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Stochastic Schrödinger equation driven by cylindrical Wiener process and fractional Brownian motion

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Abstract. In this paper we study the properties of the solution of a stochastic nonlinear equation of Schrödinger type, which is perturbed by a cylindrical Wiener process and an additive cylindrical fractional Brownian motion with Hurst parameter in the interval $(\frac{1}{2}, 1)$. The existence of the solution and the existence of the Malliavin derivative are proved.

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1. Introduction

In physics, specifically in quantum mechanics, the Schrödinger equation is an equation that describes how the quantum state of a physical system changes in time.

We describe the Schrödinger equation for a harmonic oscillator subject to a periodic electric field: a particle of mass m, electric charge Q, is displaced along the x-axis ($x \in \mathbb{R}$) and subject to a force $-m\omega_0^2 x$ (for all $t > 0$) and to an electric field $E\sin(\omega t)$ directed along the x-axis

$$i\hbar \frac{\partial}{\partial t} X(x,\, t) = \Big(-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2} m\omega_0^2 x^2 + QEx\sin(\omega t)\Big) X(x,\, t), \quad x \in \mathbb{R}, t > 0,$$

$$X(\cdot, 0) = X_0$$

where i is the imaginary unit, $-\dfrac{\hbar^2}{2m}\nabla^2$ is the kinetic energy operator, $\hbar$ is Planck's constant, the complex valued function X is the wave function at position x at time t, X_0 is the initial condition (see [8], p. 639).

Many authors investigated stochastic equations of Schrödinger type: The case of additive noise is considered in [11], [13], while the case of multiplicative noise is discussed in [2], [9], [10], [16]. In these papers the existence of a mild solution is investigated. Different approaches to linear and nonlinear stochastic Schrödinger equations perturbed by cylindrical Brownian motions are given in [14] and [15].

In this paper we study the properties of the solution of a stochastic nonlinear equation of Schrödinger type, which is perturbed by a cylindrical Wiener process and an additive cylindrical fractional Brownian motion. Consequently, this model respects as well fluctations of a Brownian motion as additive disturbances with long range dependence. This paper completes the results about stochastic equations of Schrödinger type given in [5] by considering also a cylindrical fractional Brownian motion with Hurst parameter in the interval $(\frac{1}{2}, 1)$. We use the framework of stochastic evolution equations driven by fractional noise developed by T.E. Duncan, B. Pasik-Duncan, B. Maslowski [12] and M. Röckner and Y. Wang [17]. The existence results are derived by using the properties of Schrödinger type equations developed in [5]. Smoothness properties such as the existence of the Malliavin derivative are also proved. The Malliavin derivatives can be used to calculate conditional expectations or chaos decompositions of stochastic processes (see [3], [7]).

This paper has the following structure: In Section 2 we introduce the list of assumptions and give the definition of the solution. In Section 3 we briefly present the two stochastic integrals that appear in the equation which is investigated. The existence of the solution is derived in Section 4. Section 5 contains results about infinite dimensional Malliavin derivatives and the existence of the Malliavin derivative of the solution is proved.

2. Assumptions and formulation of the problem

We consider $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ to be a filtered complete probability space. Let $(V, (\cdot,\cdot)_V)$ and $(H, (\cdot,\cdot))$ be separable complex Hilbert spaces, such that (V, H, V^*) forms a triplet of rigged Hilbert spaces. Let K be a separable real Hilbert space. We consider $\Big(W(t)\Big)_{t\geq 0}$ to be a K-valued cylindrical Wiener process adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ and $(B^h(t))_{t\geq 0}$ to be a K-valued cylindrical fractional Brownian motion with Hurst index $h \in (\frac{1}{2}, 1)$ adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

We study the properties of the *variational solution* X of the following stochastic nonlinear evolution equation of Schrödinger type

$$(X(t), v) = (X_0, v) - i\int_0^t \langle AX(s), v\rangle ds + i\int_0^t (f(s, X(s)), v) ds \qquad (2.1)$$
$$+ i(\int_0^t g(s, X(s)) dW(s), v) + i(\int_0^t b(s) dB^h(s), v)$$

for a.e. $\omega \in \Omega$ and all $t \in [0,T], v \in V$.

