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Fractional stochastic differential equations: A semimartingale approach

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Abstract. The aim of this paper is to study some class of fractional stochastic equations from the approach given in [2]. The existence and uniqueness for equations with deterministic volatility are proved. The explicit solutions of some important equations are found and the ruin probability in the asset liability management (ALM) model is investigated as well.

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1. Introduction

The first problem in the study of fractional stochastic equations is how to define in some sense the fractional stochastic integration. For this, many attempts have been made by various authors. And there are definitions obtained from some kinds of approximation approach as those of D. Nualart and al.[1], Tran Hung Thao and Christine Thomas-Agnan [12, 11] and P. Carmona, L. Coutin and G. Montseny [2, 4]. This paper is based on the results given by the last mentioned authors.

By definition, a fractional Brownian motion (fBm) W^H is a centered Gaussian process with the covariance function given by

$$R_H(t,s) := E[W_t^H W_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

In [2], the authors proved that W_t^H can be approximated by semimartingales $W_t^{H,\varepsilon}$

$$W_t^H = \int\limits_0^t K(t,s)dB_s \ , \ \ t \geq 0$$

$$W_t^{H,\varepsilon} = \int\limits_0^t K(t+\varepsilon, s) dB_s \,,$$

where B is a standard Brownian montion and the kernel $K(t,s)$ is given by

$$K(t,s) = C_H \left[\frac{t^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}} (t-s)^{H-\frac{1}{2}} - (H - \frac{1}{2}) \int\limits_s^t \frac{u^{H-\frac{3}{2}}}{s^{H-\frac{1}{2}}} (u-s)^{H-\frac{1}{2}} du \right].$$

Under suitable conditions on the function f, they proved that the integral $\int\limits_0^t f_s dW_s^{H,\varepsilon}$ converges in $L^2(\Omega)$ as $\varepsilon \to 0$, and then the fractional stochastic integral $\int\limits_0^t f_s dW_s^H$ is defined as a limit of $\int\limits_0^t f_s dW_s^{H,\varepsilon}$.

In this paper we are interested in a class of fractional stochastic differential equations with deterministic volatility of the following form

$$\begin{cases} dX_t = a(t, X_t)\, dt + \sigma(t)\, dW_t^H \\ X_t|_{t=0} = X_0 \ , \ \ t \in [0,T] \,. \end{cases} \tag{1.1}$$

The existence and uniqueness of the solution of (1.1) are established via a study of its corresponding approximation equation.

The organization of the paper is as follows: Section 2 contains some basic results on the semimartingale approach given in [2, 4]. In Section 3, we prove the existence, uniqueness and Lipschitzian continuity of the solution of the approximation equations, one of main results of this paper is formulated in Theorem 3.5. In Section 4, the explicit solutions for the equation of Ornstein-Uhlenbeck type and for the fractional stochastic differential equation with polynomial drift are found. Finally, in Section 5 we study the ruin probability in the ALM model.

2. Preliminaries

For the sake of convenience, we recall some important results from [2, 4] which will be the basis of this paper.

Theorem 2.1. *For every $\varepsilon > 0$, $W_t^{H,\varepsilon}$ is a $\mathcal{F}_t$-semimartingale with the following decomposition*

$$W_t^{H,\varepsilon} = \int\limits_0^t K(s+\varepsilon, s) dB_s + \int\limits_0^t \varphi_s^\varepsilon ds, \tag{2.1}$$

where $(\mathcal{F}_t, 0 \le t \le T)$ is the natural filtration associated to B or W^H and

$$\varphi_s^\varepsilon = \int\limits_0^s \partial_1 K(s+\varepsilon, u) dB_u \,,$$

$$\partial_1 K(t,s) = \frac{\partial K(t,s)}{\partial t} = C_H \frac{t^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}}(t-s)^{H-\frac{3}{2}} .$$

Hypothesis (H): *Assume that f is an adapted process belonging to the space $\mathbf{D}_B^{1,2}\big(L^2([0,T],\mathbb{R},du)\big)$ and that there exists β fulfilling $\beta + H > 1/2$ and $p > 1/H$ such that*

(i) $\sup\limits_{0<s<u<T} \dfrac{E\Big[(f_u-f_s)^2+\int_0^T (D_r^B f_u - D_r^B f_s)^2 dr\Big]}{|u-s|^{2\beta}}$ *is finite,*

(ii) $\sup\limits_{0<s<T} f_s$ *belongs to* $L^p(\Omega)$.

