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Uniform approximation in weighted spaces using some positive linear operators

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Abstract. We characterize the functions defined on a weighted space, which are uniformly approximated by the Post-Widder, Gamma, Weierstrass and Picard operators and we obtain the range of the weights which can be used for uniform approximation. We give, also, an estimation of the rate of the approximation in terms of the usual modulus of continuity. Some well-known results are obtained, as limit cases.

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1. Introduction

In the survey paper [2], the authors present some ideas related to the approximation of functions in weighted spaces and enounced some unsolved problems in weighted approximation theory. Three such problems are:
1. Let $\mathcal{F}$ be a linear subspace of $\mathbb{R}^I$ and $A_n \colon \mathcal{F} \to C(I)$ a sequence of positive linear operators. For which weights ρ, does A_n map $C_\rho(I) \cap \mathcal{F}$ onto $C_\rho(I)$ with uniformly bounded norms?
2. For which functions $f \in C_\rho(I)$ do we have $\|A_n - f\|_\rho \to 0$, as $n \to \infty$?
3. Which moduli of smoothness are appropriate for weighted approximation?

In [6], we presented a result that give an answer to this questions. Below, in Theorem 1.1 we recall this result. In the same paper, we analized the particular cases of Szász-Mirakjan and Baskakov operators. In this paper, we continue the applications of the general result in the case of some integral-type positive linear operators, namely: the Post-Widder, Gamma, Gauss-Weierstrass and Picard operators. Firstly, we introduce the basic notations.

Let $I \subseteq \mathbb{R}$ be a noncompact interval and let $\rho \colon I \to [1, \infty)$ be an increasing and differentiable function called weight. Let $B_\rho(I)$ be the space of all functions $f \colon I \to \mathbb{R}$ such that $|f(x)| \leq M \cdot \rho(x)$, for all $x \in I$, where $M > 0$ is a constant depending on f and ρ, but independent of x. The space

$B_\rho(I)$ is called weighted space and it is a Banach space endowed with the ρ-norm

$$\|f\|_\rho = \sup_{x\in I} \frac{|f(x)|}{\rho(x)}.$$

Let $C_\rho(I) = C(I) \cap B_\rho(I)$ be the subspace of $B_\rho(I)$ containing continuous functions.

Let $(A_n)_{n\geq 1}$ be a sequence of positive linear operators defined on the weighted space $C_\rho(I)$. It is known (see [4]) that A_n maps $C_\rho(I)$ onto $B_\rho(I)$ if and only if $A_n\rho \in B_\rho(I)$.

Theorem 1.1. *Let $A_n : C_\rho(I) \to B_\rho(I)$ be positive linear operators reproducing constant functions and satisfying the conditions*

$$\sup_{x\in I} A_n(|\varphi(t) - \varphi(x)|, x) = a_n \to 0, \quad (n \to \infty) \tag{1.1}$$

$$\sup_{x\in I} \frac{A_n(|\rho(t) - \rho(x)|, x)}{\rho(x)} = b_n \to 0. \quad (n \to \infty) \tag{1.2}$$

If $A_n(f, x)$ is continuously differentiable and there is a constant $K(f, \rho, n)$ such that

$$\frac{|(A_nf)'(x)|}{\varphi'(x)} \leq K(f, \rho, n) \cdot \rho(x), \quad \textit{for every } x \in I, \tag{1.3}$$

and ρ and φ are such that there exists a constant $\alpha > 0$ with the property

$$\frac{\rho'(x)}{\varphi'(x)} \leq \alpha \cdot \rho(x), \quad \textit{for every } x \in I, \tag{1.4}$$

then, the following statements are equivalent

(*i*) $\|A_nf - f\|_\rho \to 0$ *as* $n \to \infty$.

(*ii*) $\dfrac{f}{\rho} \circ \varphi^{-1}$ *is uniformly continuous on* J.

