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Multifractional Brownian motion in vehicle crash tests

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Abstract. Different crash tests are carried out in the car industry to measure the acceleration dependent on time. With the aim of improving the airbag-system a discussion of crash processes was raised. Experimental studies approve the modelling of the crash tests as a multifractional Brownian motion which will be introduced as a generalisation of the fractional case (including the Wiener process). Based on the ideas of Coeurjolly [1] an estimation of the significant time-dependent Hurst parameter $H(t)$ will be developed. Its interpretation as a measure of deformation of the crash car leads to interesting results. So the Hurst index' value is important for supporting the fire-decision [4].

Mathematics Subject Classification (2010): 60H05, 60H30, 62P30.

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1. Motivation of the model

The car industry has performed extensive crash tests for sensitizing and improving the airbag-system. They have measured the acceleration dependent on time with different sensors installed on characteristic positions in the vehicle, especially in the front part of the cars. The activation of the restraint-system is implemented in the airbag-control-unit which is mounted on the middle tunnel. On the basis of mechanical models in a crash situation the airbag-algorithms will be specifically adapted and optimized for each new car. To further improve the accident detection a more general mathematical discussion of the crash process should be conducted.

Currently the crucial criterion for activating the airbags is the velocity calculated by the integral over the acceleration. But these results are not sufficient for a distinction between different crash cases and situations. The aim is to identify the type of crash so that selected airbags will fire only if they are necessary.

The researches are premised on data like in FIGURE 1 whose character changes in time. Here a head-on collision with 56 km/h against a solid wall is presented. There arises the question whether crash test situations suffice a stochastic process. This assumption can be affirmed because the progress of acceleration is significant: wild fluctuations at the beginning which rapidly decrease after 50 ms. These fluctuations can be described by the fractional Brownian motion with a Hurst index H greater than 0 but less than 1/2. If H converges to 1 the fractional Brownian motion will tend to a random variable. This supports the interpretation of the crash process as a multifractional Brownian motion with a time-dependent Hurst parameter $H(t)$.

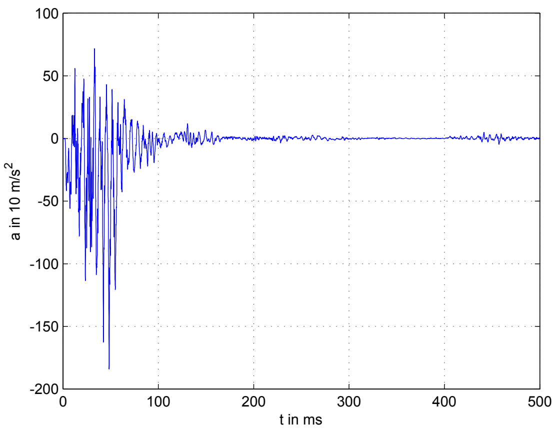

FIGURE 1. Head-on collision with 56 km/h against a solid wall

2. The multifractional Brownian motion

2.1. Definition and representation

First the fractional Brownian motion will be defined as a Brownian motion with a constant parameter H:

Definition 2.1. *A real-valued random process* $(B_H(t), t \geq 0)$ *is called fractional Brownian motion with Hurst parameter* $H \in (0,1)$ *provided that*

(i) $B_H(t)$ *is a Gaussian process;*
(ii) $B_H(0) = 0$ *a.s.;*
(iii) $I\!E(B_H(t)) = 0, \ \forall t \geq 0$*, that means the process is centered;*
(iv) $I\!E(B_H(t)B_H(s)) = \frac{1}{2}\operatorname{Var}(B_H(1))\left[|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right]$.

Especially the case $H = 0.5$ leads to the Brownian motion also known as Wiener process [4]. A generalisation of the fractional Brownian motion is the multifractional Brownian motion where the constant Hurst index H will be substituted by a time-dependent Hurst exponent $H(t)$:

Definition 2.2. *A real-valued random process* $(B_{H_t}(t), t \geq 0)$ *is said to be a multifractional Brownian motion if the following conditions are fulfilled*

(i) $B_{H_t}(t)$ *is a Gaussian process;*
(ii) $B_{H_0}(0) = 0$ *a.s.;*
(iii) $\mathbb{E}(B_{H_t}(t)) = 0,\ \forall\, t \geq 0$, *that means the process is centered;*
(iv) $\mathbb{E}(B_{H_t}(t) B_{H_s}(s)) = \frac{1}{2} C(H_t, H_s) \left[|t|^{H_t+H_s} + |s|^{H_t+H_s} - |t-s|^{H_t+H_s} \right]$, *with* $C(H_t, H_s) =$ *const. dependent on* H_t *and* H_s*;*
(v) $H : [0, \infty) \mapsto (0,1)$ *is Hölder continuous with exponent* $\beta > 0$.

