Stud. Univ. Babeş-Bolyai Math.
Volume LVI, Number 1
March 2011, pp. 179–193

Boundary integral equations for the problem of 2D Brinkman flow past several voids

Elena-Maria Ului

Abstract. In this paper we obtain the existence and uniqueness result for the classical solution of the boundary value problem which describes the $2D$ flow of an incompressible Newtonian fluid in a porous medium and in the presence of $N \geq 2$ voids.

Mathematics Subject Classification (2010): 76D07, 76D03, 31A10.

Keywords: Brinkman flow, potential theory, boundary integral representation, existence and uniqueness result.

1. Introduction

The problem of viscous incompressible fluid flow through porous media has various chemical, biotechnology, and geological applications, concerning: the treatment of transport and chemical reaction within catalyst particles in fixed and fluidized beds, the modeling of polymer molecules as porous particles, immobilization of cells or enzymes and perfusion chromatography for purifying proteins and other bio-molecules, the flow of various kinds of fluids past porous rocks embedded in porous soil. In [2] Kohr and Sekhar have used the potential theory, as well as the Brinkman model, in order to obtain the existence and uniqueness result of the classical solution to a boundary value problem which describes the flow of an unbounded viscous incompressible fluid in the presence of a porous body embedded in another porous medium. Also, in [3] the authors obtained an indirect boundary integral formulation for the three-dimensional viscous flow problem in a granular material with one void. The method of matched asymptotic expansions and the method of boundary integral equations have been used in [4] in order to study the two-dimensional steady flow of a viscous incompressible fluid at low Reynolds number past a porous body of arbitrary shape. In this paper we show the existence and uniqueness result for the classical solution of a boundary value problem that describes the two-dimensional flow of an incompressible Newtonian fluid in a porous medium and in the presence of $N \geq 2$ voids by using the Brinkman model for the external flow, as well as the Stokes model for the internal flow. We use a boundary integral method that reduces the flow problem to a system of Fredholm integral equations of the second kind that has a unique solution in some Banach spaces.

2. The mathematical formulation of the problem

Let us consider an otherwise unbounded homogeneous granular material in which $N \geq 2$ fluid obstacles (voids) are given. The k-th void occupies the bounded domain $D_k \subset \mathbb{R}^2$ whose boundary Γ_k is a closed Lypaunov curve in the class $C^{1,\alpha}$, $\alpha \in (0,1]$, $k = 1, \dots, N$. Let us denote by D_0 the set given by $D_0 = \cup_{k=1}^N D_k$. We denote by D_e the unbounded domain with the boundary $\Gamma = \cup_{k=1}^N \Gamma_k$, and assume that at great distances, i.e., far from the voids, the fluid flow is uniform with velocity and pressure fields $\mathbf{U}_\infty$ and p_∞, respectively.

Let us now assume that the flow in the unbounded domain D_e is described by the Brinkman model, i.e., the Brinkman and continuity equations. Thus, the non-dimensional volume averaged velocity and pressure fields $\mathbf{v}^e$ and p^e satisfy in D_e the following equations:

$$-\nabla p^e + (\nabla^2 - \chi^2)\mathbf{v}^e = \mathbf{0} \ \text{ in } D_e, \tag{2.1}$$

$$\nabla \cdot \mathbf{v}^e = 0 \ \text{ in } D_e, \tag{2.2}$$

where $\chi > 0$ is the constant having the expression $\chi = \frac{a}{\sqrt{\kappa}}\sqrt{\frac{\mu_f}{\mu_{eff}}}$, a is a characteristic length (connected to the sizes of the curves Γ_k, $k = 1, \dots, N$) and k is the permeability of the porous medium. Note that if $\mu_f = \mu_{eff}$, then χ becomes $\chi = a/\sqrt{\kappa}$.

The flow inside each void is assumed to be described by the Stokes system, i.e., by the Stokes and continuity equations:

$$-\nabla p^i + \nabla^2 \mathbf{v}^i = \mathbf{0} \ \text{ in } D_0, \tag{2.3}$$

$$\nabla \cdot \mathbf{v}^i = 0 \ \text{ in } D_0. \tag{2.4}$$

Also, we assume that the velocity and boundary traction fields are continuous across each curve Γ_k, $k = 1, \dots, N$, i.e.,

$$\mathbf{v}^i = \mathbf{v}^e, \quad \mathbf{t}^i = \mathbf{t}^e \text{ on } \Gamma_k. \tag{2.5}$$

Note that $\mathbf{t}^e$ is the boundary traction corresponding to the external fields $\mathbf{v}^e$ and p^e, and $\mathbf{t}^i$ is the boundary traction due to the internal fields $\mathbf{v}^i$ and p^i.

At large distances, the fields $\mathbf{v}^p = \mathbf{v}^e - \mathbf{U}^\infty$ and $p^p = p^e - P^\infty$ vanish such that

$$(|\mathbf{v}^p||\nabla \mathbf{v}^p|)(\mathbf{x}) = o(|\mathbf{x}|^{-1}), \quad (|\mathbf{v}^p||p^p|)(\mathbf{x}) = o(|\mathbf{x}|^{-1}) \ \text{ as } |\mathbf{x}| \to \infty, \tag{2.6}$$

where $\mathbf{U}^\infty$ and P^∞ are the non-dimensional undisturbed velocity and pressure fields.

Therefore, the considered flow problem reduces to the boundary value problem consisting of the system of equations (2.1)-(2.4) subject to the transmission and far field conditions (2.5)-(2.6) and having as unknowns the fields $\mathbf{v}^e$, p^e, $\mathbf{v}^i$ and p^i. We show that this problem has a unique classical solution $((\mathbf{v}^e, p^e), (\mathbf{v}^i, p^i)) \in ((C^2(D_e) \cap C^0(\overline{D}_e)) \times C^1(D_e)) \times ((C^2(D_0) \cap C^0(\overline{D}_0)) \times C^1(D_0))$, where $D_0 = \cup_{k=1}^N D_k$.

3. Uniqueness of the solution

First, we show the following uniqueness result:

Theorem 3.1. *The boundary value problem* (2.1)-(2.6) *has at most one classical solution* $((\mathbf{v}^e, p^e), (\mathbf{v}^i, p^i)) \in ((C^2(D_e) \cap C^0(\overline{D}_e)) \times C^1(D_e)) \times ((C^2(D_0) \cap C^0(\overline{D}_0)) \times C^1(D_0))$.

Proof. Let us assume that the boundary value problem (2.1)-(2.6) has two classical solutions and let $((\mathbf{v}_0^e, p_0^e), (\mathbf{v}_0^i, p_0^i))$ be their difference. Therefore, the pairs $(\mathbf{v}_0^e, p_0^e)$ and $(\mathbf{v}_0^i, p_0^i)$ satisfy the following equations, boundary and far field conditions:

$$-\nabla p_0^i + \nabla^2 \mathbf{v}_0^i = \mathbf{0} \ \text{ and } \ \nabla \cdot \mathbf{v}_0^i = 0 \ \text{ in } D_0, \tag{3.1}$$

$$-\nabla p_0^e + (\nabla^2 - \chi^2)\mathbf{v}_0^e = \mathbf{0} \ \text{ and } \ \nabla \cdot \mathbf{v}_0^e = 0 \ \text{ in } D_e, \tag{3.2}$$

$$\mathbf{v}_0^i = \mathbf{v}_0^e \ \text{ and } \ \mathbf{t}_0^i = \mathbf{t}_0^e \ \text{ on } \Gamma_k, \ k = 1, \dots, N, \tag{3.3}$$

$$(|\mathbf{v}_0^e||\nabla \mathbf{v}_0^e|)(\mathbf{x}) = o(|\mathbf{x}|^{-1}), \quad (|\mathbf{v}_0^e||p_0^e|)(\mathbf{x}) = o(|\mathbf{x}|^{-1}) \ \text{ as } |\mathbf{x}| \to \infty. \tag{3.4}$$

In addition, the fields $\mathbf{v}_0^e$ and p_0^e satisfy the energy identity (see e.g. [1], p.24)

$$2\int_{D_e} E_{kj}(\mathbf{v}_0^e) E_{kj}(\mathbf{v}_0^e) d\mathbf{x} = -\sum_{k=1}^N \int_{\Gamma_k} \mathbf{v}_0^e \cdot \mathbf{t}_0^e d\Gamma_k, \tag{3.5}$$

where

$$E_{kj}(\mathbf{v}_0^e) = \frac{1}{2}\left(\frac{\partial v_{0,k}^e}{\partial x_j} + \frac{\partial v_{0,j}^e}{\partial x_k}\right)$$

and $\mathbf{t}_0^e = (t_{0,1}^e, t_{0,2}^e)$ is the boundary traction due to the fields $\mathbf{v}_0^e = (v_{0,1}^e, v_{0,2}^e)$ and p_0^e, i.e.,

$$t_{0,j}^e = T_{jk}(\mathbf{v}_0^e) n_k = (-p_0^e \delta_{jk} + 2E_{jk}(\mathbf{v}_0^e)) n_k. \tag{3.6}$$

In the relations (3.5) and (3.6) and in what follows we use Einstein's repeated-index summation convention. Also we denote by $\mathbf{n} = (n_1, n_2)$ the outward unit normal to Γ.