Furthermore, we have

$$\|A_nf - f\|_\rho \leq b_n \cdot \|f\|_\rho + 2 \cdot \omega\left(\frac{f}{\rho} \circ \varphi^{-1}, a_n\right), \quad \textit{for every } n \geq 1.$$

Remark 1.2. The relation (1.4) give us the connection between the function φ and the weight ρ. We must have

$$\rho(x) \leq Me^{\alpha\cdot\varphi(x)}, \quad \text{for every } x \in I,$$

where $M, \alpha > 0$ are constants independent of x. So, we have obtained the range of the weights ρ, for which Theorem 1.1 is valid. In the case of the maximal class of weights: $\rho(x) = e^{\alpha\varphi(x)}$, instead of proving the conditions (1.1) and (1.2) we prove

$$\lim_{n\to\infty} \sup_{x\in I} A_n(|\varphi(t) - \varphi(x)|^2, x) = 0. \tag{1.5}$$

For the estimation of the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ we use the inequalities

$$a_n \le \sup_{x\in I} \sqrt{A_n(|\varphi(t)-\varphi(x)|^2, x)}$$
$$b_n \le \frac{\alpha}{2}\sqrt{\|A_n\rho^2\|_{\rho^2} + 2\,\|A_n\rho\|_\rho + 1}\cdot \sup_{x\in I} \sqrt{A_n(|\varphi(t)-\varphi(x)|^2, x)}.$$

2. Main results

The Post-Widder operators.

Lemma 2.1. *For $I = (0,\infty)$ and for $\rho(x) = 1 + x^\alpha$, for some $\alpha > 0$, the Post-Widder operators (*[9], [14]*)*

$$P_n(f,x) = \frac{1}{(n-1)!}\left(\frac{n}{x}\right)^n \int_0^\infty e^{-\frac{nu}{x}} u^{n-1} f(u)\,du, \quad x > 0,$$

have the property that $P_n f \in C_\rho(0,\infty)$ for every $f \in C_\rho(0,\infty)$.

Proof. Setting $t = nu/x$, we get

$$P_n(\rho,x) = 1 + \frac{1}{(n-1)!}\int_0^\infty e^{-t} t^{n-1}\left(\frac{xt}{n}\right)^\alpha dt = 1 + \frac{x^\alpha \Gamma(n+\alpha)}{n^\alpha (n-1)!}.$$

Using the formula (see [1, formula 6.1.46])

$$\lim_{n\to\infty} \frac{\Gamma(n+\alpha)}{n^\alpha \Gamma(n)} = 1,$$

we deduce the existence of a constant $C > 0$, independent of n, such that $\Gamma(n+\alpha) \le C n^\alpha (n-1)!$, for every $n \ge 1$. We obtain

$$P_n(\rho,x) \le C\rho(x), \quad x > 0,$$

which proves the mapping property of P_n. □

Theorem 2.2. *For $\alpha > 0$ and $\rho(x) = 1 + x^\alpha$, the Post-Widder operators $P_n\colon C_\rho(0,\infty) \to C_\rho(0,\infty)$ have the property*

$$\|P_n f - f\|_\rho \to 0, \quad \textit{whenever } n \to \infty$$

if and only if

$$f(e^x)e^{-\alpha x} \quad \textit{is uniformly continuous on } (0,\infty).$$

Moreover, for every $f \in C_\rho(0,\infty)$ and every $n \ge 2$, we have

$$\|P_n f - f\|_\rho \le \|f\|_\rho \frac{\alpha C}{\sqrt{n-1}} + 2\cdot\omega\left(f(e^t)e^{-\alpha t}, \frac{1}{\sqrt{n-1}}\right),$$

where $C = \sup_{n\in\mathbb{N}} \frac{1}{2}\sqrt{\|P_n\rho^2\|_{\rho^2} + 2\,\|P_n\rho\|_\rho + 1} < \infty$ is a constant depending only on α.

