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On the stability of the bivariate geometric composed distribution's characterization

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Abstract. Let $(X_j, Y_j), j = 1, 2...$ be nonnegative i.i.d random vectors and (N_1, N_2) be independent of (X_j, Y_j), $j = 1, 2, ...$ with Bivariate Geometric Distribution. The vector $(Z_1 = \sum_{j=1}^{N_1} X_j; Z_2 = \sum_{j=1}^{N_2} Y_j)$ is called the Bivariate Geometric Composed vector. In [3], a characterization for distribution function of this vector was showed and in this paper we shall consider the stability of this characterization.

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1. Introduction

At first, we recall a well-known characterization of the univariate geometric composed distribution. Let $X_1, X_2, ...$ be nonegative i.i.d random variables (r.v's) $P(X_j > x) = \overline{F}(x), EX_j = 1(j = 1, 2, ..)$ and let N be independent of $X_j, (j = 1, 2, ...)$ with the Geometric distribution, i.e.

$$P(N = k) = p(1 - p)^{k-1} \quad (k = 1, 2, ...)$$

The random variable $Z = \sum_{j=1}^{N} X_j$ is called the Geometric Composed random variable. We denote $\overline{G}_p(x) = P\{pZ > x\}$. In [1], Renyi has given characteristics of this Geometric Composed Distribution. In [2], some stabilities of this Renyi's characteristic theorem was considered by two Vietnamese authors. In [3] (1985), A. Kovat (Hungarian) expanded this Renyi's characteristic theorem for the case of two dimensions.

We consider the Bivariate Geometric Composed distribution as the following definition (See [3]).

Let A_1, A_2 be arbitrary events and $p = (p_1, p_2, p_{12})$, means the probabilities

$$P(A_1\overline{A_2}) = p_1; P(\overline{A_1}A_2) = p_2; P(A_1A_2) = p_{12} \tag{1.1}$$

and $q = 1 - p_1 - p_2 - p_{12} = 1 - P(\overline{A_1 \cup A_2})$.

Let N_1, N_2 be the serial numbers of necessary trials for occuring at first of the event A_1, A_2 resp. occur at first. Then we will say that the random vector (N_1, N_2) has bivariate geometric distribution and we can obtain the following distribution of (N_1, N_2):

$$P\{N_1 = k_1; N_2 = k_2\} = \begin{cases} q^{k_2-1}p_2(1-p_1-p_{12})^{k_1-k_2-1}(p_1+p_{12} & if \;\; k_1 > k_2 \\ q^{k_1-1}p_{12} & if \;\; k_1 = k_2 \\ q^{k_1-1}p_1(1-p_2-p_{12})^{k_2-k_1-1}(p_1+p_{12} & if \;\; k_1 < k_2 \end{cases} \tag{1.2}$$

Let $(X_j, Y_j), j = 1, 2, ...$ be nonegative i.i.d. random vectors, $P\{X_j > x; Y_j > y\} = \overline{F}(x, y)$, $\varphi(t_1, t_2) = E\{e^{it_1X_j+it_2Y_j}\}; EX_j = 1; EY_j = 1(j = 1, 2, ..)$

Let (N_1, N_2) be independent of (X_j, Y_j) (j=1,2,...) and (N_1, N_2) has Bivariate geometric distribution. The random vector $(Z_1 = \sum_{j=1}^{N_1} X_j; Z_2 = \sum_{j=1}^{N_2} Y_j)$ is called the Bivariate Geometric Composed random vector.

Put

$$\overline{G}_p(x, y) = P\{(p_1 + p_{12})Z_1 > x; (p_2 + p_{12})Z_2 > y\}. \tag{1.3}$$

The following characteristic theorem was showed in [3].

Theorem 1.1 $\overline{G}_p(x, y) = \overline{F}(x, y)$ *if and only if*

$$\varphi(t_1, t_2) = [1 - it_1 - it_2 + \sum_{r=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{n+k}a_{n,k}t_1^n t_2^k]^{-1}, \tag{1.4}$$

where

$$a_{1,1} = \frac{p_1 + p_2}{p_1 + p_2 + p_{12} - (p_1 + p_{12})(p_2 + p_{12})},$$

$$a_{1,2} = \frac{p_2 - a_{1,1}(p_2 + p_{12})(1 - p_1 - p_{12})}{p_1 + p_2 + p_{12} - (p_1 + p_{12})(p_2 + p_{12})^2},$$

$$a_{2,1} = \frac{p_1 - a_{1,1}(p_1 + p_{12})(1 - p_2 - p_{12})}{p_1 = p_2 + p_{12} - (p_1 + p_{12})^2(p_2 + p_{12})}, \tag{1.5}$$

$$\begin{aligned} a_{n,k} = {} & [p_1 + p_2 + p_{12} - (p_1 + p_{12})^n(p_2 + p_{12})^k]^{-1} \\ & \cdot \{a_{n-1,k-1}[(p_1 + p_{12})^{n-1}(p_2 + p_{12})^{k-1} - p_{12}] \\ & + a_{n,k-1}[(p_1 + p_{12})^n(p_2 + p_{12})^{k-1} - p_2 - p_{12}] \\ & + a_{n-1,k}[(p_1 + p_{12})^{n-1}(p_2 + p_{12})^k - p_1 - p_{12}]\} \end{aligned}$$

Now, we shall consider the stability of this characteristic theorem.