This definition of the multifractional case is equivalent to a representation as an Itô integral [5]

$$B_{H_t}(t) = \frac{1}{\Gamma(H_t + \frac{1}{2})} \left\{ \int_{-\infty}^{0} \left[(t-s)^{H_t - \frac{1}{2}} - (-s)^{H_t - \frac{1}{2}} \right] dB(s) + \int_{0}^{t} (t-s)^{H_t - \frac{1}{2}} dB(s) \right\}$$

for all $t \geq 0$ where $H : [0, \infty) \mapsto (0,1)$ is a Hölder continuous function with exponent $\beta > 0$ and B marks the ordinary two-sided Brownian motion.

A process $\left(B(t),\, t \in \mathbb{R}^1\right)$ denotes a two-sided Brownian motion if

$$B(t) = \begin{cases} B_1(t) & : \text{for } t \geq 0, \\ B_2(-t) & : \text{for } t < 0, \end{cases}$$

where $B_1(t)$ and $B_2(t)$ are two independent Brownian motions for $t \geq 0$.

2.2. Typical properties

Because of zero mean and the Itô isometry [3] of the stochastic integral all the properties listed in Definition 2.2 can be proved from the equivalent integral representation, explicitly shown in [4]. Furthermore two important theorems will be presented but not proved, only the main idea will be mentioned.

Theorem 2.3. *The multifractional Brownian motion* $B_{H_t}(t)$ *is a continuous process for all* $t \in [0, \infty)$ *with probability* 1.

It is possible to show this with the help of skilful splittings of the Itô integral representation, some fundamental inequalities and the Kolmogorov criterion [5], detailed in [4].

Theorem 2.4. *It exists a positive continuous function* $t \mapsto \sigma_t$ *so that for all* $t \geq 0$ *the following asymptotic distribution holds*

$$\frac{B_{H_{t+h}}(t+h) - B_{H_t}(t)}{h^{H_t}} \xrightarrow[h \to 0]{\mathcal{L}} N(0, \sigma_t^2).$$

Evidently the mean is 0 but the variance is harder to predict. Again skilful splittings and useful inequalities yield the result [5], explicitly in [4]. Hence a standard multifractional Brownian motion can be introduced.

2.3. Hurst index' estimation

An estimation of the significant time-dependent Hurst parameter $H(t)$ is based on the ideas of Coeurjolly [1], [2]. It is a kind of parameter estimator harking back to the asymptotic behaviour of the k-th absolute moment. Here $k \leq 2$ is considered. A particularity is that only one realisation is necessary for the estimation which actually is a well-known method for the fractional case with constant H. First the raw data have to be filtered, here with the so called Daubechies-filter. Then the procedure will be extended from the fractional Brownian motion to the multifractional one. That means the estimation does not happen over the entire time range, but rather over a defined time period so that a time-dependent $H(t)$ will be obtained (see also in the next chapter).

With the help of the trajectory filtered by the Daubechies-filter a of length $l+1$ (in detail [1], [2])

$$V^a\left(\frac{i}{n}\right) = \sum_{q=0}^{l} a_q B_H\left(\frac{i-q}{n}\right), \qquad \text{for } i = l, \ldots, n-1,$$

the covariance function π_H^a of this series will be calculated by

$$\pi_H^a(j) = I\!E\left(V^a\left(\frac{i}{n}\right) V^a\left(\frac{i+j}{n}\right)\right) = -\frac{1}{2}\sum_{q,r=0}^{l} a_q a_r \left|q-r+j\right|^{2H}.$$

The k-th empirical absolute moment of the discrete variations of the fractional Brownian motion has the following representation

$$S_n(k,a) = \frac{1}{n-l}\sum_{i=l}^{n-1}\left|V^a\left(\frac{i}{n}\right)\right|^k.$$

Finally Coeurjolly estimates the Hurst parameter H by

$$\hat{H}_n(k,a) = g_{k,a,n}^{-1}\left(S_n(k,a)\right),$$

where the function $g_{k,a,n}^{-1}(t)$ is defined as the inverse of

$$g_{k,a,n}(t) = \frac{1}{n^{kt}}\{\pi_t^a(0)\}^{\frac{k}{2}} E_k$$

and the indices k, a and n denote the order of the moment, the filter and the number of partition points. The factor E_k depends on the used order k of the moment and is explained by

$$E_k = 2^{\frac{k}{2}}\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right).$$

3. Crash test analysis

3.1. Application of the estimation

Experimental studies have shown that the Hurst index depends on time. Figure 1 represents the acceleration measured over 500 ms in 10.000 data points. That means for 1 ms 20 data points are available. But if the airbags are necessary to protect the inmates they have to fire empirically by no later than 25 ms. So it suffices to consider only the first 500 measured points.

