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A Q-fractional version of Itô's formula

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Abstract. In this paper we consider a white noise calculus for fractional Brownian motion with values in a separable Hilbert space, whereby the covariance operator Q is a kernel operator (Q-fractional Brownian motion). We prove a Q-fractional version of the Itô's formula.

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1. Introduction

Extending white noise analysis [9], Biagini and Øksendal [2] introduce fractional white noise calculus. They give the corresponding definition of stochastic integrals, a fractional Itô formula and Itô isometry, fractional differentiation and a fractional Malliavin calculus, using the results of Elliott and van der Hoek [4].

In [1] Grecksch, Roth and Anh introduce the Q-fractional Brownian motion, i.e., a Hilbert space-valued fractional Brownian motion defined by a kernel operator Q, and develop the Q-fractional Brownian motion framework for $\frac{1}{2} < h < 1$ as it was done in [9] for the standard Brownian motion case and in [2] for the fractional Brownian motion case in finite dimensions. Grecksch, Roth and Anh introduce Q-fractional test functions spaces and distribution spaces analogous to the way Hida [7] did and develop the Q-fractional chaos expansion. The corresponding stochastic integral and the Hilbert space-valued Wick scalar product are introduced. Furthermore they proved Q-fractional versions of Girsanov's theorem and of Clark-Haussmann-Ocone theorem.

In this paper we give a short overview of the most important notions and definitions for Q-fractional Brownian motion, see [1]. In Section 3 we prove a Q-fractional version of Itô's formula (see Theorem 3.1).

2. Q-fractional Brownian motion setup

Let $\mathcal{S}(\mathbb{R}^1)$ denote the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^1$ and let $\mathcal{S}'(\mathbb{R}^1)$ be its dual, usually called the space of tempered distributions.

Let K and H be two separable Hilbert spaces with scalar product $(\cdot,\cdot)_K$ and $(\cdot,\cdot)_H$, and $(\Omega, \mathcal{F}, P)$ a complete probability space. We denote by $L(K,H)$ the set of all linear bounded operators from K to H. Let $Q \in L(K,K)$ be a self-adjoint, non-negative operator on K. We call Q a kernel operator in K if

(i) there exists a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{R}^1_+ = \{x \in \mathbb{R}^1 : x \geq 0\}$ with $\lambda_n \to 0$ as $n \to \infty$;

(ii) there exists a complete orthonormal system $(e_n)_{n\in\mathbb{N}} \in K$ such that

$$Q(x) := \sum_{n=1}^{\infty} \lambda_n (x, e_n) e_n \tag{2.1}$$

for all $x \in K$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$.

Definition 2.1. *A K-valued continuous Gaussian process $B^h(t)_{t\in[0,T]}$ with Hurst parameter $h \in (0,1)$ is called a Q-fractional Brownian motion, if there exists a kernel operator Q in K such that*

1. $\forall x, y \in K,\ s,t \in [0,T]$,

$$E\left(\left(B^h(t),x\right)_K \left(B^h(s),y\right)_K\right) = \frac{1}{2}(Q(x),y)_K \left(t^{2h} + s^{2h} - |t-s|^{2h}\right); \tag{2.2}$$

2. $\forall x \in K$,

$$E\left(B^h(t),x\right)_K = 0. \tag{2.3}$$

Remark 2.2. (i) In view of (2.2) we say that B^h has the covariance operator $\frac{1}{2}Q\left(t^{2h} + s^{2h} - |t-s|^{2h}\right)$.

(ii) Eq. (2.3) is equivalent to $EB^h(t) = 0$, i.e., it is the zero element of K.

(iii) The case of long-range dependence, i.e. $\frac{1}{2} < h < 1$, is given by

$$E\left(\left(B^h(t),x\right)_K \left(B^h(s),y\right)_K\right) = (Q(x),y)_K \int_0^t \int_0^s \varphi(u,v)\,du\,dv,$$

where $\varphi(u,v) := h(2h-1)|u-v|^{2h-2}$.

(iv) The Hilbert space valued Wiener process is obtained for $h = \frac{1}{2}$.