Now, making use of the fact that the fields $\mathbf{v}_0^i$ and p_0^i satisfy the equations (3.2), we get the identity (see e.g. [1], p.15):

$$\int_{D_k} (\chi^2 |\mathbf{v}_0^i|^2 + 2E_{kj}(\mathbf{v}_0^i) E_{kj}(\mathbf{v}_0^i)) d\mathbf{x} = \int_{\Gamma_k} \mathbf{v}_0^i \cdot \mathbf{t}_0^i d\Gamma_k, \quad k = 1, \dots, N, \tag{3.7}$$

where

$$E_{jk}(\mathbf{v}_0^i) = \frac{1}{2}\left(\frac{\partial v_{0,j}^i}{\partial x_k} + \frac{\partial v_{0,k}^i}{\partial x_j}\right)$$

and $\mathbf{t}_0^i = (t_{0,1}^i, t_{0,2}^i)$ is the boundary traction due to the fields $\mathbf{v}_0^i = (v_{0,1}^i, v_{0,2}^i)$ and p_0^i, defined as in (3.6).

Taking into account the boundary conditions (3.3), as well as the identities (3.5) and (3.7), we obtain the equality

$$2\int_{D_e} E_{jk}(\mathbf{v}_0^e)E_{jk}(\mathbf{v}_0^e)d\mathbf{x} = -\sum_{k=1}^{N}\int_{D_k}\big(\chi^2|\mathbf{v}_0^i|^2 + 2E_{jk}(\mathbf{v}_0^i)E_{jk}(\mathbf{v}_0^i)\big)d\mathbf{x}, \quad (3.8)$$

where the left-hand side is non-negative and the right-hand side is less than or equal to zero. Thus, we obtain that

$$\int_{D_e} E_{jk}(\mathbf{v}_0^e)E_{jk}(\mathbf{v}_0^e)d\mathbf{x} = 0,$$

$$\int_{D_k}(\chi^2|\mathbf{v}_0^i|^2 + 2E_{jk}(\mathbf{v}_0^i)E_{jk}(\mathbf{v}_0^i))d\mathbf{x} = 0, \ k = 1, \dots, N.$$

Therefore, we find that

$$\mathbf{v}_0^i = \mathbf{0} \ \text{ in } D_k, \quad k = 1, \dots, N \quad (3.9)$$

and, due to (3.4),

$$\mathbf{v}_0^e = \mathbf{0} \ \text{ in } D_e. \quad (3.10)$$

In view of (3.1) and (3.10) it follows that $p_0^e = c_e \in \mathbb{R}$ in D_e. The decay condition of p_0^e at infinity yields that $c_e = 0$, i.e., $p_0^e = 0$ in D_e. Hence we have

$$\mathbf{v}_0^e = \mathbf{0} \ \text{ and } \ p_0^e = 0 \text{ in } D_e. \quad (3.11)$$

Using similar arguments, we obtain

$$\mathbf{v}_0^i = \mathbf{0} \ \text{ and } \ p_0^i = c_k \in \mathbb{R} \ \text{ in } D_k, \quad k = 1, \dots, N. \quad (3.12)$$

On the other hand, the properties (3.11) yield that

$$\mathbf{t}_0^e = \mathbf{0} \text{ on } \Gamma_k, \quad k = 1, \dots, N, \quad (3.13)$$

and, in view of the second of the conditions (3.3), it follows that $\mathbf{t}_0^i = -c_k\mathbf{n} = \mathbf{0}$ on Γ_k, $k = 1, \dots, N$. Therefore, we get $c_k = 0$, $k = 1, \dots, N$. Consequently, we have

$$\mathbf{v}_0^i = \mathbf{0}, \quad p_0^i = 0 \text{ in } D_0. \quad (3.14)$$

The relations (3.11) and (3.14) yield the desired uniqueness result. This completes the proof of Theorem 3.1. □

4. Potential theory for the Brinkman and Stokes equations

In this section we will present the fundemental solution for the Brimkman and Stokes equations and the main properties of the potential theory for the Brinkman system of equations (2.1)-(2.2) and respectively for the Stokes system (2.3)-(2.4).

4.1. The fundamental solutions of the Brinkman and Stokes equations

The components of the fundamental Brinkman tensor $\mathcal{G}^{\chi^2}$ and those of its associated pressure vector $\mathbf{\Pi}^{\chi^2}$, which determine the fundamental solution $(\mathcal{G}^{\chi^2}, \mathbf{\Pi}^{\chi^2})$ of the Brinkman system in $\mathbb{R}^2$, are given by (see e.g. [1, p. 81]):

$$\mathcal{G}_{jk}^{\chi^2}(\mathbf{x}-\mathbf{y}) = \delta_{jk} A_1(\chi|\mathbf{x}-\mathbf{y}|) + \frac{(x_j - y_j)(x_k - y_k)}{|\mathbf{x}-\mathbf{y}|^2} A_2(\chi|\mathbf{x}-\mathbf{y}|) \quad \text{and}$$

$$\Pi_j^{\chi^2}(\mathbf{x}-\mathbf{y}) = 2\frac{x_j - y_j}{|\mathbf{x}-\mathbf{y}|^2}, \tag{4.1}$$

where

$$\begin{aligned} A_1(z) &= 2\{K_0(z) + z^{-1}K_1(z) - z^{-2}\}, \\ A_2(z) &= 2\{-K_0(z) - 2z^{-1}K_1(z) + 2z^{-2}\}, \end{aligned} \tag{4.2}$$

and and K_ν is the modified Bessel function of the second kind and order ν.

The corresponding stress and pressure tensors $\mathbf{S}^{\chi^2}$ and $\mathbf{\Lambda}^{\chi^2}$ have the following components (see e.g. [1, p. 82, 196]):

$$\begin{aligned} S_{ijk}^{\chi^2}(\mathbf{x}-\mathbf{y}) &= -\Pi_j^{\chi^2}(\mathbf{x}-\mathbf{y})\delta_{ik} + \frac{\partial \mathcal{G}_{ij}^{\chi^2}(\mathbf{x}-\mathbf{y})}{\partial x_k} + \frac{\partial \mathcal{G}_{kj}^{\chi^2}(\mathbf{x}-\mathbf{y})}{\partial x_i} \\ = -2\Bigg\{\delta_{ik}\frac{x_j - y_j}{|\mathbf{x}-\mathbf{y}|^2} D_1(\chi|\mathbf{x}-\mathbf{y}|) &+ \left(\delta_{kj}\frac{x_i - y_i}{|\mathbf{x}-\mathbf{y}|^2} + \delta_{ij}\frac{x_k - y_k}{|\mathbf{x}-\mathbf{y}|^2}\right) D_2(\chi|\mathbf{x}-\mathbf{y}|) \\ &+ \frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)}{|\mathbf{x}-\mathbf{y}|^4} D_3(\chi|\mathbf{x}-\mathbf{y}|)\Bigg\}, \end{aligned} \tag{4.3}$$

$$\Lambda_{ik}^{\chi^2}(\mathbf{x}-\mathbf{y}) = 2\frac{\delta_{ik}}{|\mathbf{x}-\mathbf{y}|^2}\big(-\chi^2|\mathbf{x}-\mathbf{y}|^2 \ln|\mathbf{x}-\mathbf{y}| - 2\big) + 8\frac{(x_i - y_i)(x_k - y_k)}{|\mathbf{x}-\mathbf{y}|^4}, \tag{4.4}$$

where

$$\begin{aligned} D_1(z) &= 2K_2(z) + 1 - 4z^{-2}, \\ D_2(z) &= 2K_2(z) + zK_1(z) - 4z^{-2}, \\ D_3(z) &= -8K_2(z) - 2zK_1(z) + 16z^{-2}. \end{aligned} \tag{4.5}$$