Remark 2.2. *The space $\mathbf{D}_B^{1,2}\big(L^2([0,T],\mathbb{R},du)\big)$ is defined as follows:*

For $h \in L^2([0,T],\mathbb{R})$, we denote by $B(h)$ the Wiener integral

$$B(h) = \int_0^T h(t) dB_t.$$

Let $\mathcal{S}$ denote the dense subset of $L^2(\Omega, \mathcal{F}, P)$ consisting of those classes of random variables of the form

$$F = f(B(h_1), ..., B(h_n)), \tag{2.2}$$

where $n \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^n, L^2([0,T],\mathbb{R})), h_1, ..., h_n \in L^2([0,T],\mathbb{R})$. If F has the form (2.2), we define its derivative as the process $D^B F := \{D_t^B F, t \in [0,T]\}$ given by

$$D_t^B F = \sum_{k=1}^n \frac{\partial f}{\partial x_k}(B(h_1), ..., B(h_n)) h_k(t).$$

We shall denote by $\mathbf{D}_B^{1,2}\big(L^2([0,T],\mathbb{R},du)\big)$ the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,2} := \big[E|F|^2\big]^{\frac{1}{2}} + E\Bigg[\int_0^T |D_u^B F|^2 du\Bigg]^{\frac{1}{2}}.$$

Definition 2.3. *For a process f fulfilling* **Hypothesis (H)**. *The fractional stochastic integral of f with respect to W^H is defined by*

$$\int_0^t f_s\, dW_s^H = \int_0^t f_s K(t,s)\, dB_s + \int_0^t \int_s^t (f_u - f_s)\, \partial_1 K(u,s) du \delta B_s + \int_0^t du \int_0^u D_s^B f_u\, \partial_1 K(u,s) ds\,, \tag{2.3}$$

where the second integral in the right-hand side is a Skorohod integral (we refer to [10] *for more details about the Skorohod integral).*

Remark 2.4. *Suppose that f be an adapted process belonging to the space $\mathbf{D}^{1,2}_B\big(L^2([0,T],\mathbb{R},du)\big)$, then*

$$\int_0^t f_s\,dW_s^{H,\varepsilon} = \int_0^t f_s K(t+\varepsilon,s)\,dB_s + \int_0^t\int_s^t (f_u - f_s)\,\partial_1 K(u+\varepsilon,s)du\delta B_s$$
$$+ \int_0^t du \int_0^u D_s^B f_u\,\partial_1 K(u+\varepsilon,s)ds$$

and under the **Hypothesis (H)**, $\int_0^t f_s\,dW_s^{H,\varepsilon} \to \int_0^t f_s\,dW_s^H$ *in* $L^2(\Omega)$ *as* $\varepsilon \to 0$.

Remark 2.5. *If f is a deterministic function such that*

$$\int_0^t f_s^2 K^2(t,s)\,ds < \infty\,,$$

then

$$\int_0^t f_s\,dW_s^H = \int_0^t f_s K(t,s)\,dB_s + \int_0^t\int_s^t (f_u - f_s)\,\partial_1 K(u,s)dudB_s.$$

3. The main result

In this section we study the existence and uniqueness of the solution of (1.1) by considering its corresponding approximation equation which is defined immediately below.

Definition 3.1. *The stochastic differential equation*

$$\begin{cases} dX_t^\varepsilon = a(t,X_t^\varepsilon)\,dt + \sigma(t)\,dW_t^{H,\varepsilon} \\ \\ X_t^\varepsilon|_{t=0} = X_0\ \ ,\ \ t\in[0,T] \end{cases} \tag{3.1}$$

*is called the approximation equation corresponding to the fractional stochastic differential equation (*1.1*).*

Noting that (3.1) is a stochastic differential equation driven by a semimartingale, the conditions for uniqueness and existence of the solution of it is well known. For more details, from (2.1) we can rewrite the equation (3.1) as follows

$$dX_t^\varepsilon = \Big(a(t,X_t^\varepsilon) + \sigma(t)\varphi_t^\varepsilon\Big)\,dt + K(t+\varepsilon,t)\sigma(t)\,dB_t\,. \tag{3.2}$$