Proof. Using the Geometric-Logarithmic-Arithmetic Mean Inequality (see [8, p. 40])

$$\sqrt{u \cdot v} \leq \frac{u-v}{\ln u - \ln v} < \frac{u+v}{2}, \quad 0 < v < u, \tag{2.1}$$

for the function $\varphi(x) = \ln x$, we obtain

$$|\varphi(t) - \varphi(x)| \leq \left| \sqrt{\frac{t}{x}} - \sqrt{\frac{x}{t}} \right|, \quad t, x > 0.$$

Because $P_n(1, x) = 1$, $P_n(t, x) = x$ and $P_n\left(\frac{1}{t}, x\right) = \frac{n}{(n-1)x}$, $n \geq 2$, we deduce

$$\sup_{x>0} P_n(|\varphi(t) - \varphi(x)|^2, x) \leq \sup_{x>0} \left[P_n\left(\frac{t}{x}, x\right) + P_n\left(\frac{x}{t}, x\right) - 2 \right] = \frac{1}{n-1},$$

which proves (1.5)

Now, using the equality (see [12])

$$P_n((t-x)^2, x) = \frac{x^2}{n}$$

and the Cauchy-Schwarz inequality for positive linear operators, we have

$$P_n(|t-x|\rho(t), x) \leq \sqrt{P_n((t-x)^2, x) \cdot P_n(\rho^2, x)} \leq \frac{x}{\sqrt{n}} \cdot C_1 \rho(x).$$

Estimating the absolute value of the derivative

$$\begin{aligned} |(P_n f)'(x)| &= \frac{n}{x^2} \left| \int_0^\infty \left(\frac{n}{x}\right)^n \frac{1}{(n-1)!} e^{-\frac{nu}{x}} u^{n-1}(u-x) f(u)\, du \right| \\ &\leq \frac{n}{x^2} P_n(|t-x| \cdot |f(t)|, x)| \leq \frac{n}{x^2} \|f\|_\rho P_n(|t-x|\rho(t), x) \\ &\leq \|f\|_\rho \frac{\sqrt{n}}{x} C_1 \rho(x), \end{aligned}$$

we obtain

$$\frac{|(P_n f)'(x)|}{\varphi'(x)} \leq C_2 \rho(x), \quad \text{for every } x > 0,$$

which proves (1.3) The relation (1.4) is true because

$$\frac{\rho'(x)}{\varphi'(x)} = \alpha x^\alpha \leq \alpha(1 + x^\alpha) = \alpha \rho(x).$$

Using the Theorem 1.1, the convergence $\|P_n f - f\|_\rho \to 0$ is true if and only if the function $\frac{f}{\rho} \circ \varphi^{-1}$ is uniformly continuous on $(0, \infty)$. The equality

$$\frac{f(e^x)}{e^{\alpha x}} = \frac{f(e^x)}{1 + e^{\alpha x}} \cdot \left(1 + e^{-\alpha x}\right),$$

the boundedness of the function $1 \leq 1 + e^{-\alpha x} \leq 2$ and the uniform continuity of the functions $1 + e^{-\alpha x}$ and $(1 + e^{-\alpha x})^{-1}$ prove that $\frac{f}{\rho} \circ \varphi^{-1}$ is uniformly continuous, if and only if $f(e^x)e^{-\alpha x}$ is uniformly continuous. □

Remark 2.3. The result of the Theorem 2.2 for the limit case, $\alpha = 0$, was obtained in [12] and in [3].

The Gamma operators.

Lemma 2.4. *For $I = (0,\infty)$ and for $\rho(x) = 1 + x^\alpha$, for some $\alpha > 0$ the Gamma operators ([7])*

$$G_n(f,x) = \frac{x^{n+1}}{n!}\int_0^\infty e^{-xu}u^n f\left(\frac{n}{u}\right) du, \quad x > 0,\ n \geq 1,$$

have the property that $G_n f \in C_\rho(0,\infty)$ for every $f \in C_\rho(0,\infty)$ and $n \geq [\alpha]$.