The components of the fundamental tensor $\mathcal{G}$ and those of its associated pressure vector $\mathbf{\Pi}$, which determine the fundamental solution $(\mathcal{G}, \mathbf{\Pi})$ of the Stokes system in $\mathbb{R}^2$, are given by (see e.g. [1, p. 38])

$$\mathcal{G}_{jk}(\mathbf{x}-\mathbf{y}) = -\delta_{jk}\ln|\mathbf{x}-\mathbf{y}| + \frac{(x_j - y_j)(x_k - y_k)}{|\mathbf{x}-\mathbf{y}|^2}, \quad \Pi_j(\mathbf{x}-\mathbf{y}) = 2\frac{x_j - y_j}{|\mathbf{x}-\mathbf{y}|^2}, \tag{4.6}$$

and the stress and pressure tensors $\mathbf{S}$ and $\mathbf{\Lambda}$ have the components (see e.g. [1, p. 39, 132])

$$\begin{aligned} S_{ijk}(\mathbf{x}-\mathbf{y}) &= -4\frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)}{|\mathbf{x}-\mathbf{y}|^4}, \\ \Lambda_{ik}(\mathbf{x}-\mathbf{y}) &= 4\left(-\frac{\delta_{ik}}{|\mathbf{x}-\mathbf{y}|^2} + 2\frac{(x_i - y_i)(x_k - y_k)}{|\mathbf{x}-\mathbf{y}|^4}\right). \end{aligned} \tag{4.7}$$

4.2. Boundary potentials for the Brinkman and Stokes equations

Let $\mathcal{C} \in \mathbb{R}^2$ be a closed Lypaunov curve in the class $C^{1,\alpha}$, $\alpha \in (0,1]$. The *single-* and *double-layer potentials*, $\mathbf{V}_{\chi^2}(\cdot,\mathbf{g})$ and $\mathbf{W}_{\chi^2}(\cdot,\mathbf{h})$, associated with the Brinkman system and having the densities $\mathbf{g}$ and $\mathbf{h}$, respectively, are given by

$$\mathbf{V}_{\chi^2}(\mathbf{x},\mathbf{g}) = \frac{1}{4\pi}\int_{\mathcal{C}} \mathcal{G}^{\chi^2}(\mathbf{x}-\mathbf{y})\cdot\mathbf{g}(\mathbf{y})d\mathcal{C}(\mathbf{y}) \quad \text{for} \ \ \mathbf{x}\in\mathbb{R}^2\setminus\mathcal{C} \tag{4.8}$$

$$\left(\mathbf{W}_{\chi^2}\right)_k(\mathbf{x},\mathbf{h}) = \frac{1}{4\pi}\int_{\mathcal{C}} S^{\chi^2}_{jk\ell}(\mathbf{y}-\mathbf{x})n_\ell(\mathbf{y})h_j(\mathbf{y})d\mathcal{C}(\mathbf{y}) \quad \text{for} \ \ \mathbf{x}\in\mathbb{R}^2\setminus\mathcal{C}, \tag{4.9}$$

and the corresponding pressure functions $P^s_{\chi^2}(\cdot,\mathbf{g})$ and $P^d_{\chi^2}(\cdot,\mathbf{h})$ have the expressions

$$P^s_{\chi^2}(\mathbf{x},\mathbf{g}) = \frac{1}{4\pi}\int_{\mathcal{C}} \Pi^{\chi^2}_j(\mathbf{x}-\mathbf{y})g_j(\mathbf{y})d\mathcal{C}(\mathbf{y}) \quad \text{for} \ \ \mathbf{x}\in\mathbb{R}^2\setminus\mathcal{C} \tag{4.10}$$

$$P^d_{\chi^2}(\mathbf{x},\mathbf{h}) = \frac{1}{4\pi}\int_{\mathcal{C}} \Lambda^{\chi^2}_{j\ell}(\mathbf{x}-\mathbf{y})n_\ell(\mathbf{y})h_j(\mathbf{y})d\mathcal{C}(\mathbf{y}) \quad \text{for} \ \ \mathbf{x}\in\mathbb{R}^2\setminus\mathcal{C}. \tag{4.11}$$

The pairs $(\mathbf{V}_{\chi^2}(\cdot,\mathbf{g}), P^s_{\chi^2}(\cdot,\mathbf{g}))$ and $(\mathbf{W}_{\chi^2}(\cdot,\mathbf{h}), P^d_{\chi^2}(\cdot,\mathbf{h}))$ satisfy the Brinkman system in both domains D_0 and D_e, respectively.

The *single-* and *double-layer potentials*, $\mathbf{V}(\cdot,\mathbf{g})$ and $\mathbf{W}(\cdot,\mathbf{h})$, for the Stokes system and with the densities $\mathbf{g}$ and $\mathbf{h}$, respectively, can be obtained as in (4.8) and (4.9), but with $\mathcal{G}$ and $S_{jk\ell}$ instead of $\mathcal{G}^{\chi^2}$ and $S^{\chi^2}_{jk\ell}$. Similarly, the pressure terms $P^s(\cdot,\mathbf{g})$ and $P^d(\cdot,\mathbf{h})$ can be obtained as in (4.10) and (4.11), but with Π_j and $\Lambda_{j\ell}$ instead of ${\Pi_j}^{\chi^2}$ and $\Lambda^{\chi^2}_{j\ell}$.

Let us denote by $\mathbf{H}_{\chi^2}(\cdot,\mathbf{g})$ the normal stress due to the single-layer potential $\mathbf{V}_{\chi^2}(\cdot,\mathbf{g})$ and defined in a neighborhood $U\subset\mathbb{R}^2$ of $\mathcal{C}$ by the relation

$$\left(\mathbf{H}_{\chi^2}\right)_k(\mathbf{x},\mathbf{g}) = T_{k\ell}(\mathbf{V}_{\chi^2}(\mathbf{g}))(\mathbf{x})n_\ell(\tilde{\mathbf{x}}), \quad \mathbf{x}\in\widetilde{\mathcal{U}}\setminus\mathcal{C},$$

where $\tilde{\mathbf{x}}$ is the orthogonal projection of $\mathbf{x}\in U$ onto $\mathcal{C}$. On the components, we have

$$\left(\mathbf{H}_{\chi^2}\right)_k(\mathbf{x},\mathbf{g}) = \frac{1}{4\pi}\int_{S} S^{\chi^2}_{kj\ell}(\mathbf{x}-\mathbf{y})n_\ell(\bar{\mathbf{x}})g_j(\mathbf{y})d\mathcal{C}(\mathbf{y}), \quad \mathbf{x}\in U\setminus\mathcal{C},\ k=1,2. \tag{4.12}$$

The stress field due to the single-layer potential $\mathbf{V}(\cdot,\mathbf{g})$ is defined in U by the relation:

$$t_j(\mathbf{V}(\mathbf{g}))(\mathbf{x}) = T_{j\ell}(\mathbf{V}(\mathbf{g}))(\mathbf{x})n_\ell(\tilde{\mathbf{x}}), \quad \mathbf{x}\in U\setminus\mathcal{C},\ j=1,2. \tag{4.13}$$

Let $\mathbf{K}^{\chi^2}(\mathbf{y},\mathbf{x})$ be the kernel of the double-layer potential $\mathbf{W}_{\chi^2}(\cdot,\mathbf{h})$, whose components are given by $K^{\chi^2}_{jk}(\mathbf{y},\mathbf{x}) = S^{\chi^2}_{jk\ell}(\mathbf{y}-\mathbf{x})n_\ell(\mathbf{y})$. Similarly, the components of the kernel of the double-layer potential $\mathbf{W}(\cdot,\mathbf{h})$ are denoted by $K_{jk}(\mathbf{y},\mathbf{x})$, and are given by the relation $K_{jk}(\mathbf{y},\mathbf{x}) = S_{jk\ell}(\mathbf{y}-\mathbf{x})n_\ell(\mathbf{y})$.