The stochastic process $\sigma(t)\varphi_t^\varepsilon$ is not bounded. However, we can establish the existence and uniqueness of the solution of equation (3.2) by considering the sequence of stopped times

$$\tau_M = \inf\{t \in [0,T] : \int_0^t (\varphi_s^\varepsilon)^2 ds > M\} \wedge T\,, \tag{3.3}$$

and consider the sequence of corresponding stopped equations. The existence and uniqueness of the solution of the stopped equations is well known (see, for instance, [7, 8]). Then by taking limit when $M \to \infty$, we have the following theorem

Theorem 3.2. *Assume that the functions $a : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}\,, \sigma : [0,T] \longrightarrow \mathbb{R}$ are measurable with respect to all their arguments and the following conditions hold:*
(A1) There exists a constant $K > 0$ such that for $x, y \in \mathbb{R}$ and $t \in [0,T]$

$$|a(t,x) - a(t,y)| \le K|x-y| \ \ , \ \ |a(t,x)| \le K(1+|x|)\,. \tag{3.4}$$

(A2) For all $t \in [0,T]$

$$\int_0^t \sigma_s^2 K^2(t,s)\, ds < \infty\,, \tag{3.5}$$

(A3) The initial value X_0 is square-integrable random variable and it is independent of W.

Then equation (3.1) has unique solution $\sigma(W_s, 0 \le s \le t)$-adapted X_t^ε on $[0,T]$. Moreover, in the case $H > 1/2$

$$\sup_{0 \le t \le T} E|X_t^\varepsilon|^2 \le C\,, \tag{3.6}$$

where C is some positive constant not depending on ε.

Proposition 3.3. *Assume that conditions for the existence and uniqueness of the solutions of both fractional stochastic differential equation (1.1) and approximation equation (3.1) hold. Then the sequence of solutions of the approximation equation (3.1) converges in $L^2(\Omega)$ to the solution of (1.1) as $\varepsilon \to 0$.*

Proof. We have

$$E|X_t^\varepsilon - X_t|^2 \le 2E|\int_0^t a(s, X_s^\varepsilon)\, ds - \int_0^t a(s, X_s)\, ds|^2$$
$$+ 2E|\int_0^t \sigma(s)\, dW_s^{H,\varepsilon} - \int_0^t \sigma(s)\, dW_s^H|^2 \tag{3.7}$$

According to Remark 2.4 we can see that

$$E|\int_0^t \sigma(s)\, dW_s^{H,\varepsilon} - \int_0^t \sigma(s)\, dW_s^H|^2 := C(t,\varepsilon) \to 0 \ \text{ as } \ \varepsilon \to 0\,.$$

Now, using the Lipschitz continuity assumption (3.4) we get

$$\int_0^t E|a(s,X_s^\varepsilon)-a(s,X_s)|^2ds \le K^2\int_0^t E|X_t^\varepsilon - X_t|^2ds.$$

Thus, the conclusion of this proposition is easily achieved by applying Gronwall's lemma. $\square$

Due to the above Proposition, the solution of equation (1.1) can be considered as the limit in $L^2(\Omega)$ of the solutions of the equations (3.1), and so, if this limit exists then the equation (1.1) has an unique solution.

Let $\varepsilon = \frac{1}{n}$, $n \ge 1$, we recall from Remark 2.4 and Remark 2.5 that

$$\int_0^t \sigma_s\, dW_s^{H,\frac{1}{n}} = \int_0^t \sigma_s K(t+\frac{1}{n},s)\, dB_s + \int_0^t\int_s^t (\sigma_u-\sigma_s)\,\partial_1 K(u+\frac{1}{n},s)du dB_s$$

Let us now consider a sequence of approximation equations

$$\begin{cases} dX_t^n = a(t,X_t^n)\,dt + \sigma(t)\,dW_t^{H,\frac{1}{n}} \\ X_t^n|_{t=0} = X_0\ ,\ \ t\in[0,T] \end{cases} \tag{3.8}$$

or

$$X_t^n = X_0 + \int_0^t a(s,X_s^n)\,ds + \int_0^t \sigma_s K(t+\frac{1}{n},s)\,dB_s + \int_0^t\int_s^t (\sigma_u-\sigma_s)\,\partial_1 K(u+\frac{1}{n},s)du dB_s\,. \tag{3.9}$$

Theorem 3.4. *Let $H\in(\frac{1}{2},1)$ and the coefficients of equation (3.8) satisfy the assumptions (A1), (A2), (A3) from Theorem 3.1. Then*
I. The solution of (3.8) is Lipschitz continuous in $L^2(\Omega)$, i.e

$$E|X_t^n - X_s^n|^2 \le C|t-s|\,. \tag{3.10}$$

II. For every $t\in[0,T]$, the sequence $\{X_t^n\}_{n\ge1}$ of the solutions of the equations (3.8) is a fundamental sequence in $L^2(\Omega)$.