Theorem 2.3. *Let*

(i) *$(e_n)_{n\in\mathbb{N}}$ be a complete orthonormal system in K;*

(ii) *$(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{R}^1_+$, $\sum_{n=1}^{\infty} \lambda_n < \infty$;*

(iii) *$(\beta_n^h(t))_{t\in[0,T]}$, $n = 1, 2, \ldots$ be independent real fractional Brownian motions with*

$$E\left(\beta_n^h(t)\beta_k^h(s)\right) = \frac{1}{2}\delta_{nk}\left(t^{2h} + s^{2h} - |t-s|^{2h}\right),$$

where δ_{nk} is the Kronecker delta function.

Then $\left(B^h(t)\right)_{t\in[0,T]}$ *is a Q-fractional Brownian motion if and only if*

$$B^h(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n^h(t)e_n = \sum_{n=1}^{\infty} Q^{1/2}(e_n)\beta_n^h(t). \tag{2.4}$$

Proof. See Grecksch and Anh [6], or Duncan, Maslowski and Pasic-Duncan [3].

We write $B_n^h(t) = \sqrt{\lambda_n}\beta_n^h(t)$.

In the following we will discuss (a two-sided) Q-fractional Brownian motion with help of fractional white noise calculus. Therefore we assume that the underlying probability spaces for the independent real fractional Brownian motions $B_1^h(\cdot)$, $B_2^h(\cdot)$, ... are $\Omega_1 = \mathcal{S}'(\mathbb{R}^1)$, $\Omega_2 = \mathcal{S}'(\mathbb{R}^1)$, ..., that is $B^h(\cdot)$ is defined on $\Omega = \prod_{i=1}^{\infty}\Omega_i$.

We now introduce the fundamental operator $M_h(t)$ according to Elliott and van der Hoek [4].

For $0 < h < \frac{1}{2}$ and $f \in \mathcal{S}(\mathbb{R}^1)$,

$$M_h f(x) := \left(2\Gamma\left(h-\frac{1}{2}\right)\cos\left(\frac{\pi}{2}\left(h-\frac{1}{2}\right)\right)\right)^{-1}\int_{\mathbb{R}^1}\frac{f(x-t)-f(x)}{|t|^{\frac{3}{2}-h}}\,dt. \tag{2.5}$$

For $\frac{1}{2} < h < 1$ and $f \in \mathcal{S}(\mathbb{R}^1)$,

$$M_h f(x) := \left(2\Gamma\left(h-\frac{1}{2}\right)\cos\left(\frac{\pi}{2}\left(h-\frac{1}{2}\right)\right)\right)^{-1}\int_{\mathbb{R}^1}\frac{f(t)}{|t-x|^{\frac{3}{2}-h}}\,dt. \tag{2.6}$$

For $h = \frac{1}{2}$ we put $M_h f(x) = f(x)$, the identity map.

When $f(x) = I(0,t)(x)$ we write

$$M_h f(x) = M_h(0,t)(x). \tag{2.7}$$

Now we want to characterize the Hilbert space valued fractional Brownian motion with white noise calculus. We define

$$\tilde{B}_h(t,\omega) = \sum_{n=1}^{\infty}\sqrt{\lambda_n} < M_h(0,t),\omega_n > e_n, \tag{2.8}$$

with $< M_h(0,t),\omega_n > = \int_{\mathbb{R}^1} M_h(0,t)(s)\,d\beta_n(s)$ and β_n are independent real Brownian motions.

Again, $\tilde{B}_h(t)$ is a Gaussian random variable with

$$E\left[\left(\tilde{B}_h(t),x\right)_K\right] = 0 \tag{2.9}$$

and for $s < t$, we get using the independence of ω_i

$$\begin{aligned} & E\left[\left(\tilde{B}_h(t),x\right)_K\left(\tilde{B}_h(s),y\right)_K\right] \\ &= E\left[\sum_{i=1}^{\infty}\sqrt{\lambda_i} < M_h(0,t),\omega_i > (x,e_i)_K\sum_{k=1}^{\infty}\sqrt{\lambda_k} < M_h(0,s),\omega_k > (y,e_k)_K\right] \\ &= C_h\left(|t|^{2h}+|s|^{2h}-|t-s|^{2h}\right)(Qx,y). \end{aligned} \tag{2.10}$$

The process $\tilde{B}^h(t)$ has a continuous version in K, which we denote by $B^h(t)$.