We assume that:

[I] X_0 is $\mathcal{F}_0$-measurable, $X_0 \in L^2(\Omega; V)$;

[A] $A : V \to V^*$ has the following properties:

• A is linear and continuous $\|Au\|_{V^*} \le c_A \|u\|_V$ for all $u \in V$;

• $\langle Au, v\rangle = \overline{\langle Av, u\rangle}$ for all $u, v \in V$;

• there exists constants $\alpha_1 \in \mathbb{R}$ and $\alpha_2 > 0$, such that for all $v \in V$ it holds

$$\langle A(v), v\rangle \ge \alpha_1 \|v\|^2 + \alpha_2 \|v\|_V^2.$$

• Let $(h_n)_n \subset H$ be the eigenvectors of the operator A, for which we assume that $Ah_n \in H$ for all $n \in \mathbb{N}$ and $(h_n)_n$ is a complete orthonormal system in H.

[f] $f : \Omega \times [0,T] \times H \to H$ is a measurable function, which is $\mathcal{F}_t$-adapted for each $t \in [0,T]$:

(1) there exists a constant $c_f > 0$ such that for a.e. $\omega \in \Omega$ it holds

$$\|f(t,u) - f(t,v)\|^2 \le c_f \|u - v\|^2 \quad \text{for all } t \in [0,T], u, v \in H;$$

(2) for a.e. $\omega \in \Omega$ and all $t \in [0,T], u \in V$ we have $f(t,u) \in V$ and there exists $k_f > 0$ such that

$$\|f(t,u)\|_V^2 \le k_f (1 + \|u\|_V^2);$$

[g] $g : \Omega \times [0,T] \times H \to L_2(K,H)$ is a measurable function, which is $\mathcal{F}_t$-adapted for each $t \in [0,T]$:

(1) there exists a constant $c_g > 0$ such that for a.e. $\omega \in \Omega$ it holds

$$\|g(t,u) - g(t,v)\|_{L_2(K,H)}^2 \le c_g \|u - v\|^2 \text{ for all } t \in [0,T], u, v \in H;$$

(2) for a.e. $\omega \in \Omega$ and all $t \in [0,T]$, $u \in V$ we have $g(t,u) \in L_2(K,V)$ and there exists $k_g > 0$ such that

$$\|g(t,u)\|_{L_2(K,V)}^2 \le k_g (1 + \|u\|_V^2);$$

[b] $b : [0,T] \to L_2(K,V)$ and for each $u \in K$ we have $b(\cdot)u \in L^p([0,T]; V)$ for some $p > \frac{1}{h}$ and it holds

$$\int_0^T \int_0^T \|b(r)\|_{L_2(K,V)} \|b(s)\|_{L_2(K,V)} |r - s|^{2h-2} dr ds < \infty.$$

3. The stochastic integrals

In this section we briefly present the definitions of the stochastic integrals we considered in (2.1). Let $(e_n)_n$ be an orthonormal basis in K.

For the K-valued cylindrical Wiener process $(W(t))_{t \ge 0}$ and for $g : \Omega \times [0,T] \times H \to L_2(K,H)$ satisfying [g]-(1) the stochastic integral

$\int_0^T g(s,v)dW(s)$ ($v \in H$ fixed) is defined as a zero mean H-valued Gaussian random variable given by

$$\int_0^T g(s,v)dW(s) := \sum_{n=1}^{\infty} \int_0^T g(s,v)e_n dw_n(s),$$

where the series above converges in $L^2(\Omega; H)$ and $((w_n(t))_{t\geq 0})_n$ is a sequence of mutually independent real-valued Brownian motions. One can prove that

$$E\Big\| \int_0^T g(s,v)dW(s)\Big\|^2 = \sum_{n=1}^{\infty} E\Big\| \int_0^T g(s,v)e_n dw_n(s)\Big\|^2$$

$$= \sum_{n=1}^{\infty} E \int_0^T \|g(s,v)e_n\|^2 ds = E\int_0^T \|g(s,v)\|^2_{L_2(K,H)} ds < \infty.$$

For $0 < r < 1/(2-2h)$ the function $\phi : [0,T] \to \mathbb{R}$ defined by $\phi(u) = h(2h-1)|u|^{2h-2}$ belongs to the space $L^r([0,T];\mathbb{R})$.