Proof. Setting $xu = t$, we get

$$G_n(\rho,x) = 1 + \frac{1}{n!}\int_0^\infty e^{-t}t^n\left(\frac{nx}{t}\right)^\alpha dt = 1 + \frac{(nx)^\alpha\Gamma(n+1-\alpha)}{n!}.$$

Using the formula (see [1, formula 6.1.46])

$$\lim_{n\to\infty}\frac{n^\alpha\Gamma(n+1-\alpha)}{\Gamma(n+1)} = 1,$$

we deduce the existence of a constant $C > 0$, independent of n, such that $n^\alpha\Gamma(n+1-\alpha) \leq Cn!$, for every $n \geq [\alpha]$. We obtain

$$G_n(\rho,x) \leq C\rho(x), \quad x > 0,$$

which proves the property of G_n stated in the lemma. □

Theorem 2.5. *For $\alpha > 0$ and $\rho(x) = 1 + x^\alpha$, the Gamma operators $G_n\colon C_\rho(0,\infty) \to C_\rho(0,\infty)$ have the property*

$$\|G_n f - f\|_\rho \to 0, \quad \text{whenever } n \to \infty$$

if and only if

$$f(e^x)e^{-\alpha x} \quad \text{is uniformly continuous on } (0,\infty).$$

Moreover, for every $f \in C_\rho(0,\infty)$ and every $n \geq [2\alpha]$, we have

$$\|G_n f - f\|_\rho \leq \|f\|_\rho\frac{\alpha C}{\sqrt{n}} + 2\cdot\omega\left(f(e^t)e^{-\alpha t}, \frac{1}{\sqrt{n}}\right),$$

where $C = \sup_{n\in\mathbb{N}}\frac{1}{2}\sqrt{\|G_n\rho^2\|_{\rho^2} + 2\|G_n\rho\|_\rho + 1} < \infty$ is a constant depending only on α.

Proof. As in the proof of the Theorem 2.2, let $\varphi(x) = \ln x$. We have

$$|\ln t - \ln x| \leq \left|\sqrt{\frac{t}{x}} - \sqrt{\frac{x}{t}}\right|, \quad t, x > 0.$$

Because $G_n(e_0,x) = 1,\ G_n(t,x) = x$ and

$$G_n\left(\frac{1}{t},x\right) = \frac{n+1}{nx},$$

we deduce

$$\sup_{x>0} G_n(|\varphi(t)-\varphi(x)|^2,x) \leq \sup_{x>0}\left[G_n\left(\frac{t}{x},x\right) + G_n\left(\frac{x}{t},x\right) - 2\right] = \frac{1}{n},$$

which proves (1.5).

Estimating the derivative

$$\begin{aligned} |(G_n f)'(x)| &= \left| \frac{n+1}{x} G_n(f,x) - \frac{n+1}{x} G_{n+1}\left(f\left(\frac{nt}{n+1}\right), x\right) \right| \\ &\leq \frac{n+1}{x} \|f\|_\rho \, |G_n(\rho,x) + G_{n+1}(\rho,x)| \\ &\leq \|f\|_\rho \frac{n+1}{x} C_1 \rho(x), \end{aligned}$$

we deduce

$$\frac{|(G_n f)'(x)|}{\varphi'(x)} \leq C_2 \rho(x), \quad \text{for every } x > 0,$$

which proves (1.3). The relation (1.4) is true, because

$$\frac{\rho'(x)}{\varphi'(x)} = \alpha x^\alpha \leq \alpha(1 + x^\alpha) = \alpha\rho(x).$$

Using the Theorem 1.1, the convergence $\|P_n f - f\|_\rho \to 0$ is true if and only if the function $\frac{f}{\rho} \circ \varphi^{-1}$ is uniformly continuous on $(0, \infty)$. The equality

$$\frac{f(e^x)}{e^{\alpha x}} = \frac{f(e^x)}{1 + e^{\alpha x}} \cdot \left(1 + e^{-\alpha x}\right),$$

the boundedness of the function $1 \leq 1 + e^{-\alpha x} \leq 2$ and the uniform continuity of the functions $1 + e^{-\alpha x}$ and $(1 + e^{-\alpha x})^{-1}$ prove that $\frac{f}{\rho} \circ \varphi^{-1}$ is uniformly continuous, if and only if $f(e^x)e^{-\alpha x}$ is uniformly continuous. □

Remark 2.6. The result of the Theorem 2.5 for the limit case, $\alpha = 0$, was obtained in [11].