Now the described method to estimate the Hurst index $H(t)$ can be applied using the Daubechies-filter of order 6 and a time period of 10 ms containing 200 data points. Practically the first approximation of H results from considering the interval $(1, 200)$. Then all intervals from $(2, 201)$ to $(301, 500)$ will be examined. Because the fire-decision is usually made after 25 ms there are 15 ms available for interpretation.

The Hurst parameter is a measure of deformation of the crash car with a small H corresponding to a big deformation and a big one to a small deformation. Please note $0 < H < 1$. If the passenger cabin is affected by deformation there will be a high risk of injury for the occupants. That is why the activation of the airbags is essential.

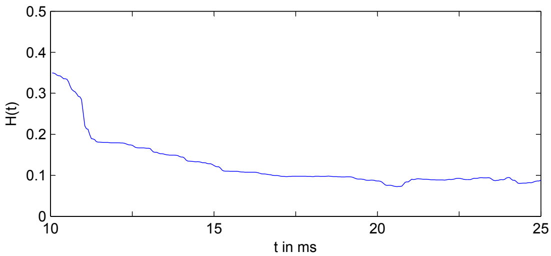

Figure 2. Corresponding Hurst parameter to head-on collision

The corresponding Hurst index to the head-on collision in Figure 1 is illustrated in Figure 2, the first estimation after 10 ms and the last one after 25 ms. With small values of $H(t)$ the airbags have to activate because a big deformation is associated and the inmates are in jeopardy.

3.2. Introduction and evaluation of the test cases

Four different crash cases depicted in Figure 3 were investigated. The first one is the head-on collision against a solid wall with velocities between 16 and 56 km/h. This crash situation will be abbreviated with *frontal*. In the picture at the top on the right a car is overlapping a barrier by only 40 %. The barrier is a deformable obstacle (that is where the name *deform* comes from) and the car collides with the obstacle with 40 to 64 km/h. The third one is called *angle10* and illustrates the crash with only 15 km/h against a solid wall at an angle of 10 degrees and a 40 % overlap. Finally at the bottom on the right there is a collision against a solid wall at an angle of 30 degrees with velocities of 32 or 40 km/h which will be abbreviated with *angle30*.

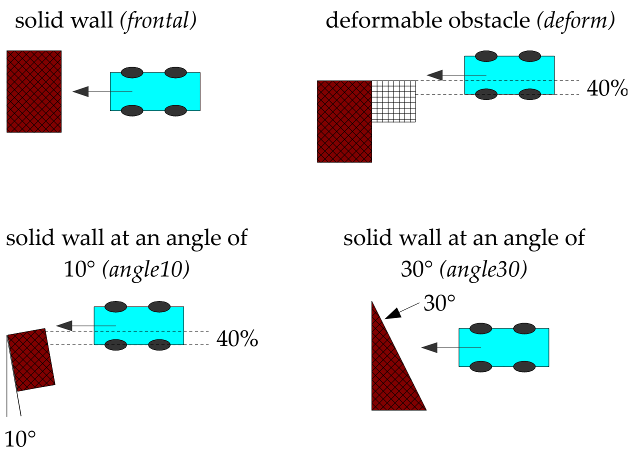

FIGURE 3. Distinction between crash cases

As a measure of deformation of the crash car the Hurst index will be considered for each situation and velocity. This leads to very interesting results. FIGURE 4 shows the Hurst parameters for some selected cases estimated with the method above using the Daubechies-filter of order 6 and a time period of 10 ms realised in 200 data points.

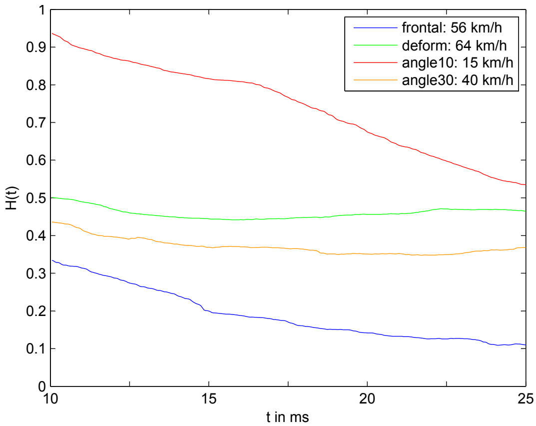

FIGURE 4. Hurst parameters for selected cases

The red line at the top represents a collision at an angle of 10 degrees and 15 km/h against a solid wall (overlapping 40 %). With a big monotonically decreasing Hurst index between 1 and 0.5 the deformation of the car body is very small. There is only an almost unnoticeable danger for the inmates and therefore the airbags are unnecessary. It is the biggest Hurst index of the four observed cases in FIGURE 4, thus the lowest damage. In consideration of a velocity of only 15 km/h this result is easily comprehensible.