If $p > 1/h$, then by Theorem 3.9.4 in [4], there exists $C_T > 0$ such that for any function $\eta, \varphi \in L^p([0,T];\mathbb{R})$ it holds

$$\int_0^T \int_0^T |\eta(u)\varphi(v)\phi(u-v)|dudv \leq C_T \|\varphi\|_{L^p([0,T];\mathbb{R})} \|\eta\|_{L^p([0,T];\mathbb{R})}.$$

If $(\beta^h(t))_{t\geq 0}$ is a real-valued fractional Brownian motion with Hurst index $h \in (\frac{1}{2}, 1)$, and $\varphi \in L^p([0,T];\mathbb{R})$, then the stochastic integral $\int_0^T \varphi(s)d\beta^h(s) \in L^2(\Omega;\mathbb{R})$ is defined as a zero mean real-valued Gaussian random variable, such that

$$E\left(\int_0^T \varphi(s)d\beta^h(s) \int_0^T \varphi(s)d\beta^h(s)\right) = E\int_0^T\int_0^T \varphi(u)\varphi(v)\phi(u-v)dudv.$$

If $\varphi \in L^p([0,T];\mathbb{R})$ with $p > \frac{1}{h}$, then the process $\Big(\int_0^t \varphi(s)d\beta^h(s)\Big)_{t\geq 0}$ has P-a.s. continuous sample paths (see [18] Lemma 2.0.17).

Let $(k_n)_n$ be an orthonormal basis in K.

For the K-valued cylindrical fractional Brownian motion $(B^h(t))_{t\geq 0}$ and for $b : [0,T] \to L_2(K,V)$ satisfying assumption [b] the stochastic integral $\int_0^T b(s)dB^h(s)$ is defined as a zero mean V-valued Gaussian random variable given by

$$\int_0^T b(s)dB^h(s) := \sum_{n=1}^{\infty} \int_0^T b(s)k_n d\beta_n^h(s),$$

where the series above converges in $L^2(\Omega;V)$ and $\left((\beta_n^h(t))_{t\geq 0}\right)_n$ is a sequence of mutually independent real-valued fractional Brownian motions each with Hurst parameter h. One can prove that

$$E\Big\|\int_0^T b(s)dB^h(s)\Big\|_V^2 = \sum_{n=1}^{\infty} E\Big\|\int_0^T b(s)k_n d\beta_n^h(s)\Big\|_V^2$$

$$= \sum_{n=1}^{\infty}\int_0^T\int_0^T (b(r)k_n, b(s)k_n)_V \phi(r,s)drds$$

$$\leq \int_0^T\int_0^T \|b(r)\|_{L_2(K,V)}\|b(s)\|_{L_2(K,V)}\phi(r,s)drds < \infty.$$

For more details see for example [12],[18].

For a.e. $\omega \in \Omega$ and for each $t \in [0,T]$ we denote by

$$Z(t) := \int_0^t b(s)dB^h(s),$$

which is obviously a V-valued process adapted to $(\mathcal{F}_t)_{t\geq 0}$.

Proposition 3.1. [18, Corollary 2.0.16, Lemma 2.0.17] *The process* $(Z(t))_{t\in[0,T]}$ *has a continuous version in* V *and in* H *and*

$$E\int_0^T \|Z(s)\|_V^2 ds < \infty.$$

Remark 3.2. *The stochastic integral* $Z(t)$ *can also be represented by a stochastic integral with respect to the cylindrical Wiener process* W *(see* [3], [6]*). For* $f:\mathbb{R}\to\mathbb{C}$ *and* $\frac{1}{2} < h < 1$ *we introduce the operator*

$$(M^h f)(x) = c_h \int_{\mathbb{R}} \frac{f(t)}{|t-x|^{3/2-h}}dt,$$

where $c_h = [2\Gamma(h-1/2)\cos(1/2\pi(h-1/2))]^{-1}(\Gamma(2h+1)\sin(\pi h))^{1/2}$ *and* f *is chosen in such a manner that* $(M^h f) \in L^2(\mathbb{R})$*. If* f *is concentrated on* $[0,T]$*, then we consider* $[0,T]$ *instead of* $\mathbb{R}$*. If*

$$\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\int_0^T \left(\left(M^h\left(b(\cdot)k_n, h_j\right)\right)(s)\right)^2 ds < \infty,$$

then

$$\int_0^t b(s)dB^h(s) = \sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\int_0^t \left(M^h\left(b(\cdot)k_n, h_j\right)\right)(s)dw_n(s)h_j.$$

4. Existence of the solution

Theorem 4.1. *Assume that* [I], [A], [f], [g], [b] *are satisfied. Equation* (2.1) *admits a unique solution* $X \in L^2(\Omega \times [0,T]; V) \cap L^2(\Omega; C([0,T]; H))$.