The Gauss-Weierstrass operators.

Lemma 2.7. *For $I = \mathbb{R}$ and for $\rho(x) = e^{\alpha x}$, for some $\alpha > 0$, the Gauss-Weierstrass operators (*[13]*)*

$$W_n f(x) = \frac{\sqrt{n}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-n\frac{(u-x)^2}{2}} f(u)\, du, \quad x \in (-\infty, \infty),$$

have the property that $W_n f \in C_\rho(\mathbb{R})$ for $f \in C_\rho(\mathbb{R})$.

Proof. We have

$$\begin{aligned} \frac{W_n(\rho,x)}{\rho(x)} &= \frac{\sqrt{n}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-n\frac{(u-x)^2}{2} + \alpha(u-x)}\, du \\ &= \frac{\sqrt{n}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{n}{2}\left(u-x-\frac{\alpha}{n}\right)^2} \cdot e^{\frac{\alpha^2}{2n}}\, du = e^{\frac{\alpha^2}{2n}} \leq e^{\frac{\alpha^2}{2}}, \end{aligned}$$

which proves the statement from the lemma. □

Theorem 2.8. *For $\alpha > 0$ and for $\rho(x) = e^{\alpha x}$ the Gauss-Weierstrass operators $W_n\colon C_\rho(\mathbb{R}) \to C_\rho(\mathbb{R})$ have the property*

$$\|W_n f - f\|_\rho \to 0, \quad \textit{whenever } n \to \infty,$$

if and only if

$$f(x)e^{-\alpha x} \quad \text{is uniformly continuous on } \mathbb{R}.$$

Moreover, for every $f \in C_\rho(\mathbb{R})$ *and for every* $n \geq 1$*, we have*

$$\|W_n f - f\|_\rho \leq \|f\|_\rho \frac{\alpha C}{\sqrt{n}} + 2 \cdot \omega \left(f(t)e^{-\alpha t}, \frac{1}{\sqrt{n}} \right),$$

where $C = e^{\frac{\alpha^2}{2}} \sqrt{1 + \frac{\alpha^2}{4}\left(1 + e^{\frac{\alpha^2}{2}}\right)^2}$.

Proof. Set $\varphi(x) = x$. Because $W_n(e_0, x) = 1$ and $W_n((t-x)^2, x) = \frac{1}{n}$ (see [10]), we get

$$W_n(|\varphi(t) - \varphi(x)|^2, x) = W_n((t-x)^2, x) = \frac{1}{n},$$

which proves (1.5). Using the relation

$$W_n(e^{\alpha t}, x) = e^{\alpha x} \cdot e^{\frac{\alpha^2}{2n}}$$

we deduce

$$\begin{aligned} b_n &= \sup_{x \in I} \frac{W_n(|\rho(t) - \rho(x)|, x)}{\rho(x)} = \frac{W_n\left(\left|e^{\alpha t} - e^{\alpha x}\right|, x\right)}{e^{\alpha x}} \\ &\leq \frac{\sqrt{W_n(e^{2\alpha t}, x) - 2e^{\alpha x} W_n(e^{\alpha t}, x) + e^{2\alpha x}}}{\rho(x)} \\ &= \frac{\sqrt{e^{2\alpha x} \cdot e^{\frac{4\alpha^2}{2n}} - 2e^{2\alpha x} \cdot e^{\frac{\alpha^2}{2n}} + e^{2\alpha x}}}{e^{\alpha x}} = \sqrt{e^{\frac{4\alpha^2}{2n}} - 2e^{\frac{\alpha^2}{2n}} + 1}. \end{aligned}$$