Beneath, the crash case with the deformable obstacle proceeds nearly constantly at 0.5 and the velocity of 64 km/h suggests the use of the airbags. It is the biggest test velocity and a huge deformation is accompanied by a high risk of injury for the vehicle occupants. To grant the best possible protection the airbags have to fire.

The orange Hurst index belongs to a car which collides with a solid wall at an angle of 30 degrees and a velocity of 40 km/h. The car slides along the wall because of the angle of contingence. With values of about 0.4 the Hurst parameter is smaller than in the previous cases. That means the deformation is greater due to the rough impact. So the airbags are essential because of the imminent danger.

Last but not least the blue line characterises a head-on collision against a solid wall with 56 km/h. Monotonically decreasing values between 0.35 and 0.1 illustrate the crash situation with the smallest Hurst index. Hence the biggest deformation of the vehicle takes place and the occupants could be seriously injured. Such a head-on collision can entail severe consequences and therefore require the airbags to be deployed.

In FIGURE 4 two of the curves are monotonically decreasing while the other two are nearly constant. Perhaps more information to support the fire-decision are conceivable by use of the monotonicity of the trajectories. Moreover the estimation of the Hurst index in the case *angle10* is much greater than in all the other cases. The airbags do not have to fire because there is only a small deformation contrary to the three other cases. That is why the airbags are necessary to guarantee the safety of the passengers.

Looking at the mentioned figure a boundary at about 0.5 seperating the case with airbags from these without can be supposed. This boundary is well-motivated since $H = 0.5$ forms the characteristic change between wild fluctuations of the acceleration and the levelling values which tend to a random variable. The special case $H = 0.5$ realises the Brownian motion.

3.3. Further results in detail

Considering the four presented crash situations and averaging over the Hurst parameters of these crashes with the same case and the same velocity there are the following outcomes.

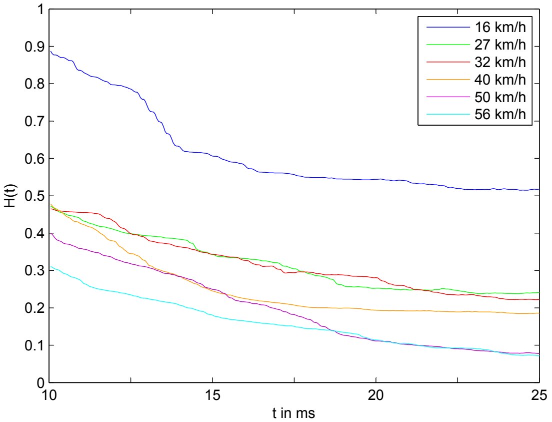

FIGURE 5. Hurst parameter for the case *frontal*

In FIGURE 5 the estimation on top (head-on collision with a velocity of 16 km/h) differs with values greater than 0.5 from all the other velocities. A big Hurst index is interpreted as a small deformation of the car body and a small risk for the occupants. That is why the airbags are unnecessary. This result is very catchy because a velocity of 16 km/h is so slow that big damages are unbelievable. But the tests with all the other velocities show with values less than 0.5 that the deformation is getting greater and so the risk of injury is growing. The airbags have to activate to protect the inmates optimally.

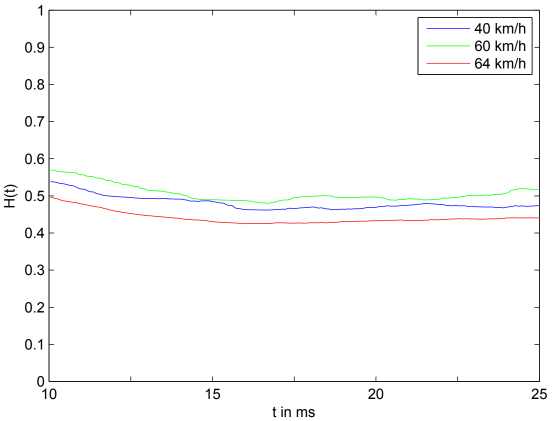

FIGURE 6. Hurst parameter for the case *deform*

The estimations of the case *deform* are close together and their progress is nearly identically. But it is conspicuous that the Hurst parameter is decreasing with growing velocities. That means the deformation keeps on entering into the passenger cabin and the occupants are increasingly threatened. To give maximum shelter to the inmates the use of the airbags is essential at all presented velocities.