Proof. I. We consider $E|X_{t+\tau}^n - X_t^n|^2$ for $0\le t\le t+\tau\le T$:

$$E|X_{t+\tau}^n - X_t^n|^2 \le 3E\Big(\int_t^{t+\tau} a(s,X_s^n)\,ds\Big)^2$$

$$+3E\Big(\int_0^{t+\tau}\sigma(s)K(t+\tau+\frac{1}{n},s)\,dB_s - \int_0^t \sigma(s)K(t+\frac{1}{n},s)\,dB_s\Big)^2$$

$$+3E\Big(\alpha\int_0^{t+\tau}\int_s^{t+\tau}[\sigma(u)-\sigma(s)]\partial_1K(u+\frac{1}{n},s)dudB_s$$

$$-\alpha\int_0^t\int_s^t[\sigma(u)-\sigma(s)]\partial_1K(u+\frac{1}{n},s)dudB_s\Big)^2 := 3I_1+3I_2+3I_3.$$

First, it follows from (3.4), (3.6) that

$$I_1\le K^2\int_t^{t+\tau}E(1+X_s^n)^2\,ds\le 2K^2(1+C)\tau\,. \tag{3.11}$$

Next, we can estimate I_2 as

$$I_2\le 2E\Big(\int_0^t\sigma(s)[K(t+\tau+\frac{1}{n},s)-K(t+\frac{1}{n},s)]\,dB_s\Big)^2$$

$$+\,2E\Big(\int_t^{t+\tau}\sigma(s)K(t+\tau+\frac{1}{n},s)\,dB_s\Big)^2 \tag{3.12}$$

$$=2\int_0^t\sigma^2(s)[K(t+\tau+\frac{1}{n},s)-K(t+\frac{1}{n},s)]^2\,ds$$

$$+\,2\int_t^{t+\tau}\sigma^2(s)K^2(t+\tau+\frac{1}{n},s)\,ds$$

$$\le 2\|\sigma\|_\infty^2E|W^H_{t+\tau+\frac{1}{n}}-W^H_{t+\frac{1}{n}}|^2+2\int_t^{t+\tau}\sigma^2(s)K^2(t+\tau+\frac{1}{n},s)\,ds$$

$$=2\|\sigma\|_\infty^2\tau^{2H}+2\int_t^{t+\tau}\sigma^2(s)K^2(t+\tau+\frac{1}{n},s)\,ds\le C_1\tau$$

where $\|\sigma\|_\infty=\sup\limits_{0\le s\le T}|\sigma(s)|$, C_1 is a positive finite constant depending on σ.

Similarly, for I_3 we have

$$I_3\le 2E\Big(\int_0^t\int_t^{t+\tau}[\sigma(u)-\sigma(s)]\partial_1K(u+\frac{1}{n},s)dudB_s\Big)^2+$$

$$+2E\Big(\int_t^{t+\tau}\int_s^{t+\tau}[\sigma(u)-\sigma(s)]\partial_1K(u+\frac{1}{n},s)dudB_s\Big)^2$$

$$= 2\int_0^t \Big(\int_t^{t+\tau} [\sigma(u)-\sigma(s)]\partial_1 K(u+\frac{1}{n},s)du\Big)^2 ds$$

$$+2\int_t^{t+\tau} \Big(\int_s^{t+\tau} [\sigma(u)-\sigma(s)]\partial_1 K(u+\frac{1}{n},s)du\Big)^2 ds$$

$$\leq 8\|\sigma\|_\infty^2 \int_0^t [K(t+\tau+\frac{1}{n},s)-K(t+\frac{1}{n},s)]^2 ds$$

$$+8\|\sigma\|_\infty^2 \int_t^{t+\tau} [K(t+\tau+\frac{1}{n},s)-K(s+\frac{1}{n},s)]^2 ds$$

$$= 8\|\sigma\|_\infty^2 \tau^{2H} + 8\|\sigma\|_\infty^2 \int_t^{t+\tau} [K(t+\tau+\frac{1}{n},s)-K(s+\frac{1}{n},s)]^2 ds\,.$$