Using the equality $x^4 - 2x + 1 = (x-1)[(x-1)(x+1)^2 + 2x]$ and the inequality $e^t - 1 \leq te^t$, for $t = \frac{\alpha^2}{2n}$, we obtain

$$\begin{aligned} b_n &\leq \sqrt{\left(e^{\frac{\alpha^2}{2n}} - 1\right)} \cdot \sqrt{\left(e^{\frac{\alpha^2}{2n}} - 1\right)\left(e^{\frac{\alpha^2}{2n}} + 1\right)^2 + 2e^{\frac{\alpha^2}{2n}}} \\ &\leq \frac{\alpha}{\sqrt{2n}} e^{\frac{\alpha^2}{2}} \sqrt{2 + \frac{\alpha^2}{2}\left(1 + e^{\frac{\alpha^2}{2}}\right)^2} \leq \frac{\alpha C}{\sqrt{n}}. \end{aligned}$$

The estimation of the derivative

$$\begin{aligned} |(W_n f)'(x)| &= n|W_n((t-x)f(t), x)| \leq n\|f\|_\rho W_n(|t-x|\rho(t), x) \\ &\leq n\|f\|_\rho \sqrt{W_n((t-x)^2, x)}\sqrt{W_n(e^{2\alpha t}, x)} \\ &= \sqrt{n}\|f\|_\rho e^{\frac{2\alpha^2}{n}} \rho(x) \end{aligned}$$

proves the relation

$$\frac{|(W_n f)'(x)|}{\varphi'(x)} \leq C_1 \rho(x), \quad \text{for every } x \in \mathbb{R}.$$

□

Remark 2.9. The result of the Theorem 2.8 for the limit case, $\alpha = 0$, was obtained in [5] and partially in [10].

The Picard Operators.

Lemma 2.10. *For $I = \mathbb{R}$ and for $\rho(x) = e^{\alpha x}$, for some $\alpha > 0$, the Picard operators*

$$\mathcal{P}_n(f,x) = \frac{n}{2}\int_{-\infty}^{\infty} e^{-n|u-x|} f(u)\,du, \quad x \in \mathbb{R},\ n \geq [\alpha]+2,$$

have the property that $\mathcal{P}_n\rho \in C_\rho(\mathbb{R})$ for every $f \in C_\rho(\mathbb{R})$.

Proof. The evaluation

$$\begin{aligned}\frac{\mathcal{P}_n(\rho,x)}{\rho(x)} &= \frac{n}{2}\int_{-\infty}^{x} e^{\alpha u - nx + nu - \alpha x}\,du + \frac{n}{2}\int_{x}^{\infty} e^{\alpha u + nx - nu - \alpha x}\,du \\ &= \frac{n}{2}e^{-nx-\alpha x}\left.\frac{e^{u(\alpha+n)}}{\alpha+n}\right|_{-\infty}^{x} + \frac{n}{2}e^{nx-\alpha x}\left.\frac{e^{u(\alpha-n)}}{\alpha-n}\right|_{x}^{\infty} \\ &= \frac{n^2}{n^2-\alpha^2} \leq 1+\alpha,\end{aligned}$$

proves the statement from the lemma. □

Theorem 2.11. *For $\alpha > 0$ and for $\rho(x) = e^{\alpha x}$ the Picard operators $\mathcal{P}_n \colon C_\rho(\mathbb{R}) \to C_\rho(\mathbb{R})$, $n \geq [2\alpha]+2$, have the property*

$$\|\mathcal{P}_n f - f\|_\rho \to 0, \quad \textit{whenever } n \to \infty,$$

if and only if

$$f(x)e^{-\alpha x} \quad \textit{is uniformly continuous on } \mathbb{R}.$$

Furthermore, for every $f \in C_\rho(\mathbb{R})$ and for every $n \geq [2\alpha]+2$, it is true the estimation

$$\|\mathcal{P}_n f - f\|_\rho \leq \|f\|_\rho \frac{\alpha C}{n} + 2\cdot\omega\left(f(t)e^{-\alpha t}, \frac{\sqrt{2}}{n}\right),$$

where $C > 0$ is a constant dependening on α, but independent of n.