Let us now consider the following decomposition of the tensors $\mathcal{G}^{\chi^2}$ and $\mathbf{S}^{\chi^2}$:

$$\begin{aligned} \mathcal{G}^{\chi^2}_{kj}(\mathbf{x}-\mathbf{y}) &= \mathcal{G}_{kj}(\mathbf{x}-\mathbf{y}) + \mathcal{G}^c_{kj}(\mathbf{x}-\mathbf{y}), \\ S^{\chi^2}_{jk\ell}(\mathbf{y}-\mathbf{x})n_\ell(\mathbf{y}) &= S_{jk\ell}(\mathbf{y}-\mathbf{x})n_\ell(\mathbf{y}) + S^c_{jk\ell}(\mathbf{y}-\mathbf{x})n_\ell(\mathbf{y}), \end{aligned} \tag{4.14}$$

where the matrix kernel $\mathcal{G}^c$ with the components $\mathcal{G}^c_{kj}$ and the matrix kernel $\mathbf{S}^c\mathbf{n}$ with the components $S^c_{jk\ell}n_l$ are continuous. Thus, one obtains the following result which shows the continuity behaviour and the jump formulas for the single- and double-layer potentials associated to the Brinkman system (for e.g. [4]):

Theorem 4.1. *a) Let $\mathcal{C}$ be a closed Lyapunov curve in $\mathbb{R}^2$, i.e., $\mathcal{C} \in C^{1,\alpha}$, $\alpha \in (0,1]$, and let densities $\mathbf{g} \in C^0(\mathcal{C})$ and $\mathbf{h} \in C^0(\mathcal{C})$ be given. Also let $\mathbf{V}_{\chi^2}(\cdot,\mathbf{g})$, $\mathbf{W}_{\chi^2}(\cdot,\mathbf{h})$ and $\mathbf{H}_{\chi^2}(\cdot,\mathbf{g})$ be the boundary potentials given by* (4.8), (4.9) *and* (4.12). *Then on $\mathcal{C}$ we have:*

$$\left(\mathbf{V}_{\chi^2}\right)^+(\cdot,\mathbf{g}) = \left(\mathbf{V}_{\chi^2}\right)^-(\cdot,\mathbf{g}) = \mathbf{V}_{\chi^2}(\cdot,\mathbf{g}), \tag{4.15}$$

$$\left(\mathbf{W}_{\chi^2}\right)^+(\cdot,\mathbf{h}) - \left(\mathbf{W}_{\chi^2}\right)^*(\cdot,\mathbf{h}) = \frac{1}{2}\mathbf{h} = \left(\mathbf{W}_{\chi^2}\right)^*(\cdot,\mathbf{h}) - \left(\mathbf{W}_{\chi^2}\right)^-(\cdot,\mathbf{h}), \tag{4.16}$$

$$\left(\mathbf{H}_{\chi^2}\right)^+(\cdot,\mathbf{g}) - \left(\mathbf{H}_{\chi^2}\right)^*(\cdot,\mathbf{g}) = -\frac{1}{2}\mathbf{g} = \left(\mathbf{H}_{\chi^2}\right)^*(\cdot,\mathbf{g}) - \left(\mathbf{H}_{\chi^2}\right)^-(\cdot,\mathbf{g}). \tag{4.17}$$

In addition, if $\mathbf{h} \in C^{1,\beta}(\mathcal{C})$, $\beta \in (0,\alpha)$, then there exist the limiting values of the boundary traction due to the double-layer potential $\mathbf{W}_{\chi^2}(\cdot,\mathbf{h})$ on both sides of $\mathcal{C}$, $\mathbf{T}^+(\mathbf{W}_{\chi^2}(\mathbf{h}))$ and $\mathbf{T}^-(\mathbf{W}_{\chi^2}(\mathbf{h}))$, and they are equal, i.e.,

$$\mathbf{T}^+(\mathbf{W}_{\chi^2}(\mathbf{h})) = \mathbf{T}^-(\mathbf{W}_{\chi^2}(\mathbf{h})) \equiv \mathbf{T}(\mathbf{W}_{\chi^2}(\mathbf{h})) \quad \text{on } \mathcal{C}. \tag{4.18}$$

The superscript $+$ $(-)$ is used for the limiting value of a field evaluated from the external side (the internal side) of $\mathcal{C}$, and the symbol $*$ refers to the principal value of a double-layer integral on $\mathcal{C}$. The relations (4.15)-(4.18) also hold for the boundary potentials associated with the Stokes system.

The functions $\mathbf{V}_{\chi^2}(\cdot,\mathbf{g})$, $\mathbf{W}_{\chi^2}(\cdot,\mathbf{h})$, $P^s_{\chi^2}(\cdot,\mathbf{g})$, $P^d_{\chi^2}(\cdot,\mathbf{h})$ satisfy the relations

$$\mathbf{V}_{\chi^2}(\mathbf{x},\mathbf{g}) = O(|\mathbf{x}|^{-2}), \quad \mathbf{W}_{\chi^2}(\mathbf{x},\mathbf{h}) = O(|\mathbf{x}|^{-1}) \ \text{ as } |\mathbf{x}| \to \infty, \tag{4.19}$$

$$P^s_{\chi^2}(\mathbf{x},\mathbf{g}) = O(|\mathbf{x}|^{-1}), \quad P^d_{\chi^2}(\mathbf{x},\mathbf{h}) = O(\ln|\mathbf{x}|) \ \text{ as } |\mathbf{x}| \to \infty, \tag{4.20}$$

and in the case $\chi = 0$, we have:

$$\mathbf{V}(\mathbf{x},\mathbf{g}) = O(\ln|\mathbf{x}|), \quad P^s(\mathbf{x},\mathbf{h}) = O(|\mathbf{x}|^{-1}) \ \text{ as } |\mathbf{x}| \to \infty, \tag{4.21}$$

$$\mathbf{W}(\mathbf{x},\mathbf{h}) = O(|\mathbf{x}|^{-1}), \quad P^d(\mathbf{x},\mathbf{h}) = O(|\mathbf{x}|^{-2}) \ \text{ as } |\mathbf{x}| \to \infty. \tag{4.22}$$

4.3. Complementary integral operators

For $\lambda \in (0,\alpha)$, let $\mathcal{V}_{\chi^2} : C^{\lambda}(\mathcal{C}) \to C^{1,\lambda}(\mathcal{C})$ and $\mathbf{K}_{\chi^2} : C^{1,\lambda}(\mathcal{C}) \to C^{1,\lambda}(\mathcal{C})$ be the single- and double-layer integral operators for the Brinkman system, i.e.,

$$\mathcal{V}_{\chi^2}\mathbf{g} = \mathbf{V}_{\chi^2}(\cdot,\mathbf{g}), \quad \mathbf{K}_{\chi^2}\mathbf{h} = \mathbf{W}^*_{\chi^2}(\cdot,\mathbf{h}), \quad \forall\ \mathbf{g} \in C^{\lambda}(\mathcal{C}),\ \mathbf{h} \in C^{1,\lambda}(\mathcal{C}),$$

Similarly, $\mathcal{V} : C^{\lambda}(\mathcal{C}) \to C^{1,\lambda}(\mathcal{C})$ and $\mathbf{K} : C^{1,\lambda}(\mathcal{C}) \to C^{1,\lambda}(\mathcal{C})$ are the corresponding integral operators for the Stokes system.