In order to prove the existence of the solution of (2.1), we first transform it equivalently into an equation of Schrödinger type studied in [5]. For a.e. $\omega \in \Omega$ and for each $t \in [0,T], v \in H$ we denote by

- $U(t) := X(t) - iZ(t)$.
- $F(\omega, t, v) := f(\omega, t, v + iZ(\omega, t))$,
- $G(\omega, t, v) := g(\omega, t, v + iZ(\omega, t))$.

Observe that for a.e. $\omega \in \Omega$ and all $t \in [0,T], u, v \in H$ it holds

$$\|F(t,u) - F(t,v)\|^2 \le c_f \|u - v\|^2$$

$$\|G(t,u) - G(t,v)\|^2_{L_2(K,H)} \le c_g \|u - v\|^2$$

and for all $u \in V$

$$\|F(t,u)\|_V^2 \le 2k_f(1 + \|u\|_V^2 + \|Z(t)\|_V^2);$$

$$\|G(t,u)\|^2_{L_2(K,V)} \le 2k_g(1 + \|u\|_V^2 + \|Z(t)\|_V^2).$$

We rewrite (2.1) equivalently as

$$\begin{aligned} (U(t), v) \;=\; & (X_0, v) - i\int_0^t \langle AU(s), v\rangle ds + i\int_0^t (F(s, U(s)), v) ds \\ & + i\Big(\int_0^t G(s, U(s)) dW(s), v\Big) + i\int_0^t \langle AZ(s), v\rangle ds \text{ for all } v \in V. \end{aligned} \tag{4.1}$$

(2.1) admits a unique solution $X \in L^2(\Omega \times [0,T]; V) \cap L^2(\Omega; C([0,T]; H))$ if and only if (4.1) admits a unique solution $U \in L^2(\Omega \times [0,T]; V) \cap L^2(\Omega; C([0,T]; H))$.

The proof of the existence of a unique solution U for (4.1) is similar to the proof of Theorem 1 in [5]. For this reason one introduces Galerkin approximations: For each $n \in \mathbb{N}$ we consider the finite dimensional spaces $H_n := \text{sp}\{h_1, h_2, \ldots, h_n\}$ (equipped with the norm induced from H) and $K_n := \text{sp}\{e_1, e_2, \ldots, e_n\}$ (equipped with the norm induced from K). We define $\pi_n : H \to H_n$ the orthogonal projection of H on H_n by $\pi_n h := \sum_{j=1}^{n} (h, h_j) h_j$. Let $A_n : H_n \to H_n$, $F_n : \Omega \times [0,T] \times H_n \to H_n, G_n : \Omega \times [0,T] \times H_n \to L(K_n, H_n)$ be defined respectively by

$$A_n u = \sum_{j=1}^{n} \langle Au, h_j\rangle h_j, \quad F_n(t,u) = \sum_{j=1}^{n} (F(t,u), h_j) h_j,$$

$$G_n(t,u)v = \sum_{j=1}^{n} (G(t,u)v, h_j) h_i \text{ for } v \in K_n$$

$$Z_n(t) = \sum_{j=1}^{n} (Z(t), h_j) h_j$$

and we denote $X_{0n} = \pi_n X_0$ and $W_n(t) = \sum_{j=1}^{n} e_j w_j(t) \in K_n$. For a.e. $\omega \in \Omega$ and all $t \in [0,T]$ and all $j = \overline{1,n}$ we consider the finite dimensional equations corresponding to (4.1)

$$\begin{aligned}(U_n(t), h_j) &= (X_{0n}, h_j) - i\int_0^t (A_n U_n(s), h_j)ds \\ &+i\int_0^t (F_n(s, U_n(s)), h_j)ds + i(\int_0^t G_n(s, U_n(s))dW_n(s), h_j) \\ &+i\int_0^t (A_n(s)Z_n(s), h_j)ds.\end{aligned} \tag{4.2}$$