Hence,

$$I_3 \leq C_2\tau\,. \tag{3.13}$$

Finally, (3.10) follows from the inequalities (3.11)-(3.13).
II. We now are ready to prove the rest of the theorem. Consider $E|X_t^n - X_t^m|^2$:

$$\begin{aligned}
E|X_t^n - X_t^m|^2 \leq 3\int_0^t E[a(s,X_s^n)-a(s,X_s^m)]^2\,ds \\
+ 3E\Big(\int_0^t [\sigma(s)K(t+\frac{1}{n},s)-\sigma(s)K(t+\frac{1}{m},s)]\,dB_s\Big)^2 \\
+ 3E\Big(\int_0^t\int_s^t \big\{[\sigma(u)-\sigma(s)]\partial_1 K(u+\frac{1}{n},s) \\
- [\sigma(u)-\sigma(s)]\partial_1 K(u+\frac{1}{m},s)\big\} du dB_s\Big)^2 \\
:= 3(J_1+J_2+J_3)\,.
\end{aligned} \tag{3.14}$$

$$J_1 \leq K^2 \int_0^t E|X_s^n - X_s^m|^2 ds\,. \tag{3.15}$$

$$J_2 = \int_0^t [\sigma(s)K(t+\frac{1}{n},s) - \sigma(s)K(t+\frac{1}{m},s)]^2\,ds$$

$$\leq \|\sigma\|_\infty^2 \int_0^T [K(t+\frac{1}{n},s) - K(t+\frac{1}{m},s)]^2\,ds$$

$$= \|\sigma\|_\infty^2 \{R(t+\frac{1}{n},t+\frac{1}{n}) + R(t+\frac{1}{m},t+\frac{1}{m}) - 2R(t+\frac{1}{n},t+\frac{1}{m})\}$$

$$\leq C_3|\frac{1}{n} - \frac{1}{m}|^{2H-1} := c_1(m,n), \quad (3.16)$$

where C_3 is a finite positive constant depending on σ, and $R(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$ is the covariance function of the fBm W^H. We have

$$J_3 = \int_0^t \left(\int_s^t [\sigma(u) - \sigma(s)][\partial_1 K(u+\frac{1}{n},s) - \partial_1 K(u+\frac{1}{m},s)]du \right)^2 ds \quad (3.17)$$

$$\leq 8\|\sigma\|_\infty^2 \int_0^t \left(K(t+\tfrac{1}{n},s) - K(t+\tfrac{1}{m},s) \right)^2 ds$$

$$+ 8\|\sigma\|_\infty^2 \int_0^t \left(K(s+\frac{1}{n},s) - K(s+\frac{1}{m},s) \right)^2 ds \leq 16c_1(m,n).$$

Put $g(t) = E|X_t^n - X_t^m|^2$, then combining (3.14)-(3.17) yields

$$g(t) \leq 3K^2 \int_0^t g(s)ds + c(m,n)\,, \quad (3.18)$$

where $c(m,n) = 3(c_1(m,n) + 16c_1(m,n)) \to 0$ as $m \to \infty, n \to \infty$.

From (3.18) and by applying Gronwall's lemma we get

$$g(t) \leq c(m,n)\,e^{3K^2t}\,,$$

or

$$E|X_t^n - X_t^m|^2 \leq c(m,n)\,e^{3K^2t}\,.$$

And, as a consequence, the solutions $\{X_t^n, 0 \leq t \leq T\}_{n\geq 1}$ of equations (3.8) form a fundamental sequence in $L^2(\Omega)$. $\square$

Now we can state the following theorem

Theorem 3.5. *Suppose that $H \in (\frac{1}{2}, 1)$. Consider the fractional stochastic differential equation*

$$\begin{cases} dX_t = a(t, X_t)\, dt + \sigma(t)\, dB_t \\ X_t|_{t=0} = X_0 \ , \ \ t \in [0, T], \end{cases} \tag{3.19}$$

where $\sigma(t)$ is a deterministic function. If the coefficients $a(t,x)\,, \sigma(t)$ satisfy the assumptions (A1), (A2) from Theorem 3.1, then (3.19) has a unique solution. Moreover, this solution is Lipschitz continuous in $L^2(\Omega)$, i.e

$$E|X_t - X_s|^2 \leq C|t-s|\,.$$

Remark 3.6. *If $a(t,x)$ is Lipschitzian with respect to x and under assumption (A2) then the existence of the solution (3.19) can be proved by applying the fixed point theorem in some Banach space after constructing an appropriate contraction operator in this space. For the uniqueness, it suffices to use Gronwall's lemma.*

4. Explicit solution for some important classes of stochastic differential equations

From practical point of view, it is important to find the explicit expression for the solution of each specific model. In the rest of this paper, we will see that the semimatingale approach has more advantages for this.