Also, let $\mathbf{D}_{\chi^2} : C^{1,\lambda}(\mathcal{C}) \to C^{\lambda}(\mathcal{C})$ be the operator given in (4.18), i.e.,

$$\left(\mathbf{D}_{\chi^2}\mathbf{h}\right)_j(\mathbf{x}) = \text{p.f.} \int_{\mathcal{C}} D^{\chi^2}_{j\ell}(\mathbf{x},\mathbf{y}) h_\ell(\mathbf{y}) d\mathcal{C}(\mathbf{y}), \tag{4.23}$$

where

$$D^{\chi^2}_{j\ell}(\mathbf{x},\mathbf{y}) = -\Lambda^{\chi^2}_{\ell k}(\mathbf{x}-\mathbf{y}) n_k(\mathbf{y}) n_j(\mathbf{x})$$
$$+\Big(\frac{\partial}{\partial x_j} S^{\chi^2}_{\ell i k}(\mathbf{y}-\mathbf{x}) + \frac{\partial}{\partial x_i} S^{\chi^2}_{\ell j k}(\mathbf{y}-\mathbf{x})\Big) n_i(\mathbf{x}) n_k(\mathbf{y}).$$

The corresponding operator for the Stokes system is denoted by $\mathbf{D}_0$. The operators $\mathbf{D}_{\chi^2}$ and $\mathbf{D}_0$ belong to the class of hypersingular operators.

Let us introduce the notations

$$\Lambda^c_{\ell k}(\mathbf{x}-\mathbf{y}) = \Lambda^{\chi^2}_{\ell k}(\mathbf{x}-\mathbf{y}) - \Lambda_{\ell k}(\mathbf{x}-\mathbf{y}), \quad K^c_{jk}(\mathbf{y},\mathbf{x}) = K^{\chi^2}_{jk}(\mathbf{y},\mathbf{x}) - K_{jk}(\mathbf{y},\mathbf{x}), \tag{4.24}$$

in view of which we are now able to define the complementary integral operators for the Stokes-Brinkman-coupled system.

The complementary single- and double-layer operators $\mathcal{V}_{\chi^2,0} : C^{\lambda}(\mathcal{C}) \to C^{1,\lambda}(\mathcal{C})$ and $\mathbf{K}_{\chi^2,0} : C^{1,\lambda}(\mathcal{C}) \to C^{1,\lambda}(\mathcal{C})$ are given by

$$\mathcal{V}_{\chi^2,0} = \mathcal{V}_{\chi^2} - \mathcal{V}, \quad \mathbf{K}_{\chi^2,0} = \mathbf{K}_{\chi^2} - \mathbf{K}, \tag{4.25}$$

and the adjoint of the complementary double-layer operator $\mathbf{K}'_{\chi^2,0} : C^{\lambda}(\mathcal{C}) \to C^{\lambda}(\mathcal{C})$ has the expression $\mathbf{K}'_{\chi^2,0} = \mathbf{K}'_{\chi^2} - \mathbf{K}'$, where $\mathbf{K}'_{\chi^2}$ is the adjoint operators of $\mathbf{K}_{\chi^2}$.

In addition, the complementary hypersingular operator

$$\mathbf{D}_{\chi^2,0} : C^{1,\lambda}(\mathcal{C}) \to C^{\lambda}(\mathcal{C})$$

is given by $\mathbf{D}_{\chi^2,0} = \mathbf{D}_{\chi^2} - \mathbf{D}_0$.

We have following compactness result whose proof can be consulted in [4]:

Theorem 4.2. *If $\mathcal{C}$ is a closed Lyapunov curve in $\mathbb{R}^2$, i.e., $\mathcal{C} \in C^{1,\alpha}$, $\alpha \in (0,1]$, and $\lambda \in (0,\alpha)$, then the complementary boundary integral operators*

$$\mathcal{V}_{\chi^2,0} : C^{\lambda}(\mathcal{C}) \to C^{1,\lambda}(\mathcal{C}), \quad \mathbf{K}_{\chi^2,0} : C^{1,\lambda}(\mathcal{C}) \to C^{1,\lambda}(\mathcal{C}),$$
$$\mathbf{K}'_{\chi^2,0} : C^{\lambda}(\mathcal{C}) \to C^{\lambda}(\mathcal{C}), \quad \mathbf{D}_{\chi^2,0} : C^{1,\lambda}(\mathcal{C}) \to C^{\lambda}(\mathcal{C})$$

are compact.

In addition, if $\mathbf{h} \in C^{1,\lambda}(\mathcal{C})$, then $\mathbf{T}^{+}(\mathbf{W}_{\chi^2,0}(\mathbf{h}))$ and $\mathbf{T}^{-}(\mathbf{W}_{\chi^2,0}(\mathbf{h}))$ exist everywhere on $\mathcal{C}$ and they are equal.

5. The boundary integral formulation of the problem

In order to prove that the boundary value problem (2.1)-(2.6) has a unique classical solution, we consider the following boundary integral representations:

$$\begin{aligned} v_k^e(\mathbf{x}) = U_k^\infty + \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} K_{jk}^{\chi^2}(\mathbf{y},\mathbf{x})\phi_j(\mathbf{y})d\Gamma_l(\mathbf{y}) \\ + \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} \mathcal{G}_{kj}^{\chi^2}(\mathbf{x}-\mathbf{y})\Big(h_j(\mathbf{y}) \\ - \frac{1}{|\Gamma_l|}\int_{\Gamma_l} h_j(\mathbf{z})d\Gamma_l(\mathbf{z})\Big)d\Gamma_l(\mathbf{y}), \quad \mathbf{x}\in D_e \end{aligned} \tag{5.1}$$

$$\begin{aligned} p^e(\mathbf{x}) = P^\infty(\mathbf{x}) + \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} \Lambda_{jk}^{\chi^2}(\mathbf{x}-\mathbf{y})n_k(\mathbf{y})\phi_j(\mathbf{y})d\Gamma_l(\mathbf{y}) \\ + \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} \Pi_j^{\chi^2}(\mathbf{x}-\mathbf{y})\Big(h_j(\mathbf{y}) \\ - \frac{1}{|\Gamma_l|}\int_{\Gamma_l} h_j(\mathbf{z})d\Gamma_l(\mathbf{z})\Big)d\Gamma_l(\mathbf{y}),\ \mathbf{x}\in D_e, \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} v_k^i(\mathbf{x}) = \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} K_{jk}(\mathbf{y},\mathbf{x})\phi_j(\mathbf{y})d\Gamma_l(\mathbf{y}) \\ + \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} \mathcal{G}_{kj}(\mathbf{x}-\mathbf{y})\Big(h_j(\mathbf{y}) \\ - \frac{1}{|\Gamma_l|}\int_{\Gamma_l} h_j(\mathbf{z})d\Gamma_l(\mathbf{z})\Big)d\Gamma_l(\mathbf{y}) \\ - \int_{\Gamma_m} h_j(\mathbf{y})d\Gamma_m(\mathbf{y}), \quad \mathbf{x}\in D_m, m=1,\dots,N \end{aligned} \tag{5.3}$$

$$\begin{aligned} p^i(\mathbf{x}) = \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} \Lambda_{jk}(\mathbf{x}-\mathbf{y})n_k(\mathbf{y})\phi_j(\mathbf{y})d\Gamma_l(\mathbf{y}) \\ + \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} \Pi_j(\mathbf{x}-\mathbf{y})\Big(h_j(\mathbf{y}) \\ - \frac{1}{|\Gamma_l|}\int_{\Gamma_l} h_j(\mathbf{z})d\Gamma_l(\mathbf{z})\Big)d\Gamma_l(\mathbf{y}), \quad \mathbf{x}\in D_0, \end{aligned} \tag{5.4}$$

where $\mathbf{\Phi} = (\phi_1,\phi_2) \in C^{1,\lambda}(\Gamma)$ and $\mathbf{h} = (h_1,h_2) \in C^{\lambda}(\Gamma)$ are unknown densities, $\lambda \in (1,\alpha)$, and $|\Gamma_k| = \int_{\Gamma_k} d\Gamma_k$ is the length of Γ_k, $k=1,\dots,N$.

Let us observe that the boundary integral representations (5.1)-(5.4) satisfy the system of equations (2.1)-(2.4), as well as far field conditions (2.5)-(2.6).

