and relation (2.9) becomes the well-known estimate of Mond [11].

## 3. Estimates for the weight $\varphi(x) = \sqrt{x}$

**Theorem 3.1.** *Let $W \subset \mathcal{F}([0,\infty))$ be a linear subspace, such that $\Pi_2 \in W$. If $L : W \to \mathcal{F}((0,\infty))$ is a positive linear operator, then for any $f \in W$, any $x \in (0,\infty)$ and any $h > 0$ we have*

$$\begin{aligned} &|L(f,x) - f(x)| \leq |f(x)| \cdot |L(e_0,x) - 1| \\ &\qquad + \left(L(e_0,x) + \frac{4}{h^2 x} L((e_1 - xe_0)^2, x)\right)\omega^{\varphi}(f,h). \quad (3.1) \end{aligned}$$

*In the particular case $L(e_0) = e_0$ and $h = \sqrt{\frac{L((e_1 - xe_0)^2, x)}{x}}$ we have*

$$|L(f,x) - f(x)| \leq 5 \cdot \omega^{\varphi}\left(f, \sqrt{\frac{L((e_1 - xe_0)^2, x)}{x}}\right). \quad (3.2)$$

*Proof.* We apply Corollary 2.8 by taking into account the estimate:

$$\left(\int_x^t \frac{\mathrm{d}u}{\sqrt{u}}\right)^2 = (2(\sqrt{t} - \sqrt{x}))^2 = 4 \cdot \left(\frac{t - x}{\sqrt{x} + \sqrt{t}}\right)^2 \leq \frac{4(t - x)^2}{x}.$$

□

In the following theorem we give the connections between the modulus $\omega^{\varphi}(f, \bullet)$, for $\varphi(x) = \sqrt{x}$ and the usual modulus of function $f(x^2)$.

**Theorem 3.2.** *For any $f \in \mathcal{F}([0,\infty))$ and $h > 0$ we have*

$$\omega^{\varphi}(f, \sqrt{2}h) \le \omega(f \circ e_2, h) \le \omega^{\varphi}(f, 2h). \tag{3.3}$$

*Proof.* Let $x, y \in [0,\infty)$, such that $|x^2 - y^2| \le \sqrt{2}h\sqrt{\frac{x^2+y^2}{2}}$, which is equivalent to the inequality $|x-y| \le \frac{h\sqrt{x^2+y^2}}{x+y}$. But $\sqrt{x^2+y^2} \le x+y$. Hence $|x-y| \le h$. It follows $|f(x^2) - f(y^2)| \le \omega(f \circ e_2, h)$. Therefore

$$\sup_{x,y,\ |x^2-y^2| \le \sqrt{2}h\sqrt{\frac{x^2+y^2}{2}}} |f(x^2) - f(y^2)| \le \omega(f \circ e_2, h).$$

But

$$\sup_{x,y,\ |x^2-y^2| \le \sqrt{2}h\sqrt{\frac{x^2+y^2}{2}}} |f(x^2) - f(y^2)| = \sup_{u,v,\ |u-v| \le \sqrt{2}h\sqrt{\frac{u+v}{2}}} |f(u) - f(v)|$$
$$= \omega^{\varphi}(f, \sqrt{2}h).$$

Therefore

$$\omega^{\varphi}(f, \sqrt{2}h) \le \omega(f \circ e_2, h).$$

Conversely, let $x, y \in [0,\infty)$, such that $|\sqrt{x} - \sqrt{y}| \le h$, which is equivalent to $|x-y| \le h(\sqrt{x}+\sqrt{y})$. But $\sqrt{x}+\sqrt{y} \le 2\sqrt{\frac{x+y}{2}}$. Hence $|x-y| \le 2\sqrt{\frac{x+y}{2}}$ and consequently $|f(y) - f(x)| \le \omega^{\varphi}(f, 2h)$. Since $x, y$ are arbitrarily chosen, we have

$$\sup_{x,y,\ |\sqrt{x}-\sqrt{y}| \le h} |f(y) - f(x)| \le \omega^{\varphi}(f, 2h).$$