One can show similar as in the proof of Theorem 1 in [5] (see also Remark 3 in [5]) that for all $t \in [0,T]$ it holds

$$\lim_{n\to\infty} E\|U_n(t) - U(t)\|^2 = 0$$

and

$$\lim_{n\to\infty} E\int_0^t \|U_n(s) - U(s)\|^2 ds = 0.$$

5. The existence of Malliavin derivative of the solution

We briefly present some results about infinite dimensional Malliavin derivatives: We consider the random variable Y with values in a complex Hilbert space H. Y with $E\|Y\|^2 < \infty$ is called a smooth random variable and we denote $Y \in \mathcal{S}$, if

$$Y = \sum_{j=1}^{n} f_j \left(\int_0^T (\gamma_{1,j}(s), dW(s))_K, \ldots, \int_0^T \left(\gamma_{n_j,j}(s), dW(s)\right)_K \right) h_j,$$

where $\gamma_{1,j}, \ldots, \gamma_{n_j,j} \in L^2([0,T];K)$ for $j = 1, \ldots n$, $h_j \in H$, $f_j \in C^\infty(R^{n_j})$ and f_j and all its derivatives have polynomial growth for $j = 1, \ldots, n$.

The Malliavin derivative D_tY, $(t \in [0,T])$ of $Y \in \mathcal{S}$ is a random variable with values in $L_2(K,H)$ defined by

$$D_tY = \sum_{j=1}^{n}\sum_{k=1}^{n_j} \frac{\partial f_j}{\partial x_k} \left(\int_0^T (\gamma_{1,j}(s), dW(s))_K, \ldots, \int_0^T \left(\gamma_{n_j,j}(s), dW(s)\right)_K \right) \cdot \\ \cdot h_j \otimes \gamma_{k,j}(t).$$

The Malliavin derivative D_t as defined for H-valued smooth random variables is closable on $L^2(\Omega; L_2(K,H))$ (see Proposition 5.1 in [7]).

Consequently, if Y is the $L^2(\Omega;H)$ limit of a sequence $(Y_n)_n \subset \mathcal{S}$ so that the sequence $(D_tY_n)_n$ convergences in $L^2(\Omega; L_2(K,H))$, we can define D_tY as

$$D_tY = \lim_{n\to\infty} D_tY_n.$$

We use the notation $H(K)$ for the subspace of $L^2(\Omega; H)$, where the derivative D_t can be defined. This subspace is a separable Hilbert space equipped with the graph norm

$$\|Y\|^2_{H(K)} = E\|Y\|^2 + E\|D_t Y\|^2_{L_2(K,H)}.$$

The following result is known (see Lemma 5.2 in [7]):

Lemma 5.1. *Let $Y_n \to Y$ in $L^2(\Omega; H)$ and suppose that there is a constant $C > 0$ such that for all n we have*

$$E\|D_t Y\|^2_{L_2(K,H)} < C.$$

Then, the random variable Y is in the domain $H(K)$ of the Malliavin derivative D_t.

By using Proposition 5.2 in [7] the following chain rule holds:

Proposition 5.2. *Let M be a further separable Hilbert space. Given a random variable $Y \in H(K)$ and a Fréchet differentiable function $\eta : H \to M$. Then,*

$$D_t \eta(Y) = \nabla \eta D_t Y.$$

We will use the following well-known properties of D_t (see, for example [7], [3]):

Proposition 5.3. (1) *If Y is $\mathcal{F}_s$-measurable and $Y \in H(K)$, then $D_t Y = 0$ a.e. $\omega \in \Omega$ and for all $t > s$.*

(2) *Let $a(s)$, $s \in [0, T]$ an $\mathcal{F}_s$-adapted $L_2(K, H)$-valued process which fulfills the assumptions of the Skorochod integral definition in* [7]*. Then, for all $r > t$ it holds*

$$D_t \int_0^r a(s)dW(s) = a(t) + \int_t^r D_t a(s)dW(s).$$

Further in this section we assume:

1. The assumption in Remark 3.2 is valid for the process b.
2. The functions f and g are deterministic.
3. The functions f and g are Fréchet differentiable with respect to $x \in H$ for all $t \in [0, T]$ and the Fréchet derivatives $\nabla_x f(t, x)$ and $\nabla_x g(t, x)$ are bounded in the following sense: There exists a positive constant c such that
$$\|\nabla_x f(t, x)\|_{L(H,H)},\ \|\nabla_x g(t, x)\|_{L(H,L_2(K,H))} \leq c$$
for all $t \in [0, T]$, $x \in H$.
4. The initial condition X_0 is deterministic.