We extend the definition of M_h to Hilbert space valued functions $f : \mathbb{R}^1 \to K$. Then M_h is defined by

$$M_h f(x) := \sum_{n=1}^{\infty} e_n M_h \left(f, e_n\right)_K (x) \tag{2.11}$$

for all $x \in \mathbb{R}^1$ and all

$$f \in L_h^2(\mathbb{R}^1, K) := \left\{ f : \mathbb{R} \to K, M_h f = \sum_{i=1}^{\infty} M_h \left((f, e_i)_K\right) e_i \in L^2(\mathbb{R}^1, K) \right\}, \tag{2.12}$$

where $M_h \left(f, e_i\right)_K$ is defined by applying (2.5) and (2.6) to the real functions $(f(\cdot), e_i)_K$.

The Hermite functions $\{\xi_n\}_{n=1}^{\infty}$, i.e.

$$\xi_n = \pi^{-\frac{1}{4}}((n-1)!)^{-\frac{1}{2}} h_{n-1}(\sqrt{2}x) e^{\frac{x^2}{2}}, \tag{2.13}$$

where $h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} \left(e^{\frac{-x^2}{2}} \right)$ form a basis of $L^2(\mathbb{R}^1, \mathbb{R}^1)$. Define

$$\eta_n(x) = M_h^{-1} \xi_n(x); \;\; n = 1, 2... \tag{2.14}$$

Then it follows from [4]

$$(f(x), e_n) = \sum_{j=1}^{\infty} c_{jn} \eta_j(x) \tag{2.15}$$

that η_j is an orthonormal basis of $L_h^2(\mathbb{R}^1, \mathbb{R}^1)$. Consequently $\eta_j(x) e_n$, ($j = 1, 2, ...,$ $n = 1, 2...$) defines an orthonormal basis of $L_h^2(\mathbb{R}^1, K)$.

Let $\mathcal{H}_r$, $r = 1, 2, ...$, be the Hermite polynomials of order r. Evidently we have

$$\mathcal{H}_1(\langle B^h, \eta_j e_n \rangle) = \frac{1}{2} \langle B^h, \eta_j e_n \rangle = \frac{1}{2} \langle B_n^h, \eta_j \rangle = \frac{1}{2} \langle \sqrt{\lambda_n} \beta_n^h, \eta_j \rangle.$$

Furthermore we define

$$\mathcal{H}_\alpha \left(B_n^h \right) := \mathcal{H}_{\alpha_1} \left(B_n^h \left(\eta_1\right) \right) \cdot ... \cdot \mathcal{H}_{\alpha_j} \left(B_n^h \left(\eta_j\right) \right),$$

and α is a multi-index, that is, $\alpha = (\alpha_1, ..., \alpha_j)$, $\alpha_i \in \mathbb{N}$. In particular $\varepsilon^{(n)}$ denotes the multi-index with 1 at the place n and 0 else.

Remark 2.4. In view of the representation Theorem 2.3, Eq. (2.4) for Q-fractional Brownian motions, we have for a deterministic function F with values in $L^2[0, T]$

$$\int_0^T F(s)\, dB^h(s) \;\; = \;\; \sum_{n=1}^{\infty} \int_0^T \sqrt{\lambda_n} F(s) e_n \, d\beta_n^h(s) \tag{2.16}$$

in mean square in H.

We can write the expansion of $B^h(t)$ as

$$B^h(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n^h(t)e_n = \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} \int_0^t \eta_j(s)\, ds \mathcal{H}_{\varepsilon(j)}(B_n^h)e_n. \quad (2.17)$$

We introduce the notation

$$B^h(\eta_j e_n) := \langle B^h, \eta_j e_n \rangle e_n = \int_{\mathbb{R}^1} \eta_j(x)\, dB_n^h(x)e_n. \quad (2.18)$$

Furthermore $\int_0^T \eta_j(t)\, dB_n^h(t)e_n$ is defined by $\int_{\mathbb{R}^1} I_{[0,T]}(t)\eta_j(t)\, dB_n^h(t)e_n$.