Now, imposing the transmission condition (2.5), we obtain the equations

$$\begin{aligned}\phi_k(\mathbf{x}_0) &+ \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} K^c_{jk}(\mathbf{y},\mathbf{x}_0)\phi_j(\mathbf{y})d\Gamma_l(\mathbf{y}) \\ &+ \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} \mathcal{G}^c_{kj}(\mathbf{x}_0-\mathbf{y})\Big(h_j(\mathbf{y}) - \frac{1}{|\Gamma_m|}\int_{\Gamma_m} h_j(\mathbf{y})d\Gamma_m(\mathbf{y})\Big)\, d\Gamma_l(\mathbf{y}) \\ &+ \int_{\Gamma_m} h_j(\mathbf{y})d\Gamma_m(\mathbf{y}) = -U^\infty_k,\ \mathbf{x}_0 \in \Gamma_m,\ m = 1,\dots,N. \qquad (5.5)\end{aligned}$$

Taking into account the second of the boundary conditions (2.5), we obtain the boundary integral equations

$$\begin{aligned}-h_k(\mathbf{x}_0) &+ \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} K^c_{kj}(\mathbf{x}_0,\mathbf{y})\Big(h_j(\mathbf{y}) - \frac{1}{|\Gamma_l|}\int_{\Gamma_l} h_j(\mathbf{z})d\Gamma_l(\mathbf{z})\Big)d\Gamma_l(\mathbf{y}) \\ &+ T_{kj}(\mathbf{W}^c(\mathbf{\Phi}))(\mathbf{x}_0)n_j(\mathbf{x}_0) = -t^\infty_k(\mathbf{x}_0),\ \mathbf{x}_0 \in \Gamma_m,\ m = 1,\dots,N, \qquad (5.6)\end{aligned}$$

where t^∞_k are the components of the stress field associated to the velocity field $\mathbf{U}^\infty$, i.e.,

$$t^\infty_k(\mathbf{x}) = p^\infty n_k(\mathbf{x}).$$

We mention that the integrals on Γ_m who appears in (5.5) and (5.6) are understood in the sense of principal value.

Therefore, the boundary value problem (2.1)-(2.6) reduces to the system of boundary integral equations (5.5) and (5.6). In view of Theorem 4.2 it follows that all operators that appear in the boundary integral equations (5.5) and (5.6) are compact, as mappings into one of the spaces $C^{1,\lambda}(\Gamma)$, $C^\lambda(\Gamma)$. Thus, these equations are Fredholm integral equations of the second kind with the unknowns $(\mathbf{\Phi},\mathbf{h}) \in C^{1,\lambda}(\Gamma)\times C^\lambda(\Gamma)$.

We have the following existence and uniqueness result (see also [2]):

Theorem 5.1. *Let Γ_k be closed Lyapunov curves of class $C^{1,\alpha}$ in $\mathbb{R}^2$, $\alpha \in (0,1]$, $k = 1,\dots,N$, $\Gamma = \cup_{k=1}^N \Gamma_k$, and let $\lambda \in (0,\alpha)$. Then the system of Fredholm integral equations of the second kind* (5.5) *and* (5.6) *has a unique solution $(\mathbf{\Phi},\mathbf{h}) \in C^{1,\lambda}(\Gamma)\times C^\lambda(\Gamma)$. In addition, the boundary integral representations* (5.1)-(5.4)*, obtained with the densities $\mathbf{\Phi}$ and $\mathbf{h}$, determine the unique classical solution $((\mathbf{v}^e,p^e),(\mathbf{v}^i,p^i)) \in ((C^2(D_e)\cap C^1(\overline{D}_e))\times(C^1(D_e)\cap C^0(\overline{D}_e)))\times((C^2(D_0)\cap C^1(\overline{D}_0))\times(C^1(D_0)\cap C^0(\overline{D}_0)))$ to the boundary value problem consisting of the equations* (2.1)-(2.4) *and the boundary and far field conditions* (2.5)-(2.6).

Proof. Let us consider the following homogeneous system of integral equations

$$\phi_k^0(\mathbf{x}_0) + \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} K_{jk}^c(\mathbf{y},\mathbf{x}_0)\phi_j^0(\mathbf{y})d\Gamma_l(\mathbf{y})$$
$$+\frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_k} \mathcal{G}_{kj}^c(\mathbf{x}_0-\mathbf{y})\Big(h_j^0(\mathbf{y}) - \frac{1}{|\Gamma_l|}\int_{\Gamma_l} h_j^0(\mathbf{z})d\Gamma_l(\mathbf{z})\Big)d\Gamma_l(\mathbf{y})$$
$$+\int_{\Gamma_m} h_j^0(\mathbf{y})d\Gamma_m(\mathbf{y}) = 0,\ \mathbf{x}_0 \in \Gamma_m,\ m = 1,\dots,N, \tag{5.7}$$

$$-h_k^0(\mathbf{x}_0) + \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} K_{kj}^c(\mathbf{x}_0,\mathbf{y})\Big(h_j^0(\mathbf{y}) - \frac{1}{|\Gamma_l|}\int_{\Gamma_l} h_j^0(\mathbf{z})d\Gamma_l(\mathbf{z})\Big)d\Gamma_l(\mathbf{y})$$
$$+T_{kj}(\mathbf{W}^c(\mathbf{\Phi^0}))(\mathbf{x}_0)n_j(\mathbf{x}_0) = 0,\ \mathbf{x}_0 \in \Gamma_m,\ m = 1,\dots,N. \tag{5.8}$$

Also let $(\mathbf{\Phi}^0, \mathbf{h}^0) \in C^{1,\lambda}(\Gamma) \times C^{\lambda}(\Gamma)$ be an arbitrary solution to this system, and let $(\mathbf{u}^e, q^e)$ and $(\mathbf{u}^i, q^i)$ be the fields given by the following boundary integral representations:

$$u_k^e(\mathbf{x}) = \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} K_{jk}^{\chi^2}(\mathbf{y},\mathbf{x})\phi_j^0(\mathbf{y})d\Gamma_l(\mathbf{y}) + \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} \mathcal{G}_{kj}^{\chi^2}(\mathbf{x}-\mathbf{y})\Big(h_j^0(\mathbf{y})$$
$$-\frac{1}{|\Gamma_l|}\int_{\Gamma_l} h_j^0(\mathbf{z})d\Gamma_l(\mathbf{z})\Big)d\Gamma_l(\mathbf{y}),\quad \mathbf{x}\in\mathbb{R}^2\setminus\Gamma_m, m=1,\dots,N \tag{5.9}$$

$$q^e(\mathbf{x}) = \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} \Lambda_{jk}^{\chi^2}(\mathbf{x}-\mathbf{y})\phi_j^0(\mathbf{y})d\Gamma_l(\mathbf{y})$$
$$+\frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} \Pi_j^{\chi^2}(\mathbf{x}-\mathbf{y})\Big(h_j^0(\mathbf{y})$$
$$-\frac{1}{|\Gamma_l|}\int_{\Gamma_l} h_j^0(\mathbf{z})d\Gamma_l(\mathbf{z})\Big)d\Gamma_l(\mathbf{y}),\quad \mathbf{x}\in\mathbb{R}^2\setminus\Gamma_m, m=1,\dots,N \tag{5.10}$$

and

$$u_k^i(\mathbf{x}) = \frac{1}{4\pi}\int_{\Gamma_m} K_{jk}(\mathbf{y},\mathbf{x})\phi_j^0(\mathbf{y})d\Gamma_m(\mathbf{y}) + \frac{1}{4\pi}\int_{\Gamma_m} \mathcal{G}_{kj}(\mathbf{x}-\mathbf{y})\Big(h_j^0(\mathbf{y})$$
$$-\frac{1}{|\Gamma_m|}\int_{\Gamma_m} h_j^0(\mathbf{z})d\Gamma_m(\mathbf{z})\Big)d\Gamma_m(\mathbf{y})$$
$$-\int_{\Gamma_m} h_j^0(\mathbf{y})d\Gamma_m(\mathbf{y}),\quad \mathbf{x}\in\mathbb{R}^2\setminus\Gamma_m, m=1,\dots,N \tag{5.11}$$

$$q^i(\mathbf{x}) = \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l}\Lambda_{jk}(\mathbf{x}-\mathbf{y})n_k(\mathbf{y})\phi_j^0(\mathbf{y})d\Gamma_l(\mathbf{y})$$
$$+\frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l}\Pi_j(\mathbf{x}-\mathbf{y})\Big(h_j^0(\mathbf{y})$$
$$-\frac{1}{|\Gamma_l|}\int_{\Gamma_l}h_j^0(\mathbf{z})d\Gamma_l(\mathbf{z})\Big)d\Gamma_l(\mathbf{y}),\quad \mathbf{x}\in\mathbb{R}^2\setminus\Gamma_m, m=1,\ldots,N. \quad (5.12)$$