Theorem 5.4. *There exists $D_r U(t)$ as an $L_2(K, H)$-valued random variable for all $r, t \in [0, T]$.*

Proof. We process the proof in two steps:
Step 1: It follows from the above assumption 3 that the functions f and g are globally Lipschitz continuous. Consequently, we can consider directly

the Galerkin equations (4.2). Similar to Remark 3 in [5] we have for the variational solution U

$$\lim_{n\to\infty} E\|U_n(t)-U(t)\|^2 = 0 \text{ and } \lim_{n\to\infty} E\int_0^t \|U_n(s)-U(s)\|^2 ds = 0 \quad (5.1)$$

for all $t\in[0,T]$. Equation (4.2) is an Itô equation in V_n and H_n and its solution can be approximated by the method of successive approximations

$$\begin{aligned} U_n^{m+1}(t) &= X_{0n} - i\int_0^t A_n U_n^m(s)ds \\ &\quad + i\int_0^t F_n(s,U_n^m(s))ds + i\int_0^t G_n(s,U_n^m(s))dW_n(s) \\ &\quad + i\int_0^t A_n(s)Z_n(s)ds. \end{aligned} \quad (5.2)$$

for $m=0,1,\dots$ with $U^0(s)\equiv X_{0n}$.

The finite dimensional theory shows

$$\lim_{m\to\infty} E\|U_n^m(t)-U_n(t)\|^2 = 0. \quad (5.3)$$

Now we calculate $D_rU_n^{m+1}(t)$. Since U_n^{m+1} is $\mathcal{F}_t$-measurable we get also the $\mathcal{F}_r$-measurability for $r\geq t$. In this case it follows from Proposition 5.3 $D_rU_n^{m+1}(t)=0$. We now consider $r<t$. Then, by Proposition 5.2, Proposition 5.3 and Remark 3.2 we get

$$\begin{aligned} D_rU_n^{m+1}(t) &= -i\int_r^t A_n D_rU_n^m(s)ds \\ &\quad + i\int_r^t \nabla_x F_n(s,U_n^m(s))D_rU_n^m(s)ds \\ &\quad + i\int_r^t \nabla_x F_n(s,U_n^m(s))D_rZ_n(s)ds \\ &\quad + i\int_r^t \nabla_x G_n(s,U_n^m(s))D_rU_n^m(s)dW_n(s) \\ &\quad + i\int_r^t \nabla_x G_n(s,U_n^m(s))D_rZ_n(s)dW_n(s) \\ &\quad + iG_n(r,U_n^m(r)) + i\int_r^t A_n(s)D_rZ_n(s)ds \end{aligned} \quad (5.4)$$

where $D_rZ_n(t): K_n \to H_n$ is the linear operator defined by

$$(D_rZ_n(t)x,y) = \left(M^h\left(b_n(\cdot)x,y\right)\right)(s).$$

$D_rZ_n(t)$ has values in $L(K_n,V_n)$ and $L(K_n,H_n)$. Since the spaces are finite dimensional, the operators are also Hilbert-Schmidt operators. If we use the energy equality in the space $L_2(K_n,H_n)$, then we get by the assumptions of this section and by Gronwall's lemma that there is a positive constant C with

$$E\|D_rU_n^m(t)\|^2_{L_2(K,H)} \leq C$$

for all m, r, t and fixed n, since from equation (5.3) the boundedness of $E\|U_n^m(t)\|^2$ follows for all m, r, t and fixed n. The constant C does not depend on n. Then we get by Lemma 5.1, from the last inequality and from equation (5.3) that $D_rU_n(t)$ exists and

$$E\|D_rU_n(t)\|^2_{L_2(K,H)} \leq C. \tag{5.5}$$

Step 2: Since the relations (5.5) and (5.1) hold, we can use again Lemma 5.1 and get

$$E\|D_rU(t)\|^2_{L_2(K,H)} \leq C.$$

□

Theorem 5.5. *Consider that the assumptions of this section hold. Then, for $t > r$ we have*

$$D_rX(t) = D_rU(t) + i(M^hb(\cdot))(r),$$

where $(M^hb(\cdot))(r) \in L_2(K, H)$ is defined by the bilinearform

$$(M^h(b(\cdot)x, y))(r) \text{ for all } x \in K,\ x \in H.$$

Proof. Theorem 5.4 shows the existence of $D_rU(t)$ and it holds $D_rX(t) = D_rU(t) + iD_rZ(t)$. Since b is deterministic, we get by Proposition 5.3 and Remark 3.2 for $t > r$

$$D_rZ(t) = (M^hb(\cdot))(r).$$

□

Remark 5.6. The Malliavin derivative is used for example to define Skorochod integrals [12] and in the optimal control theory [1]. Optimal control problems for stochastic Schrödinger equations are under preparation.

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