Therefore we have

$$E\left(B^h(\eta_j e_n)\right)^2 = \int_{\mathbb{R}^1} \lambda_n |M_h(\eta_j(t))|^2\, dt = \lambda_n. \quad (2.19)$$

Remark 2.5. (i) Let $F(s)$ be a deterministic operator function. Then we get

$$\int_0^T (F(s)e_n, h_k)_K\, dB_n^h(s) = \sum_{j=1}^{\infty} c_{knj}\sqrt{\lambda_n}\mathcal{H}(\beta_n^h(I_{[0,T]}\eta_j)). \quad (2.20)$$

(ii) Especially, if $H = \mathbb{R}^1$ and $F(s) = \gamma(s) \in L_h^2([0,T], K)$ and $\|\gamma(s)\| \le C$ $\forall\, s \in [0,T]$. Then

$$\int_0^T \left(\gamma(s), dB^h(s)\right)_K = \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} c_{nj}\mathcal{H}_1(B_n^h(I_{[0,T]}\eta_j)). \quad (2.21)$$

(iii) Using the properties of Hermite polynomials the expansion of $Exp\{b_j\eta_j\}$ ($b_j \in \mathbb{R}^1$) is given by

$$\begin{aligned} Exp\{b_j\eta_j\} &= exp\left\{ b_j \int_{\mathbb{R}^1} \sqrt{\lambda_n}\eta_j(t)\, d\beta_n^h(t) - \frac{b_j^2\lambda_n}{2}\|M_h\eta_j\|^2_{L^2(\mathbb{R})} \right\} \\ &= \sum_{l=1}^{\infty} \frac{b_j^l}{l!}\mathcal{H}_l(B_n^h(\eta_j)) = \sum_{l=1}^{\infty} \frac{b_j^l}{l!}\mathcal{H}_l(B_n^h(\eta_j))), \end{aligned} \quad (2.22)$$

(see [7], [8] or [10]).

Example 2.6. Now let us consider the expansion of $Exp\{\gamma\}$ for $\gamma \in L_Q\left(\mathbb{R}^1, K\right)$ with respect to $e_n\eta_j(t)$, $j = 1, 2...$, $n = 1, 2, ...$ see (2.21). We can write the exponential of γ as

$$\begin{aligned} Exp\{\gamma\} &= \exp\left\{ \int_{\mathbb{R}^1} \left(\gamma(t),\, dB^h(t)\right) - \frac{1}{2}\|M_h\gamma\|^2_{L_Q^2(\mathbb{R}^1,K)} \right\} \\ &= \exp\left\{ \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \sqrt{\lambda_n}c_{nj}\mathcal{H}_1(\beta_n^h(\eta_j)) - \frac{1}{2}\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\lambda_n c_{nj}^2\|M_h\eta_j\|^2_{L^2(\mathbb{R})} \right\} \\ &=: \sum_{\alpha\in\mathcal{I}}\prod_{n,j=1}^{\infty} c_{\alpha nj}\mathcal{H}_\alpha\left(B_n^h(\eta_j)\right), \end{aligned} \quad (2.23)$$

where $\mathcal{H}_\alpha(B_n^h) := \mathcal{H}_{\alpha_1}(B_n^h(\eta_j)) \cdot \ldots \cdot \mathcal{H}_{\alpha_j}(B_n^h(\eta_j))$ and

$$c_{\alpha nj} := \prod_{l=1}^{\infty} \frac{(c_{nj})^{\alpha_l}}{\alpha_l!}, \ \alpha = (\alpha_1, ..., \alpha_j).$$

Here, $\mathcal{I}$ denotes the set of all multi-indices α, $\mathcal{I} = \{(\alpha_1, ..., \alpha_n) : \alpha_1, ..., \alpha_n \in \mathbb{N}_0, n \in \mathbb{N}\}$.

