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On uniform exponential stability of backwards evolutionary processes

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Abstract. The exponential stability of a special class of evolution families is analyzed. Extensions of the well-known theorems due to Datko and Barbashin are obtained, in both continuous and discrete-time.

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1. Introduction

In 1967 E. A. Barbashin [1, Th. 5.1] obtained a stability result for exponentially bounded evolution families generated by differential systems in Banach spaces, a result that remains true in the case of evolution families with exponential growth. In 1970 R. Datko [4] proved that the C_0-semigroup $\{T_t\}_{t\geq 0}$ is exponentially stable if and only if its trajectories $(T(\cdot)x)$ are in L^2 for all x in X. This result was generalized by A. Pazy [12], who proved that the exponential stability property is equivalent with $T(\cdot)x \in L^p$, for $1 \leq p < \infty$ and for all x in X, where X is a Banach space.

Later, a well-known Datko result from 1972 [5] states that an exponentially bounded, strongly continuous evolution family $\mathbf{U} = \{U(t,t_0)\}_{t\geq t_0\geq 0}$ with exponential growth is exponentially stable if and only if there exist $k, p > 0$ such that

$$\left(\int_t^{\infty} ||U(\tau,t)x||^p d\tau\right)^{\frac{1}{p}} \leq k||x||, \text{ for all } t \geq 0, \text{ and } x \in X.$$

This result was extended by J.L. Daleckij and M.G. Krein [3] for evolutionary processes generated by differential systems in Banach spaces and instead of R. Datko's method, it has been used a characterization theorem for the exponential stability of differential systems [3, Th. 6.1, pg 132]. S. Rolewicz [13] noticed that the theorem used by J.L. Daleckij and M.G. Krein in [3] remains true in the case of evolutionary processes with exponential growth (without stating the proof, though). This theorem, along with the Baire Cathegory Principle allowed S. Rolewicz [13] to extend Datko's result from 1972 to the fact that $\{U(t,t_0)\}_{t\geq t_0\geq 0}$ is exponentially stable if and only if there exists $N:(0,\infty)\times(0,\infty)\to\mathbb{R}_+$ with the property that $N(\alpha,u)$ is continuous and increasing for all α, and $N(\alpha,u)$ is increasing for all u, $N(\alpha,0)=0$ for all $\alpha>0$, $N(\alpha,u)>0$ for all $u>0$ and for all $x\in X$ there exists $\alpha(x)>0$ such that

$$\sup_{t\geq 0}\int_t^{\infty} N(\alpha(x),||U(\tau,t)||)d\tau<\infty.$$

Another extension of the result due to Datko [4] and Pazy [12] was obtained by W. Littman [7] in 1989. V. Pata [11] came with a new proof and a generalization of the result due to Datko [5] for the case of strongly continuous semigroups of bounded linear operators.

The classical ideas of J. L. Massera and J. J. Schäffer ([8],[9]) on exponential stability and other asymptotic properties of the solutions of differential equations have also been developed in the last years. Other results for the stability of nonlinear evolution families were obtained by A. Ichikawa [6] and in 2007, a strong variant of a result due to E. A. Barbashin [1] was obtained by C. Buşe, M. Megan, M. S. Prajea and P. Preda [2] on the dual space of the Banach space X. Some Datko [5] type results for the asymptotic behavior of skew-evolution semiflows in Banach spaces were given by M. Megan and C. Stoica [10] in 2008.

The purpose of the present paper is to give a characterization for the exponential stability of a special class of evolution families, called the backwards evolution families, and thus to reformulate the result due to E. A. Barbashin [1].

2. Preliminaries

Let us consider X a Banach space, $\mathcal{B}(X)$ the Banach algebra of all linear and bounded operators acting on X and $\Delta=\{(t,t_0)\in\mathbb{R}^2: t\geq t_0\geq 0\}$. We denote the norm of vectors on X and operators on $\mathcal{B}(X)$ by $||\cdot||$.