4.1. The Ornstein-Uhlenbeck type equations

The fractional Ornstein-Uhlenbeck processes are studied in [3]. Let us use semimartingale approach to find the solution for a class of Ornstein-Uhlenbeck type equations of following form:

$$\begin{cases} dX_t = (\alpha(t)X_t + \beta(t))\, dt + \sigma(t)\, dW_t^H \\ X_t|_{t=0} = X_0 \ , \ \ t \in [0, T], \end{cases} \tag{4.1}$$

where $\alpha(t), \beta(t), \sigma(t)$ are deterministic functions.

The approximation equation corresponding to (4.1) is

$$\begin{cases} dX_t^\varepsilon = (\alpha(t)X_t^\varepsilon + \beta(t))\, dt + \sigma(t)\, dW_t^{H,\varepsilon} \ , \ \varepsilon > 0 \\ X_t^\varepsilon|_{t=0} = X_0 \ , \ \ t \in [0, T] \end{cases}$$

or equivalently,

$$\begin{cases} dX_t^\varepsilon = (\alpha(t)X_t^\varepsilon + \beta(t) + \sigma(t)\varphi_t^\varepsilon)\, dt + K(t+\varepsilon, t)\sigma(t)\, dB_t \\ X_t^\varepsilon|_{t=0} = X_0 \ , \ \ t \in [0, T]\,. \end{cases} \tag{4.2}$$

This is a semilinear stochastic differential equation.Therefore, its solution is given by

$$X^\varepsilon(t) = e^{\int_0^t \alpha(u)du}\Big(X_0 + \int_0^t \beta(s)\, e^{-\int_0^s \alpha(u)du} ds + \int_0^t \sigma(s)\varphi_s^\varepsilon\, e^{-\int_0^s \alpha(u)du} ds + \int_0^t K(s+\varepsilon, s)\sigma(s)\, e^{-\int_0^s \alpha(u)du} dB_s\Big).$$

Using (2.1) we can rewrite the solution $X^\varepsilon(t)$ into the following form

$$X^\varepsilon(t) = e^{\int_0^t \alpha(u)du}\Big(X_0 + \int_0^t \beta(s)\, e^{-\int_0^s \alpha(u)du} ds + \int_0^t \sigma(s)\, e^{-\int_0^s \alpha(u)du} dW_s^{H,\varepsilon}\Big). \tag{4.3}$$

By taking limit when $\varepsilon \to 0$ we get the following theorem.

Theorem 4.1. *Suppose that X_0 is a square-integrable random variable independent of W^H. Then the solution of (4.1) is unique and given by*

$$X_t = e^{\int_0^t \alpha(u)du}\big(X_0 + \int_0^t \beta(s) e^{-\int_0^s \alpha(u)du} ds + \sigma \int_0^t e^{-\int_0^s \alpha(u)du} dW_t^H\big).$$

4.2. Fractional stochastic differential equations with polynomial drift

Let us consider the fractional stochastic differential equation in a complete probability space $(\Omega, \mathcal{F}, P)$

$$\begin{cases} dX_t = \big(a\, X_t^n + b\, X_t\big)\, dt + c\, X_t\, dW_t^H \\ X_t|_{t=0} = X_0. \end{cases} \tag{4.4}$$

The initial value X_0 is a measurable random variable independent of $\{B_t : 0 \le t \le T\}$.

This equation is a generalization of many important equations such as the Black-Sholes model in mathematical finance ($a = 0$), the Ginzburg-Landau equation in the theoretical physics ($n = 3$), the Verlhust equation in population study ($n = 2$).