We obtain for $Exp\{\gamma(t)\}$

$$Exp\{\gamma(t)\} = \sum_{\alpha\in\mathcal{I}} \prod_{n,j=1}^{\infty} c_{\alpha nj}\mathcal{H}_\alpha\left(B_n^h(I_{[0,T]}\eta_j)\right). \tag{2.24}$$

Now we want to develop a fractional white noise integration theory for $h \in (0,1)$. Grecksch, Roth and Anh [1] define the Q-fractional version of the Hida test function space and the Hida distribution space for $h \in (\frac{1}{2}, 1)$. Inspired by (2.23) we make the definitions as follows:

Let V be a separable Hilbert space with a complete orthonormal system $(v_k) \subseteq V$.

Definition 2.7. *The Q-fractional test function space $S_Q^h(V)$ is the space of all V-valued random functions with expansion*

$$\Psi(\omega) = \sum_{k=1}^{\infty} \left[\sum_{\alpha\in\mathcal{I}} \prod_{n,j=1}^{\infty} c_{\alpha nj}^{(k)} \mathcal{H}_\alpha(B_n^h)\right] v_k,$$

for which

$$\|\Psi\|_{h,r} := \sum_{k=1}^{\infty}\sum_{\alpha\in\mathcal{I}} \prod_{n,j=1}^{\infty} \alpha!(c_{\alpha nj}^{(j)})^2(2\mathbb{N})^{r\alpha} < \infty, \ \forall r \in \mathbb{N},$$

and $(2\mathbb{N})^\alpha := \prod_{j=1}^{\infty}(2j)^{\alpha_j}$ *if* $\alpha = (\alpha_1, ..., \alpha_m)$.

Definition 2.8. *The Q-fractional distribution space $(S_Q^h(V))^*$ is the space of all V-valued random functions with expansion*

$$G(\omega) = \sum_{k=1}^{\infty} \left[\sum_{\beta\in\mathcal{I}} \prod_{n,j=1}^{\infty} b_{\beta nj}^{(k)} \mathcal{H}_\beta(B_n^h)\right] v_k,$$

for which

$$\|G\|_{h,-q} := \sum_{k=1}^{\infty}\sum_{\beta\in\mathcal{I}} \prod_{n,j=1}^{\infty} \beta!(b_{\beta nj}^{(k)})^2(2\mathbb{N})^{-q\beta} < \infty \ \text{ for some } q \in \mathbb{N}.$$

Remark 2.9. If $V = \mathbb{R}^1$, then $\Psi(\omega) \in S_Q^h(V)$ (or $\Psi(\omega) \in (S_Q^h(V))^*$) has the following representation

$$\Psi(\omega) = \sum_{\alpha\in\mathcal{I}} \prod_{n,j=1}^{\infty} c_{\alpha nj}\mathcal{H}_\alpha(B_n^h).$$

Furthermore if the fractional noise is only one-dimensional, we find the well-known representation

$$\Psi(\omega) = \sum_{\alpha\in\mathcal{I}} c_\alpha \mathcal{H}_\alpha(B^h).$$

Consider the following duality relation between $S_Q^h(V)$ and $(S_Q^h(V))^*$. For $G \in (S_Q^h(V))^*$ and $\psi \in S_Q^h(V) \subset L_V^2(\Omega)$ we define

$$\langle\langle G, \psi\rangle\rangle := \sum_{k=1}^{\infty} \sum_{\alpha\in\mathcal{I}} \prod_{n,j=1}^{\infty} \alpha! c_{\alpha nj}^{(k)} b_{\alpha nj}^{(k)}. \tag{2.25}$$

Example 2.10. If $G \in L_V^2(\Omega)$ and $\psi \in S_Q^h(V) \subset L_V^2(\Omega)$, then we have

$$\langle\langle G, \psi\rangle\rangle = E(G,\psi)_V = (G,\psi)_{L_V^2(\Omega)}. \tag{2.26}$$

Definition 2.11. *Let $Z : [0,T] \to (S_Q^h(V))^*$ with*

$$\int_0^T |\langle\langle Z(t), \psi\rangle\rangle|\, dt < \infty, \quad \forall \psi \in S_Q^h(V).$$

Then $\int_0^T Z(t)\, dt \in (S_Q^h(V))^*$ is uniquely determined by the relation

$$\langle\langle \int_0^T Z(t)\, dt, \psi\rangle\rangle = \int_0^T \langle\langle Z(t), \psi\rangle\rangle\, dt.$$

We say that Z is $(S_Q^h(V))^*$-integrable.