Because the pairs $(\mathbf{V}_{\chi^2}(\cdot,\mathbf{g}), P^s_{\chi^2}(\cdot,\mathbf{g}))$ and $(\mathbf{W}_{\chi^2}(\cdot,\mathbf{h}), P^d_{\chi^2}(\cdot,\mathbf{h}))$ satisfy the Brinkman system in both domains D_0 and D_e, respectively, and the pairs $(\mathbf{V}(\cdot,\mathbf{g}), P^s(\cdot,\mathbf{g}))$ and $(\mathbf{W}(\cdot,\mathbf{h}), P^d(\cdot,\mathbf{h}))$ satisfy the Stokes system in both domains D_0 and D_e, respectively, we obtain that:

$$\nabla\cdot\mathbf{u}^e = 0,\quad -\nabla q^e + (\nabla^2-\chi^2)\mathbf{u}^e = \mathbf{0} \text{ in } \mathbb{R}^2\setminus\Gamma, \quad (5.13)$$
$$\nabla\cdot\mathbf{u}^i = 0,\quad -\nabla q^i + \nabla^2\mathbf{u}^e = \mathbf{0} \text{ in } \mathbb{R}^2\setminus\Gamma. \quad (5.14)$$

Taking into account the relations (4.21) we have that:

$$(|\mathbf{u}^e||\nabla\mathbf{u}^e|)(\mathbf{x}) = o(|\mathbf{x}|^{-1}),\quad (|\mathbf{u}^e||q^e|)(\mathbf{x}) = o(|\mathbf{x}|^{-1}) \text{ as } |\mathbf{x}|\to\infty. \quad (5.15)$$

Therefore, the fields $\mathbf{u}^e$ and q^e satisfy the identity

$$\int_{D_e}(2E_{jk}(\mathbf{u}^e)E_{jk}(\mathbf{u}^e)+\chi^2|\mathbf{u}^e|^2)d\mathbf{x} = -\sum_{l=1}^{N}\int_{\Gamma_l}u_k^{e+}(\mathbf{x})t_k^+(\mathbf{u}^e)(\mathbf{x})d\Gamma_l(\mathbf{x}), \quad (5.16)$$

where $t_k^{\pm}(\mathbf{u}^e) = T_{kj}^{\pm}(\mathbf{u}^e)n_j$, and

$$T_{kj}(\mathbf{u}^e) = -q^e\delta_{kj} + 2E_{kj}(\mathbf{u}^e),\quad E_{jk}(\mathbf{u}^e) = \frac{1}{2}\left(\frac{\partial u_j^e}{\partial x_k}+\frac{\partial u_k^e}{\partial x_j}\right).$$

Similarly, the fields $\mathbf{u}^i$ and q^i satisfy the identity (see e.g. [1], p. 15)

$$2\int_{D_0}E_{jk}(\mathbf{u}^i)E_{jk}(\mathbf{u}^i)d\mathbf{x} = \sum_{l=1}^{N}\int_{\Gamma_l}u_k^{i-}(\mathbf{x})t_k^-(\mathbf{u}^i)(\mathbf{x})d\Gamma_l(\mathbf{x}), \quad (5.17)$$

where $t_k^{\pm}(\mathbf{u}^i) = T_{kj}^{\pm}(\mathbf{u}^i)n_j$, and

$$T_{kj}(\mathbf{u}^i) = -q^i\delta_{kj} + 2E_{kj}(\mathbf{u}^i),\quad E_{jk}(\mathbf{u}^i) = \frac{1}{2}\left(\frac{\partial u_j^i}{\partial x_k}+\frac{\partial u_k^i}{\partial x_j}\right).$$

Now, taking into account the formulas (4.15)-(4.17), we obtain the properties

$$u_k^{e+} = u_k^{i-},\quad t_k^+(\mathbf{u}^e) = t_k^-(\mathbf{u}^i) \text{ on } \Gamma_l, l=1\ldots,N \quad (5.18)$$

which yield the equality

$$\sum_{l=1}^{N}\int_{\Gamma_l}u_k^{e+}(\mathbf{x})t_k^+(\mathbf{u}^e)(\mathbf{x})d\Gamma_l = \sum_{k=1}^{N}\int_{\Gamma_l}u_k^{i-}(\mathbf{x})t_k^-(\mathbf{u}^i)(\mathbf{x})d\Gamma_l. \quad (5.19)$$

From the properties (5.16), (5.17) and (5.19) we deduce that

$$\int_{D_e}(2E_{jk}(\mathbf{u}^e)E_{jk}(\mathbf{u}^e)+\chi^2|\mathbf{u}^e|^2)d\mathbf{x} = -2\int_{D_0}E_{jk}(\mathbf{u}^i)E_{jk}(\mathbf{u}^i)d\mathbf{x}, \quad (5.20)$$

and hence

$$\mathbf{u}^e = \mathbf{0} \text{ in } D_e, \tag{5.21}$$

$$E_{jk}(\mathbf{u}^i) = \mathbf{0} \text{ in } D_m, j,k = 1,2, \; m = 1,\dots,N. \tag{5.22}$$

Using Killing's theorem, we deduce that there exists some real constants a_0^k and b_0^k such that

$$\mathbf{u}^i = a_0^m + b_0^m \times \mathbf{x} \text{ in } \; D_m, \quad m = 1,\dots,N. \tag{5.23}$$

But,

$$0 = u^{e+} = u^{i-} \text{ on } \Gamma_m \, m = 1,\dots,N$$

thus we obtain that

$$a_0^m = b_0^m = 0.$$

Then , we have:

$$\mathbf{u}^i = \mathbf{0} \text{ in } \; D_m, m = 1,\dots,N. \tag{5.24}$$

In addition, in view of the second of equations (5.13) and from the fact that the pressure field q^e vanishes at infinity, we obtain

$$q^e = 0 \text{ in } D_e. \tag{5.25}$$

Similarly, we deduce that $q^i = c_m^0 \in \mathbb{R}$ in D_m. On the other hand, from the relations (5.18), (5.21) and (5.25) we get

$$t_k^-(\mathbf{u}^i) = t_k^+(\mathbf{u}^e) = 0 \text{ on } \; \Gamma_m, \quad m = 1,\dots,N \tag{5.26}$$

and hence the constant c_m^0 must be equal to zero, i.e.,

$$\mathbf{u}^i = \mathbf{0}, \quad q^i = 0 \text{ in } D_0. \tag{5.27}$$

Now, using the jump formula

$$\mathbf{u}^{e+} - \mathbf{u}^{e-} = \mathbf{\Phi}^{\mathbf{0}} \text{ on } \Gamma_m, \quad m = 1,\dots,N$$

(see the properties (4.15) and (4.16)) as well as the result (5.21), we deduce that

$$\mathbf{u}^{e-} = -\mathbf{\Phi}^0 \text{ on } \Gamma_m, \quad m = 1,\dots,N. \tag{5.28}$$

Similarly, from the jump formula

$$\mathbf{u}^{i+} - \mathbf{u}^{i-} = \mathbf{\Phi}^0 \text{ on } \Gamma_m, \quad m = 1,\dots,N$$

as well as the result (5.27), we find that

$$\mathbf{u}^{i+} = \mathbf{\Phi}^0 \text{ on } \Gamma_m, \quad m = 1,\dots,N. \tag{5.29}$$

On the other hand, from the relations (4.17) we deduce that the boundary traction due to the fields $\mathbf{u}^e$ and q^e has a jump across every curve Γ_k given by the formula

$$\mathbf{t}^+(\mathbf{u}^e) - \mathbf{t}^-(\mathbf{u}^e) = -\Big(\mathbf{h}^0 - \frac{1}{|\Gamma_m|}\int_{\Gamma_m} \mathbf{h}^{\mathbf{0}} d\Gamma_m\Big) \text{ on } \Gamma_m, \quad m = 1,\dots,N. \tag{5.30}$$