We consider now a corresponding approximation equation with the same initial condition $X_t^\varepsilon|_{t=0} = X_0$

$$dX_t^\varepsilon = \big(a\,(X_t^\varepsilon)^n + b\, X_t^\varepsilon\big)\, dt + c\, X_t^\varepsilon\, dW_t^{H,\varepsilon}\ ,\ \varepsilon > 0. \tag{4.5}$$

Using (2.1) again we get

$$dX_t^\varepsilon = \big(a\,(X_t^\varepsilon)^n + b\, X_t^\varepsilon + c\varphi_t^\varepsilon\, X_t^\varepsilon\big)\, dt + c\, X_t^\varepsilon\, dB_t\,. \tag{4.6}$$

In order to find the explicit expression for the solution of the equation (4.5) we will carry out several steps.

Step 1. Put $Y_t^\varepsilon = e^{-U_t}, U_t = \int\limits_0^t cK(s+\varepsilon,s)ds$. According to the Itô formula we have:

$$dY_t^\varepsilon = \frac{1}{2}Y_t^\varepsilon c^2K^2(t+\varepsilon,t)dt - Y_t^\varepsilon cK(t+\varepsilon,t)dB_t\,. \tag{4.7}$$

Step 2. We consider $Z_t^\varepsilon = X_t^\varepsilon Y_t^\varepsilon$ and then the integration-by-part formula gives us

$$dZ_t^\varepsilon = X_t^\varepsilon dY_t^\varepsilon + Y_t^\varepsilon dX_t^\varepsilon - d[X^\varepsilon, Y^\varepsilon]_t$$

or

$$dZ_t^\varepsilon = \left\{ [-\frac{1}{2}c^2K^2(t+\varepsilon,t)+b+c\varphi_t^\varepsilon]\,Z_t^\varepsilon + a\,e^{(n-1)\int\limits_0^t cK(s+\varepsilon,s)ds}(Z_t^\varepsilon)^n \right\}dt\,. \tag{4.8}$$

For every fixed $\omega \in \Omega$, the equation (4.8) is an ordinary Bernoulli equation of the form:

$$(Z_t^\varepsilon)' = P(t)(Z_t^\varepsilon)^n + Q(t)Z_t^\varepsilon$$

and the solution Z_t^ε is given by

$$Z_t^\varepsilon = e^{\int\limits_0^t Q(u)du}\left(Z_0^{1-n} + \int\limits_0^t (1-n)P(s)e^{(n-1)\int\limits_0^s Q(u)du}ds\right)^{\frac{1}{1-n}}$$

where $P(t) = a\,e^{(n-1)\int\limits_0^t cK(s+\varepsilon,s)ds}, Q(t) = -\frac{1}{2}c^2K^2(t+\varepsilon,t) + b + c\varphi_t^\varepsilon$, the initial condition $Z_0^\varepsilon = X_0^\varepsilon Y_0^\varepsilon = X_0$.

Finally, the solution $X_t^\varepsilon = \frac{Z_t^\varepsilon}{Y_t^\varepsilon}$ of the equation (4.5) is given by

$$X_t^\varepsilon = e^{bt - \frac{1}{2}\int\limits_0^t c^2K^2(s+\varepsilon,s)ds + c\,W_t^{H,\varepsilon}}$$
$$\times\left(X_0^{1-n} + (1-n)a\int\limits_0^t e^{(n-1)\left(bs - \frac{1}{2}\int\limits_0^s c^2K^2(u+\varepsilon,u)du + c\,W_s^{H,\varepsilon}\right)}ds\right)^{\frac{1}{1-n}}. \tag{4.9}$$

Noting that the solution of (4.4) is a limit in $L^2(\Omega)$ of the solution of (4.5). Hence, by taking limit when $\varepsilon \to 0$ we get the following theorem.

Theorem 4.2. *Suppose that X_0 is a random variable independent of W^H such that $E[X_0^2] < \infty$. Then the solution of (4.4) exists and is unique and given by*

$$X_t = e^{bt + c\,W_t^H}\left(X_0^{1-n} + (1-n)a\int\limits_0^t e^{(n-1)\,(bs + c\,W_s^H)}ds\right)^{\frac{1}{1-n}}.$$

5. The ruin probability in the Asset Liability Management model

In this section, we consider the asset X_t and the liability Y_t satisfing the following stochastic differential equations

$$\begin{cases} dX_t = \mu_1 X_t dt + \sigma_1 X_t dW_t^{(1)}, \\ dY_t = \mu_2 Y_t dt + \sigma_2 Y_t dW_t^{(2)}, \\ X|_{t=0} = X_0 \ , \ Y|_{t=0} = Y_0 < X_0, \end{cases} \tag{5.1}$$

where $\mu_1, \mu_2, \sigma_1, \sigma_2$ are non-negative parameters, $W_t^{(1)} = \int\limits_0^t K(t,s)dB_t^{(1)}, W_t^{(2)} = \int\limits_0^t K(t,s)dB_t^{(2)}$ are two fractional Brownian motions with correlation coefficient $|\rho| \le 1$.