Definition 2.12. (Wick scalar product)
Let F, $G \in (S_Q^h(K))^*$ with

$$\begin{aligned} F(\omega) &= F(B^h) = \sum_{k=1}^{\infty} \left[\sum_{\alpha\in\mathcal{I}} \prod_{n,j=1}^{\infty} a_{\alpha nj}^{(k)} \mathcal{H}_\alpha(B_n^h) \right] v_k, \\ G(\omega) &= G(B^h) = \sum_{k=1}^{\infty} \left[\sum_{\beta\in\mathcal{I}} \prod_{l,m=1}^{\infty} b_{\beta lm}^{(k)} \mathcal{H}_\beta(B_l^h) \right] v_k, . \end{aligned}$$

We define

$$\begin{aligned} (F,G)_{\diamond V} := & \sum_{k=1}^{\infty} \sum_{\alpha,\beta\in\mathcal{I}} \prod_{n,j=1}^{\infty} a_{\alpha nj}^{(k)} b_{\beta nj}^{(k)} \mathcal{H}_{\alpha+\beta}(B_n^h) \\ = & \sum_{k=1}^{\infty} \left[\sum_{\gamma\in\mathcal{I}} \sum_{\alpha+\beta=\gamma} \prod_{n,j=1}^{\infty} a_{\alpha nj}^{(k)} b_{\beta nj}^{(k)} \mathcal{H}_{\alpha+\beta}(B_n^h) \right]. \end{aligned} \tag{2.27}$$

Remark 2.13. If $V = \mathbb{R}^1$ then $(\cdot,\cdot)_{\diamond V}$ is the usual Wick product.

Now we introduce a fractional stochastic integral with stochastic integrands.

Definition 2.14. $Y : [0,T] \to (S_Q^h(V))^*$ *is* (dB^h-)*integrable if*

$$(Y(t), W^h(t))_{\diamond V} = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\,(Y(t), e_n)_V \diamond W_n^h(t)$$

is integrable with respect to t *in the sense of Definition* 2.11. *We define*

$$\int_0^T \big(Y(t), dB^h(t)\big) := \int_0^T (Y(t), W^h(t))_{\diamond V}\, dt.$$

3. A Q-fractional version of Itô's formula

In this section we prove a Q-fractional version of Itô's formula the way Biagini, Øksendal and al. presented it for a usual fractional Brownian motion, see [2].

$C^{1,2}([0,T] \times K, \mathbb{R}^1)$ denotes the space of all functions $f : [0,T] \times K \to \mathbb{R}^1$, such that the first Fréchet derivative $\nabla_s f(s,x)$ with respect to $s \in [0,T]$ and the first and second Fréchet derivatives $\nabla_x f(s,x)$ and $\nabla_{xx} f(s,x)$ exist continuously.

Theorem 3.1. *Let* $f(s,x) : [0,T] \times K \to \mathbb{R}$ *belong to* $C^{1,2}([0,T] \times K, \mathbb{R}^1)$. *Furthermore assume that there are constants* $C \geq 0$ *and* $0 < \lambda < \frac{1}{4T^{2h}}$ *such that for all* $(t,x) \in [0,T] \times K$

$$\max\Big\{|f(t,x)|,\ |\nabla_t f(t,x)|, \|\nabla_x f(t,x)\|_K, \|\nabla_{xx} f(t,x)\|_{L(K,K)}\Big\} \leq Ce^{\lambda x^2}. \tag{3.1}$$

Then

$$\begin{aligned} f(t, B^h(t)) &= f(0,0) + \int_0^t \nabla_s f(s, B^h(s))\, ds \\ &+ \int_0^t \big(\nabla_x f(s, B^h(s)), dB^h(s)\big)_K \\ &+ h \sum_{i=1}^{\infty} \int_0^t \big(\nabla_{xx} f(s, B^h(s)) e_i, e_i\big)_K \lambda_i s^{2h-1}\, ds, \end{aligned} \tag{3.2}$$

whereby

$$\begin{aligned} \nabla_s f(s, B^h(s)) &= \nabla_u f(u, B^h(s))\big|_{u=s}, \\ \nabla_x f(s,x) &= \nabla_x f(s,x)\big|_{x=B^h(s)}, \\ \nabla_{xx} f(s,x) &= \nabla_{xx} f(s,x)\big|_{x=B^h(s)}. \end{aligned}$$