Proof. Set $\varphi(x) = x$. Using the relations $\mathcal{P}_n(e_0,x) = 1$, $\mathcal{P}_n(e_1,x) = x$ and $\mathcal{P}_n(e_2,x) = x^2 + \frac{2}{n^2}$, we obtain

$$a_n = \sup_{x\in\mathbb{R}} \mathcal{P}_n(|\varphi(t)-\varphi(x)|,x) \leq \sup_{x\in\mathbb{R}} \sqrt{\mathcal{P}_n((t-x)^2,x)} = \frac{\sqrt{2}}{n},$$

which proves (1.1). Using the equality

$$\mathcal{P}_n(e^{\alpha t},x) = \frac{n^2 e^{\alpha x}}{n^2-\alpha^2},$$

obtained in the previous lemma, we get

$$\begin{aligned}
b_n &= \sup_{x\in\mathbb{R}} \frac{\mathcal{P}_n(|\rho(t)-\rho(x)|,x)}{\rho(x)} = \sup_{x\in\mathbb{R}} \frac{\mathcal{P}_n\left(\left|e^{\alpha t}-e^{\alpha x}\right|,x\right)}{e^{\alpha x}} \\
&\le \sup_{x\in\mathbb{R}} \frac{\sqrt{\mathcal{P}_n(e^{2\alpha t},x)-2e^{\alpha x}\mathcal{P}_n(e^{\alpha t},x)+e^{2\alpha x}}}{\rho(x)} \\
&= \sqrt{\frac{n^2}{n^2-4\alpha^2}-\frac{2n^2}{n^2-\alpha^2}+1} = \alpha\sqrt{\frac{2(n^2+2\alpha^2)}{(n^2-4\alpha^2)(n^2-\alpha^2)}} \le \frac{\alpha C}{n},
\end{aligned}$$

where

$$C^2 = \max_{n\ge[2\alpha]+2} \frac{2n^2(n^2+2\alpha^2)}{(n^2-4\alpha^2)(n^2-\alpha^2)}.$$

Using the relation

$$\mathcal{P}_n(f,x) = \frac{n}{2}\int_{-\infty}^{x} f(u)e^{-n(x-u)}\,du + \frac{n}{2}\int_{x}^{\infty} f(u)e^{-n(u-x)}\,du$$

we can compute the derivative

$$\begin{aligned}
\mathcal{P}_n'(f,x) &= \frac{n^2}{2}\left(\int_x^{\infty} f(u)e^{-n(u-x)}\,du - \int_{-\infty}^{x} f(u)e^{-n(x-u)}\,du\right) \\
&= \frac{n^2}{2}\int_0^{\infty}\left[f(x+t)-f(x-t)\right]e^{-nt}\,dt
\end{aligned}$$

and obtain the estimation

$$\begin{aligned}
|\mathcal{P}_n'(f,x)| &\le \frac{n^2}{2}\int_0^{\infty}|f(x+t)-f(x-t)|\,e^{-nt}\,dt \\
&\le \|f\|_\rho \frac{n^2}{2}\int_0^{\infty}\left[e^{\alpha(x+t)}+e^{\alpha(x-t)}\right]e^{-nt}\,dt \\
&\le e^{\alpha x}\|f\|_\rho \frac{n^3}{n^2-\alpha^2}.
\end{aligned}$$

This proves the inequality

$$\frac{|\mathcal{P}_n'(f,x)|}{\varphi'(x)} \le C_{n,\alpha}\rho(x), \qquad \text{for every } x\in\mathbb{R}.$$

□

Corollary 2.12. *For a continuous and bounded function $f:\mathbb{R}\to\mathbb{R}$, it is true the equivalence*

$$\|\mathcal{P}_n f - f\| \to 0,\ (n\to\infty) \text{ if and only if } f \text{ is uniformly continuous on } \mathbb{R}.$$

Moreover,

$$\|\mathcal{P}_n f - f\| \le 2\cdot\omega\left(f, \frac{\sqrt{2}}{n}\right),\ n\ge 2.$$

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