But

$$\sup_{x,y,\ |\sqrt{x}-\sqrt{y}| \le h} |f(y) - f(x)| = \sup_{u,v,\ |u-v| \le h} |f(u^2) - f(v^2)|$$
$$= \omega(f \circ e_2, h).$$

Therefore

$$\omega(f \circ e_2, h) \le \omega^{\varphi}(f, 2h).$$

**Corollary 3.3.** *For $\varphi(x) = \sqrt{x}$, $x \in [0,\infty)$ and a function $f \in \mathcal{F}([0,\infty))$, the following are equivalent:*

i) $\lim_{h \to 0} \omega^{\varphi}(f, h) = 0$,

ii) *the function $f(x^2)$, $x \in [0,\infty)$ is uniformly continuous.*

We exemplify for the Szász-Mirakjan operators

$$S_n(f, x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) e^{-nx} \frac{(nx)^k}{k!}, \tag{3.4}$$

$x \in [0,\infty)$, $n \in \mathbb{N}$ and $f \in W$, where $W \subset \mathcal{F}([0,\infty))$ is the linear subspace of the functions $f$ for which the series above is convergent.

We have $S_n(e_0, x) = 1$, $S_n((e_1 - xe_0)^2, x) = \frac{x}{n}$. Also we have $S_n(f, 0) = f(0)$ for any $f \in W$. Hence we obtain:

**Theorem 3.4.** *Let* $\varphi(x) = \sqrt{x}$. *Let* $f \in W$, $x \in [0, \infty)$, $n \in \mathbb{N}$. *Then*

$$|S_n(f, x) - f(x)| \leq 5 \cdot \omega^\varphi \left(f, \frac{1}{\sqrt{n}}\right). \tag{3.5}$$

□

**Remark 3.5.** In view of Corollary 3.3, relation (3.5) gives a quantitative version of a result of Totik [14] which states that, if $f(x^2)$ is a uniformly continuous function, $x \in [0, \infty)$, then the sequence of functions $(S_n f)_n$ is uniformly convergent on $[0, \infty)$ to function $f$.

## 4. Estimates for the weight $\varphi(x) = \frac{\sqrt{x}}{1+x^m}$, $m \in \mathbb{N}$, $m \geq 2$

**Theorem 4.1.** *Let* $W \subset \mathcal{F}([0, \infty))$ *be a linear subspace, such that* $\Pi_{2m} \in W$. *If* $L : W \to \mathcal{F}([0, \infty))$ *is a positive linear operator, then for any* $f \in W$, *any* $x \in (0, \infty)$ *and any* $h > 0$ *we have*

$$|L(f, x) - f(x)| \leq |f(x)| \cdot |L(e_0, x) - 1|$$
$$+ \Big(L(e_0, x) + \frac{4}{h^2 x} L((e_1 - xe_0)^2 (2e_0 + x^{2m} e_0 + e_{2m}), x)\Big) \omega^\varphi(f, h).$$

*Proof.* We apply Corollary 2.8 and use the estimate:

$$\begin{aligned}
\left(\int_x^t \frac{(1 + u^m)\mathrm{d}u}{\sqrt{u}}\right)^2 &= 4\left(\sqrt{t} - \sqrt{x} + \frac{(\sqrt{t})^{2m+1} - (\sqrt{x})^{2m+1}}{2m + 1}\right)^2 \\
&\leq 8(\sqrt{t} - \sqrt{x})^2 \left[1 + \left(\frac{\sum_{k=0}^{2m} (\sqrt{t})^k (\sqrt{x})^{2m-k}}{2m + 1}\right)^2\right] \\
&\leq 8\frac{(t - x)^2}{x} \left[1 + \left(\frac{t^m + x^m}{2}\right)^2\right] \\
&\leq 4\frac{(t - x)^2}{x} (2 + t^{2m} + x^{2m}).
\end{aligned}$$

□

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