It follows from Theorem 4.2 that

$$X_t = X_0 e^{\mu_1 t + \sigma_1 W_t^{(1)}} \ , \ Y_t = Y_0 e^{\mu_2 t + \sigma_2 W_t^{(2)}}$$

and

$$\frac{X_t}{Y_t} = \frac{X_0}{Y_0} \exp\left((\mu_1 - \mu_2)t + \sigma_1 W_t^{(1)} - \sigma_2 W_t^{(2)}\right).$$

Noting that $B^{(1)}, B^{(2)}$ have correlation coefficient ρ, because $W^{(1)}, W^{(2)}$ have correlation coefficient ρ. Hence

$$\sigma_2 B_t^{(2)} - \sigma_1 B_t^{(1)}$$

is equivalent in distribution to the process σB_t, where B_t is a standard Brownian motion and

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \tag{5.2}$$

We obtain

$$\sigma_1 W_t^{(1)} - \sigma_2 W_t^{(2)} = \int\limits_0^t K(t,s) d(\sigma_1 B_s^{(1)} - \sigma_2 B_s^{(2)})$$

$$= -\sigma \int\limits_0^t K(t,s) dB_s =: -\sigma W_t^H$$

and

$$\frac{X_t}{Y_t} = \frac{X_0}{Y_0} \exp(\mu t - \sigma W_t^H) \ , \tag{5.3}$$

where $\mu = \mu_1 - \mu_2$, W_t^H is a fractional Brownian motion with index H .

We now can study the lifetime τ of a bank or of an insurance company that is naturally defined as the first value of t such that $X_t < Y_t$:

$$\tau = \inf\{t : \ln \frac{X_t}{Y_t} < 0\}.$$

and the ruin probability on a finite time horizon $[0, t]$ is defined as

$$\varphi(X_0, Y_0, t) := P(\tau < t) = P(\ln \frac{X_s}{Y_s} < 0 \text{ for some } s < t),$$

and on an infinite time horizon,

$$\varphi(X_0, Y_0) := \lim_{t\to\infty} \varphi(X_0, Y_0, t).$$

By the relation (5.3) we obtain

$$\begin{aligned}\varphi(X_0, Y_0) &= P(\ln \frac{X_t}{Y_t} < 0 \text{ for some } t \geq 0)\\ &= P(-\mu t + \sigma W_t^H > u \text{ for some } t \geq 0)\\ &= P\big(\sup_{t\geq 0}(-\mu t + \sigma W_t^H) > u\big),\end{aligned}$$

where $u = \ln \frac{X_0}{Y_0}$. In order to estimate $\varphi(X_0, Y_0)$ we use the following result of Dębicki [5, Corollary 4.1]:

Proposition 5.1. *For* $\frac{1}{2} \leq H \leq 1$

$$\lim_{u\to\infty} \frac{1}{u^{2-2H}} \ln P\big(A(W^H, c) > u\big) = -h \tag{5.4}$$

where $A(W^H, c) = \sup\{W_t^H - ct : t \geq 0\}$ *and*

$$h = \frac{1}{2}\Big(\frac{c}{H}\Big)^{2H}\Big(\frac{1}{1-H}\Big)^{2-2H}.$$

Now we can state the following theorem

Theorem 5.2. *If* $\mu_1 \geq \mu_2$*, then the ruin probability for the ALM model* (5.1) *satisfies the following relation:*

$$\lim_{u\to\infty} \frac{\ln \varphi(X_0, Y_0)}{u^{2-2H}} = -\frac{\mu^{2H}}{2H^2\sigma^2}\Big(\frac{H}{1-H}\Big)^{2-2H}, \tag{5.5}$$

where $\mu = \mu_1 - \mu_2$, $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$ *and* $u = \ln \frac{X_0}{Y_0}$.

Proof. We have

$$\varphi(X_0, Y_0) = P\big(\sup_{t\geq 0}(W_t^H - \frac{\mu}{\sigma}t) > \frac{u}{\sigma}\big)$$

from Proposition 5.1. The theorem is completed. □

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