Proof. Define

$$g(t,x) = \exp\{(a,x)_K + \beta(t)\}, \tag{3.3}$$

whereby $a \in K$ is a constant, $\beta \in C^1([0,T], \mathbb{R}^1)$ is a deterministic function, and put

$$Y(t) = g(t, B^h(t)), \text{ i.e. } x = B^h(t). \tag{3.4}$$

With

$$(a, B^h(s))_K = \sum_{i=1}^{\infty} \sqrt{\lambda_i}(a, e_i)\beta_i^h(t)$$

we can rewrite

$$\begin{aligned} Y(t) &= \exp\left\{\sum_{i=1}^{\infty} \sqrt{\lambda_i}(a, e_i)_K \beta_i^h(t)\right\} \exp\{\beta(t)\} \\ &= \exp^{\diamond}\left\{\sum_{i=1}^{\infty} \sqrt{\lambda_i}(a, e_i)_K \beta_i^h(t) + \frac{1}{2}\sum_{i=1}^{\infty} \lambda_i (a, e_i)_K^2 t^{2h}\right\} \exp\{\beta(t)\}. \end{aligned} \tag{3.5}$$

Therefore, by applying Wick calculus, we have

$$\begin{aligned} &\frac{d}{dt} Y(t) \\ &= \exp^{\diamond}\left\{\sum_{i=1}^{\infty} \sqrt{\lambda_i}(a, e_i)_K \beta_i^h(t) + \frac{1}{2}\sum_{i=1}^{\infty} \lambda_i (a, e_i)_K^2 t^{2h}\right\} \exp\{\beta(t)\} \\ &\quad \diamond \left[(a, W^h(t))_K + h\sum_{i=1}^{\infty} \lambda_i (a, e_i)_K^2 t^{2h-1}\right] \\ &\quad + \exp^{\diamond}\left\{\sum_{i=1}^{\infty} \sqrt{\lambda_i}(a, e_i)_K \beta_i^h(t) + \frac{1}{2}\sum_{i=1}^{\infty} \lambda_i (a, e_i)_K^2 t^{2h}\right\} \exp\{\beta(t)\} \beta'(t) \\ &= Y(t) \cdot \beta'(t) + Y(t) \diamond (a, W^h(t))_K + Y(t) \cdot h \sum_{i=1}^{\infty} \lambda_i (a, e_i)_K^2 t^{2h-1}. \end{aligned} \tag{3.6}$$

Hence we have found the following representation

$$\begin{aligned} Y(t) &= Y(0) + \int_0^t Y(s) \cdot \beta'(s)\, ds + h \int_0^t Y(s) \cdot \sum_{i=1}^{\infty} \lambda_i (a, e_i)_K^2 s^{2h-1}\, ds \\ &\quad + \int_0^t Y(s) \diamond (a, W^h(s))_K\, ds. \end{aligned} \tag{3.7}$$

Remembering (3.3) this can be written as

$$\begin{aligned} g(t, B^h(t)) &= g(0,0) + \int_0^t \nabla_s g(s, B^h(s))\, ds + \int_0^t \left(\nabla_x g(s, B^h(s)), dB^h(s)\right)_K \\ &\quad + h \sum_{i=1}^{\infty} \int_0^t \left(\nabla_{xx} g(s, B^h(s)) e_i, e_i\right)_K \lambda_i s^{2h-1}\, ds, \end{aligned} \tag{3.8}$$

which is (3.2).