Nevertheless FIGURE 6 requires to raise the boundary between the cases with and without airbags from 0.5 to 0.6 because the collision with 40 and 60 km/h against a deformable obstacle - where the activation of the airbags can not be abandoned - have Hurst parameters just under 0.6. Such an enlargement does not contradict all the previous figures since in all crashes with a lower Hurst index the airbags have to fire and in all crashes with a greater Hurst parameter the airbags are not necessary.

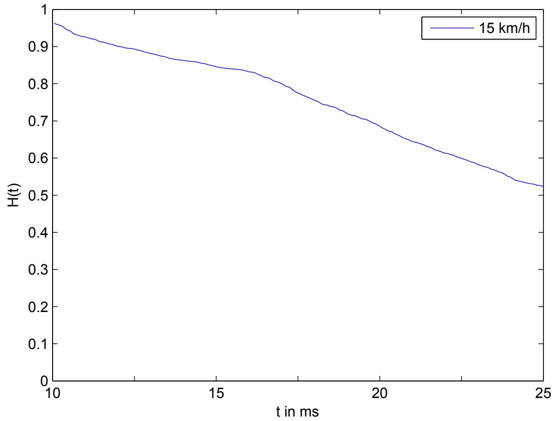

FIGURE 7. Hurst parameter for the case *angle10*

The trajectory of the estimated Hurst index of the case *angle10* with 15 km/h in FIGURE 7 is the same as in FIGURE 4 because there were no other velocities to analyse.

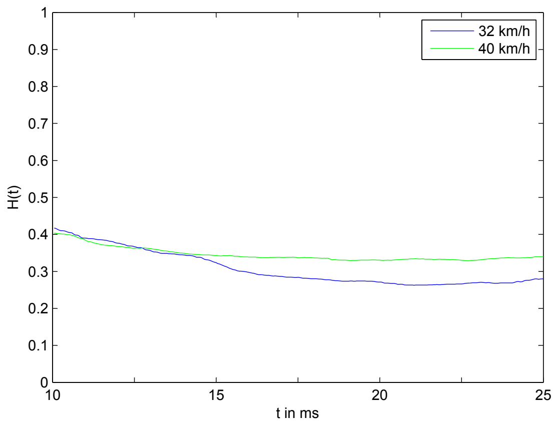

FIGURE 8. Hurst parameter for the case *angle30*

Finally the case *angle30* is mapped in FIGURE 8 whose curves are similar to the estimations of the case *frontal*. If the case is unknown one can interchange them. But considering the known velocities the distinction is easier since the estimations of the case *frontal* start with greater values at about 0.5 and finish with lower values at about 0.2 after 25 ms, the moment the fire-decision has to be made.

There exists a characteristic estimation for the Hurst index in each crash situation so that on the one hand different crash cases and situations can be distinguished due to progress and dimension of $H(t)$ and on the other hand there are some similarities. Referring to the averages of crash cases with the same situation and the same velocity a strict boundary at about 0.6 is recognisable - a boundary between cases where the airbags have to fire and those where they are unnecessary. All these results are heuristically and have to be tested with more data to cover a bigger spectrum of crash cases and velocities.

One difficulty in all well-known methods of the past was to differentiate the case *deform* from the case *angle10*. Now a distinction between these two cases is obvious. It is harder to differ between the cases *frontal* and *angle30*. Perhaps a symbiosis of old and new methods is promising.

4. Conclusion

In sum, the Hurst index' value is important for supporting the fire-decision. It exists a characteristic estimation of the Hurst parameter in progression and dimension for each crash situation so that a strict distinction is possible. In certain circumstances only special airbags have to fire. With huge values of $H(t)$ the collision at an angle of 10 degrees - requiring no activation of the airbags - contrasts with all the other cases with Hurst parameters less than 0.6. The airbags are essential for the security of the inmates. All in all there is a distinct boundary at about 0.6 between non-activating and activating the airbags. But this is only an assumption, perhaps this boundary has to be corrected by investigating more statistical series, other crash cases and velocities.

A boundary of 0.5 would be motivated very well because $H = 0.5$ is the characteristic change between wild fluctuations and the levelling values of the acceleration which tend to a random variable. It is the special case of the well-known Brownian motion.

An interesting question arises: Is it possible to make the fire-decision based only on the knowledge of the estimated Hurst index? This would be a very great result but requires any more researches.

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