Definition 2.1. *A family of linear and bounded operators*

$$\mathbf{\Phi}=\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}:\Delta\to\mathcal{B}(X)$$

is called a backwards evolutionary process if the following properties hold:

i) $\Phi(t,t)=I$, *for all* $t\geq 0$;
ii) $\Phi(\tau,t_0)\ \Phi(t,\tau)=\Phi(t,t_0)$, *for all* $t\geq\tau\geq t_0\geq 0$;
iii) $\Phi(\cdot,t_0)x:[t_0,\infty)\to X$ *is continuous for all* $t_0\geq 0$ *and* $x\in X$
$\Phi(t,\cdot)x:[0,t]\to X$ *is continuous for all* $t\geq 0$ *and* $x\in X$;

iv) there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that:

$$||\Phi(t,t_0)|| \leq Me^{\omega(t-t_0)},$$

for all $t \geq t_0 \geq 0$.

Example 2.2. *Take $X = \mathbb{R}$ and the equation:*

$$(A) \qquad \dot{x}(t) = A(t)x(t), \ t \geq 0.$$

We consider the Cauchy problem associated:

$$(B) \qquad \begin{cases} \dot{U}(t) = A(t)U(t) \\ U(0) = I. \end{cases}$$

where $A \in \mathcal{M}(2,\mathbb{R})$ and $\mathcal{M}(2,\mathbb{R})$ denotes the set of all 2-by-2 real matrices and $t \geq 0$.

The unique solution of the Cauchy problem (B) will be denoted by $U(t)$ and $\Phi(t,t_0) = U^{*-1}(t_0)U^*(t)$ represents the backwards evolutionary process generated by the equation (A).

Example 2.3. *Let $X = \mathbb{R}$. Then*

$$\Phi(t,t_0) = \frac{\sin t + 1}{\sin t_0 + 1}$$

defines a backwards evolutionary process.

Example 2.4. *Let $X = \mathbb{R}$. Then*

$$\Phi(t,t_0) = \frac{t^2 + 1}{t_0^2 + 1}$$

defines a backwards evolutionary process.

Definition 2.5. *Let $\Phi = \{\Phi(t,t_0)\}_{t \geq t_0 \geq 0}$ be a backwards evolutionary process. Φ is called uniformly exponentially stable if there exist $N, \nu > 0$ such that:*

$$||\Phi(t,t_0)|| \leq Ne^{-\nu(t-t_0)}, \ \textit{for all } t \geq t_0 \geq 0.$$

3. The main result

In order to establish sufficient conditions for the uniform exponential stability of the backwards evolutionary process, we will use a result due to J. L. Massera and J. J. Schäffer [8]:

Lemma 3.1. *Take $f, g : \mathbb{R}_+ \to \mathbb{R}_+$, g continuous, such that*

i) $f(t) \leq g(t-t_0)f(t_0)$, for all $t \geq t_0 \geq 0$;
ii) $\inf_{t \geq 0} g(t) < 1$.

Then there exist $N, \nu > 0$ such that

$$f(t) \leq Ne^{-\nu(t-t_0)}f(t_0), \ \textit{for all } t \geq t_0 \geq 0.$$

The following theorem is a strong variant of a result due to E. A. Barbashin [1], for the case of backwards evolutionary processes:

Theorem 3.2. *Let Φ be a backwards evolutionary process. Φ is uniformly exponentially stable if and only if there exist $p, k > 0$ such that:*

$$\Big(\int_{t_0}^{t} ||\Phi(t,\tau)x||^p d\tau\Big)^{\frac{1}{p}} \leq k||x||,$$

for all $x \in X$ and $t \geq t_0$.

Proof. The *necessity* is immediate, and for the *sufficiency* let $t \geq t_0 + 1$ and $r(t) = Me^{\omega t}$, where

$$||\Phi(t,t_0)x|| \leq Me^{\omega(t-t_0)}||x||, \text{ for all } t \geq t_0.$$

Then

$$||\Phi(t,t_0)x||^p \int_{t_0}^{t} r^{-p}(\tau - t_0)d\tau \leq \int_{t_0}^{t} ||\Phi(\tau,t_0)||^p \; ||\Phi(t,\tau)x||^p \; r^{-p}(\tau - t_0)d\tau$$

$$\leq \int_{t_0}^{t} ||\Phi(t,\tau)x||^p d\tau \leq k^p||x||^p.$$

But

$$\int_{t_0}^{t} r^{-p}(\tau - t_0)d\tau = \int_{0}^{t-t_0} r^{-p}(s)ds \geq \int_{0}^{1} r^{-p}(s)ds.$$