But $\mathbf{t}^+(\mathbf{u}^e) = \mathbf{0}$ on Γ_k and hence

$$\mathbf{t}^-(\mathbf{u}^e) = \mathbf{h}^0 - \frac{1}{|\Gamma_m|}\int_{\Gamma_m} \mathbf{h}^{\mathbf{0}} d\Gamma_m \text{ on } \Gamma_m, \quad m = 1,\dots,N. \tag{5.31}$$

With similar kind of arguments as before, we get the relation

$$\mathbf{t}^+(\mathbf{u}^i) = -\Big(\mathbf{h}^0 - \int_{\Gamma_m} \mathbf{h}^0 d\Gamma_m\Big) \text{ on } \Gamma_m, \quad m = 1, \ldots, N. \tag{5.32}$$

In addition, the fields $(\mathbf{u}^e, q^e)$ satisfy the identity

$$\int_{D_0} (2E_{jk}(\mathbf{u}^e)E_{jk}(\mathbf{u}^e) + \chi^2|\mathbf{u}^e|^2)d\mathbf{x} = \sum_{l=1}^{N} \int_{\Gamma_l} u_k^{e-}(\mathbf{x})t_k^-(\mathbf{u}^e)(\mathbf{x})d\Gamma_l(\mathbf{x}) \tag{5.33}$$

and, in view of the properties (5.28) and (5.31), this identity takes the form

$$\int_{D_0} (2E_{jk}(\mathbf{u}^e)E_{jk}(\mathbf{u}^e) + \chi^2|\mathbf{u}^e|^2)d\mathbf{x} = -\sum_{l=1}^{N} \int_{\Gamma_k} \mathbf{\Phi}^0 \cdot \Big(\mathbf{h}^0 - \frac{1}{|\Gamma_l|}\int_{\Gamma_l} \mathbf{h^0} d\Gamma_l\Big) d\Gamma_l. \tag{5.34}$$

Since

$$\int_{\Gamma_l} \Big(\mathbf{h}^0 - \frac{1}{\Gamma_l}\int_{\Gamma_l} \mathbf{h}^0 d\Gamma_l\Big) d\Gamma_l = 0, \tag{5.35}$$

we deduce that

$$\mathbf{V}\Big(\mathbf{x}, \mathbf{h}^0 - \frac{1}{|\Gamma_l|}\int_{\Gamma_l} \mathbf{h^0} d\Gamma_l\Big) = \mathcal{O}(|\mathbf{x}|^{-1}) \text{ as } |\mathbf{x}| \to \infty, \tag{5.36}$$

the fields $\mathbf{u}^i$ and q^i behave at infinity as follows (see also the relations (4.21)):

$$\nabla^s \mathbf{u}^i(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-1-s}), \quad q^i(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-1}) \text{ as } |\mathbf{x}| \to \infty,\ s = 0, 1, \tag{5.37}$$

and hence they satisfy the far field conditions (2.6). Consequently, we get the following identity:

$$2\int_{D_e} E_{jk}(\mathbf{u}^i)E_{jk}(\mathbf{u}^i)d\mathbf{x} = -\sum_{l=1}^{N} \int_{\Gamma_l} u_k^{i+}(\mathbf{x})t_k^+(\mathbf{u}^i)(\mathbf{x})d\Gamma_l(\mathbf{x}), \tag{5.38}$$

which, in view of the properties (5.29) and (5.32), becomes

$$2\int_{D_e} E_{jk}(\mathbf{u}^i)E_{jk}(\mathbf{u}^i)d\mathbf{x} = \sum_{l=1}^{N} \int_{\Gamma_l} \mathbf{\Phi}^0 \cdot \Big(\mathbf{h}^0 - \frac{1}{|\Gamma_l|}\int_{\Gamma_l} \mathbf{h}^0 d\Gamma_l\Big) d\Gamma_l. \tag{5.39}$$

Therefore, from the identities (5.34) and (5.39) we obtain that

$$\mathbf{u}^e = \mathbf{0} \text{ in } D_0 \tag{5.40}$$

and

$$\mathbf{u}^i = \mathbf{0} \ \text{ in } \ D_e. \tag{5.41}$$

The property (5.41), the equation $-\nabla q^i + (\nabla^2 - \chi^2)\mathbf{u}^i = \mathbf{0}$ in D_e, and the fact that the pressure field q^i vanishes at infinity lead to the additional result

$$q^i = 0 \ \text{ in } \ D_e. \tag{5.42}$$

From the relation (5.40), we get that

$$\mathbf{u}^{e-} = \mathbf{0} \ \text{ on } \ \Gamma_l, \quad l = 1, \ldots, N.$$

Using the above relation and (5.28), we obtain that:

$$\mathbf{\Phi}^0 = \mathbf{0} \ \text{ on } \ \Gamma_l, \quad l = 1, \ldots, N. \tag{5.43}$$

In addition, according to the relations (5.32), (5.41) and (5.42) we find that

$$\mathbf{t}^{+}(\mathbf{u}^{i}) = \mathbf{0} \quad \text{on} \quad \Gamma_l, \quad l = 1, \ldots, N, \tag{5.44}$$

i.e.,

$$\mathbf{h} = \frac{1}{|\Gamma_l|}\int_{\Gamma_l} \mathbf{h} d\Gamma_l := \mathbf{c}_l \in \mathbb{R}^2 \quad \text{on} \quad \Gamma_l, \quad l = 1, \ldots, N. \tag{5.45}$$

So, we obtain that

$$\mathbf{0} = \mathbf{u}^{i} = -\int_{\Gamma_l} \mathbf{h} d\Gamma_l in D_l, \quad l = 1, \ldots, N \tag{5.46}$$

and hence

$$\int_{\Gamma_l} \mathbf{h} d\Gamma_l = 0, \quad l = 1, \ldots, N. \tag{5.47}$$

Finally, from the relations(5.45) and (5.46) we find that

$$\mathbf{h}^{0} = \mathbf{0} \quad \text{on} \quad \Gamma_l, \quad l = 1, \ldots, N. \tag{5.48}$$

The relations (5.43) and (5.48) shows that the homogeneous system of equations (5.7) and (5.8) has only the trivial solution in the space $C^{1,\lambda}(\Gamma) \times C^{\lambda}(\Gamma)$. Consequently, in view of Fedholm's alternative [5] we deduce that the non-homogeneous system of Fredholm integral equations of the second kind (5.5) and (5.6) has a unique solution$(\mathbf{\Phi}, \mathbf{h}) \in C^{1,\lambda}(\Gamma) \times C^{\lambda}(\Gamma)$, as desired. □

Acknowledgements. The author is supported by the UEFISCSU - CNCSIS, Grant, PN-II-ID-525/2007.

References

[1] Kohr, M., Pop I., *Viscous Incompressible Flow for Low Reynolds Numbers*, WIT Press, Southampton, UK, 2004.

[2] Kohr, M., Raja Sekhar, G. P., *Existence and uniqueness result for two-dimensional porous media flows with porous inclusions based on Brinkman equation*, Engineering Analysis with Boundary Elements, **31**(2007), 604-613.

[3] Kohr, M., Raja Sekhar, G. P., *Existence and uniqueness result for the problem of viscous flow in a granular material with a void*, Quarterly of Applied Mathematics, **65**(2007), no. 4, 683-704.

[4] Kohr, M., Wendland, W. L., Raja Sekhar, G. P., *Boundary integral equations for two-dimensional low Reynolds number flow past a porous body*, Mathematical Methods in the Applied Sciences, Published Online, Sep 26 2008. DOI: 10.1002/mma.1074.

[5] Hsiao, G., Wendland, W.L., *Boundary Integral equations*, Springer, Heidelberg, 2008.

Elena-Maria Ului
Faculty of Mathematics and Computer Science
Babeş-Bolyai University,
1 M. Kogălniceanu Str.
400084 Cluj-Napoca, Romania
e-mail: `uluita2001@yahoo.com`