Now let $f(t, x)$ be as demanded above. Every function $f \in C^{1,2}\left([0, T] \times K, \mathbb{R}^1\right)$ can be approximated by a sequence of linear combinations of type (3.3), hence we can find a sequence of linear combinations

$f_n(t,x)$ of functions $g(t,x)$ of the form (3.3) such that

$$f_n(t,x) \to f(t,x), \;\; \nabla_t f_n(t,x) \to \nabla_t f(t,x), \;\; \nabla_x f_n(t,x) \to \nabla_x f(t,x), \\ \nabla_{xx} f_n(t,x) \to \nabla_{xx} f(t,x)$$

pointwise dominatedly as $n \to \infty$. By (3.8) we have for all n

$$f_n(t, B^h(t)) = f_n(0,0) + \int_0^t \left(\nabla_x f_n(s, B^h(s)), dB^h(s)\right)_K \\ +h \sum_{i=1}^{\infty} \int_0^t \left(\nabla_{xx} f_n(s, B^h(s)) e_i, e_i\right)_k \lambda_i s^{2h-1}\, ds + \int_0^t \nabla_s f_n(s, B^h(s))\, ds \quad (3.9)$$

Taking the limit of (3.9) in $L^2_Q(K, \mathrm{I\!R}^1)$ (and therefore also in $(S^h_Q(\mathrm{I\!R}^1))^*$) we get

$$f(t, B^h(t)) = f(0,0) + \lim_{n\to\infty} \int_0^t \left(\nabla_x f_n(s, B^h(s)), dB^h(s)\right)_K \\ +h \sum_{i=1}^{\infty} \int_0^t \left(\nabla_{xx} f(s, B^h(s)) e_i, e_i\right)_K \lambda_i s^{2h-1}\, ds + \int_0^t \nabla_s f(s, B^h(s))\, ds. \quad (3.10)$$

Since the mapping $s \to \nabla_x f(s, B^h(s))$ is continuous in $(S^h_Q(\mathrm{I\!R}^1))^*$ we get

$$\int_0^t \left(\nabla_x f_n(s, B^h(s)), dB^h(s)\right)_K = \int_0^t \left(\nabla_x f_n(s, B^h(s)), W^h(s)\right)_K ds \\ \to \int_0^t \left(\nabla_x f(s, B^h(s)), W^h(s)\right)_K ds$$

for $n \to \infty$ in $(S^h_Q(\mathrm{I\!R}^1)^*)$. The last relation and (3.10) show (3.2). □

Example 3.2. Now let $f(s,x) : [0,T] \times K \to \mathrm{I\!R}$ be defined as follows:

$$f(t,x) := \exp(t+x),$$

then we have

$$\nabla_t f(t,x) = \nabla_x f(t,x) = \nabla_{xx} f(t,x) = \exp(t+x),$$

and therefore we have by (3.2)

$$\begin{aligned} f(t,B^h(t)) &= 1+\int_0^t \exp(s+B^h(s))\,ds \\ &\quad +\int_0^t \left(\exp(s+B^h(s)),dB^h(s)\right)_K \\ &\quad +h\sum_{i=1}^{\infty}\int_0^t \left(\exp(s+B^h(s))e_i,e_i\right)_K \lambda_i s^{2h-1}\,ds \\ &= 1+\int_0^t \exp(s+B^h(s))\,ds \\ &\quad +\int_0^t \left(\exp(s+B^h(s)),W^h(s)\right)_{\diamond K}\,ds \\ &\quad +h\sum_{i=1}^{\infty}\int_0^t \left(\exp(s+B^h(s))e_i,e_i\right)_K \lambda_i s^{2h-1}\,ds. \end{aligned}$$

Example 3.3. Now let $f(s,x):[0,T]\times K\to \mathbb{R}$ be defined as follows:

$$f(t,x) := \ln\left(1+x^2\right),$$

then we have

$$\nabla_t f(t,x)=0,\quad \nabla_x f(t,x)=\frac{2x}{1+x^2} \text{ and } \nabla_{xx} f(t,x)=\frac{2-2x^2}{\left(1+x^2\right)^2},$$

and therefore we have by (3.2)

$$\begin{aligned} f(t,B^h(t)) &= 0+\int_0^t \left(\frac{2B^h(s)}{1+\left(B^h(s)\right)^2},W^h(s)\right)_{\diamond K} ds \\ &\quad +h\sum_{i=1}^{\infty}\int_0^t \left(\frac{2-2\left(B^h(s)\right)^2}{\left(1+\left(B^h(s)\right)^2\right)^2}e_i,e_i\right)_K \lambda_i s^{2h-1}\,ds. \end{aligned}$$

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