We denote by

$$\int_{0}^{1} r^{-p}(s)ds = \alpha > 0.$$

For $\sup_{||x||=1}$ it implies that

$$||\Phi(t,t_0)|| \leq \frac{k}{\alpha^{\frac{1}{p}}}, \text{ for all } t \geq t_0 + 1.$$

If $t \in [t_0, t_0 + 1]$ then

$$||\Phi(t,t_0)|| \leq Me^{\omega}$$

and therefore

$$||\Phi(t,t_0)|| \leq \max\{Me^{\omega}, \frac{k}{\alpha^{\frac{1}{p}}}\} = L, \text{ for all } t \geq t_0 \tag{3.1}$$

Take now $t \geq t_0 \geq 0$ and $\tau \in [t_0, t]$. It follows that

$$||\Phi(t,t_0)x|| = ||\Phi(\tau,t_0) \; \Phi(t,\tau)x|| \leq L||\Phi(t,\tau)x||.$$

Thus,

$$(t - t_0)||\Phi(t,t_0)x||^p \leq L^p \int_{t_0}^{t} ||\Phi(t,\tau)x||^p d\tau \leq L^p k^p ||x||^p.$$

For $\sup_{||x||=1}$ in the above inequality we obtain that

$$(t - t_0)^{\frac{1}{p}}||\Phi(t,t_0)|| \leq Lk. \tag{3.2}$$

Adding the inequalities (3.1) and (3.2) it results that

$$||\Phi(t,t_0)|| \leq \frac{(1+k)L}{1 + (t-t_0)^{\frac{1}{p}}}, \text{ for all } t \geq t_0.$$

Therefore, we have obtained that

$$||\Phi(t,t_0)|| \leq ||\Phi(\tau,t_0)||\ ||\Phi(t,\tau)|| \leq \frac{(1+k)L}{1+(t-\tau)^{\frac{1}{p}}}\ ||\Phi(\tau,t_0)||.$$

By denoting

$$f(t) = ||\Phi(t,t_0)|| \text{ and } g(t-\tau) = \frac{(1+k)L}{1+(t-\tau)^{\frac{1}{p}}},$$

from Lemma 3.1 it follows that there exist $N,\nu>0$ such that

$$||\Phi(t,t_0)|| \leq Ne^{-\nu(t-\tau)}||\Phi(\tau,t_0)||.$$

Taking $\tau = t_0$ we obtain that

$$||\Phi(t,t_0)|| \leq Ne^{-\nu(t-t_0)} \text{ for all } t \geq t_0. \qquad \square$$

Remark 3.3. *We give now another proof for the sufficiency of Theorem* 3.2*, with a direct method:*

Let $t \geq t_0+1$ *and* $\tau \in [t_0, t_0+1]$. *Then*

$$||\Phi(t,t_0)x|| \leq ||\Phi(\tau,t_0)||\ ||\Phi(t,\tau)x|| \leq Me^{\omega}||\Phi(t,\tau)x||.$$

For $\sup_{||x||=1}$ *we obtain that*

$$||\Phi(t,t_0)|| \leq Me^{\omega}||\Phi(t,\tau)||.$$

Thus,

$$\begin{aligned} ||\Phi(t,t_0)|| &\leq Me^{\omega}\left(\int_{t_0}^{t_0+1}||\Phi(t,\tau)||^p d\tau\right)^{\frac{1}{p}} \\ &\leq Me^{\omega}\left(\int_{t_0}^{t}||\Phi(t,\tau)||^p d\tau\right)^{\frac{1}{p}} \\ &\leq Me^{\omega}k. \end{aligned}$$

It follows that

$$||\Phi(t,t_0)|| \leq Me^{\omega}\max\{1,k\}, \text{ for all } t \geq t_0 \geq 0.$$

Denoting $L' = Me^{\omega}\max\{1,k\}$ *we obtain the condition (3.1) from the Theorem* 3.2*.*

The next steps in the proof of the sufficiency are as in Theorem 3.2*.*

The discrete correspondent of Theorem 3.2 is given:

Theorem 3.4. *Let* Φ *be a backwards evolutionary process.* Φ *is uniformly exponentially stable if and only if there exist* $p,l>0$ *such that:*

$$\left(\sum_{k=n_0}^{n}||\Phi(n,k)x||^p\right)^{\frac{1}{p}} \leq l||x||, \text{ for all } n \geq n_0, \text{ and } x \in X.$$

Proof. The *necessity* is immediate.
Sufficiency. From the hypothesis we have that

$$||\Phi(n,n_0)x|| \leq l||x||, \text{ for all } n \geq n_0, \text{ and } x \in X.$$

For $k \in \{n_0, n_0+1, \cdots, n\}$ it follows that

$$\begin{array}{rcl} \sum_{k=n_0}^{n} ||\Phi(n,n_0)x||^p & \leq & \sum_{k=n_0}^{n} ||\Phi(k,n_0)||^p \, ||\Phi(n,k)x||^p \\ & \leq & l^p \sum_{k=n_0}^{n} ||\Phi(n,k)x||^p \\ & \leq & l^{2p}||x||, \end{array}$$

for all $n \geq n_0$ and $x \in X$. Thus,

$$(n-n_0+1)||\Phi(n,n_0)x||^p \leq l^{2p}||x||.$$

For $\sup_{||x||=1}$ we have that

$$(n-n_0+1)||\Phi(n,n_0)||^p \leq l^{2p},$$

which implies that

$$||\Phi(n,n_0)|| \leq \frac{l^2}{(n-n_0+1)^{\frac{1}{p}}}.$$

Therefore, it follows that there exists $m_0 \in \mathbb{N}^*$ such that

$$||\Phi(n_0+m_0,n_0)|| \leq \frac{1}{2}, \text{ for all } n_0 \in \mathbb{N}.$$

For $n \geq n_0$ it results that there exist $q \in \mathbb{N}$ and $r \in \{0,1,\cdots,m_0-1\}$ such that:

$$\begin{array}{rcl} ||\Phi(n,n_0)|| & = & ||\Phi(n_0+qm_0+r,n_0)|| \\ & \leq & ||\Phi(n_0+qm_0,n_0)|| \, ||\Phi(n_0+qm_0+r,n_0+qm_0)|| \\ & \leq & L\left(\frac{1}{2}\right)^q \\ & = & L(e^{-\nu m_0})^q \\ & = & Le^{-\nu(m_0q+r)}e^{\nu r} \\ & = & Le^{\nu r}e^{-\nu(n-n_0)} \\ & \leq & Le^{\nu m_0}e^{-\nu(n-n_0)} \\ & = & 2Le^{-\nu(n-n_0)} \end{array}$$

Denoting $\nu = \frac{1}{m_0}\ln 2$ and $N = 2L$, it follows that:

$$||\Phi(n,n_0)|| \leq Ne^{-\nu(n-n_0)}, \text{ for all } n \geq n_0.$$

Let now $t \geq t_0+1, n = [t], n_0 = [t_0]$. Thus $n \geq n_0+1$ and we obtain that:

$$\begin{array}{rcl} ||\Phi(t,t_0)|| & = & ||\Phi(n_0+1,t_0)\ \Phi(n,n_0+1)\ \Phi(t,n)|| \\ & \leq & M^2e^{2\omega}||\Phi(n,n_0+1)|| \\ & \leq & M^2e^{2\omega}Ne^{-\nu(n-n_0-1)} \\ & = & M^2e^{2\omega}Ne^{-\nu(t-t_0)}e^{\nu(t-t_0-n+n_0+1)} \\ & \leq & M^2e^{2\omega}Ne^{2\nu}e^{-\nu(t-t_0)} \\ & = & M^2Ne^{2\omega+2\nu}e^{-\nu(t-t_0)}, \end{array}$$

for all $t \geq t_0+1$.

For $t_0 \leq t < t_0+1$ it results that

$$||\Phi(t,t_0)|| \leq Me^{\omega}e^{\nu}e^{-\nu(t-t_0)}.$$

Denoting $N = \max\{Me^{\omega+\nu}N, 1\}$ we obtain that:

$$||\Phi(t,t_0)|| \leq Ne^{-\nu(t-t_0)}, \text{ for all } t \geq t_0 \geq 0. \qquad \square$$

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