2. Stability theorems

Suppose that X and Y are two n-dimensional random vectors with the characteristic functions $\varphi_X(t)$ and $\varphi_Y(t)$ respectively. In [4], the metric $\lambda(X;Y)$ was defined as follows

$$\lambda(X;Y) = \lambda(\varphi_X;\varphi_Y) = \sup_{T>0}\{max\{v(X,Y;T);\frac{1}{T}\}\} \tag{2.1}$$

where

$$v(X,Y;T) = \frac{1}{2}max\{|\varphi_X(t) - \varphi_Y(t)|; ||t|| < T\} \tag{2.2}$$

and $\varphi_X(t) = Ee^{i(t,X)}$, where $(.,.)$ denotes the scalar product in the space $\mathbb{R}^n$ and $||t|| = \sqrt{(t,t)}$ with $t \in \mathbb{R}^n$.

Theorem 2.1. *Let us consider the 2-dimensional characteristic function*

$$\varphi_0(t_1,t_2) = [1 - it_1 - it_2 + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{n+k}a_{n,k}t_1^n t_2^k]^{-1}, \tag{2.3}$$

where $a_{n,k}$ was given in (1.5).

If X_j and Y_j (with $j = 1,..,n$) has the same ϵ-exponential distribution, i.e. $\exists\, T_1(\epsilon) > 0, T_2(\epsilon) > 0$ (such that $T_1(\epsilon) \to \infty$ and $T_2(\epsilon) \to \infty$ when $\epsilon \to 0$) and such that

$$|\varphi_{X_j}(t_1) - \frac{1}{1-it_1}| \leq \epsilon \quad \forall t_1, \quad |t_1| \leq T_1(\epsilon), \quad \forall j, \tag{2.4}$$

$$|\varphi_{Y_j}(t_2) - \frac{1}{1-it_2}| \leq \epsilon \quad \forall t_2, \quad |t_2| \leq T_2(\epsilon), \quad \forall j, \tag{2.5}$$

then, for every characteristic function $\varphi(t_1,t_2)$ of the random vector (X_j,Y_j), we always have the estimation

$$\lambda(\varphi;\varphi_0) = \lambda[\varphi(t_1,t_2);\varphi_0(t_1,t_2)] \leq max(C_1\epsilon;\frac{1}{T^*(\epsilon)}), \tag{2.6}$$

where $T^*(\epsilon) = min[T_1(\epsilon);T_2(\epsilon)]$ and C is a constant independent of ϵ.

Proof of the Theorem 2.1. From the proof of Theorem 2 in [3] or see [5], we have

$$\varphi(t_1,t_2) = \varphi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2][p_{12}+p_1\varphi(0,t_2)+p_2\varphi(t_1,0)+q\varphi(t_1,t_2)]$$

and

$$\varphi(t_1,t_2) = \frac{\varphi[(p_{12}+p_1)t_1,(p_{12}+p_2)t_2][p_{12}+p_1\varphi(0,t_2)+p_2\varphi(t_1,0)]}{1-q\varphi[(p_{12}+p_1)t_1,(p_{12}+p_2)t_2]}. \tag{2.7}$$

Thus, we shall have the estimation

$$|\varphi(t_1,t_2) - \varphi_0(t_1,t_2)|$$
$$= |\frac{\varphi[(p_{12}+p_1)t_1,(p_{12}+p_2)t_2][p_{12}+p_1\varphi(0,t_2)+p_2\varphi(t_1,0)]}{1-q\varphi[(p_{12}+p_1)t_1,(p_{12}+p_2)t_2]} - \varphi_0(t_1,t_2)|. \tag{2.8}$$

But from (2.4) and (2.5), $\exists\, T^*(\epsilon) = min\{T_1(\epsilon);T_2(\epsilon)]$ such that

$$\varphi(0,t_2) = \frac{1}{1-it_2} + r_2(t_2) \;\; where \;\; |r_2(t_2)| \leq \epsilon, \;\; \forall t_2, \;\; |t_2| \leq T^*(\epsilon) \tag{2.9}$$

$$\varphi(t_1,0)=\frac{1}{1-it_1}+r_1(t_1)\ \ where\ \ |r_1(t_1)|\le\epsilon,\ \ \forall t_1,\ \ |t_1|\le T^*(\epsilon). \quad (2.10)$$

On the other hand, from formula (2.8) of the proof of the Theorem 2 in [3], we obtain also the following equality

$$\varphi_0(t_1,t_2)=\frac{\varphi_0[(p_{12}+p_1)t_1,(p_{12}+p_2)t_2][p_{12}+\frac{p_1}{1-it_1}+\frac{p_2}{1-it_2}]}{1-q\varphi_0[(p_{12}+p_1)t_1,(p_{12}+p_2)t_2]}. \quad (2.11)$$

Taking into account (2.8), (2.9), (2.10) and (2.11) we get

$$|\varphi(t_1,t_2)-\varphi_0(t_1,t_2)|=|\frac{\varphi_0[(p_{12}+p_1)t_1,(p_{12}+p_2)t_2]}{1-q\varphi_0[(p_{12}+p_1)t_1,(p_{12}+p_2)t_2]}||r^*(t_1,t_2)|, \quad (2.12)$$

where $r^*(t_1,t_2)=p_1r_1(t_1)+p_2r_2(t_2)$ and from (2.9) and (2.10) we notice that

$$|r^*(t_1,t_2)|=|p_1r_1(t_1)+p_2r_2(t_2)|\le C\epsilon,$$

for all $|t_1|\le T_1(\epsilon),|t_2|\le T_2(\epsilon)$.

On the other hand, we always have the inequalities:

$$|1-qz|\ge|1-q|z||\ge 1-q \quad (2.13)$$

for all complex number $z,|z|\le 1$.
So, we have

$$|\varphi(t_1,t_2)-\varphi_0(t_1,t_2)|\le\frac{r^*(t_1,t_2)}{1-q}\le\frac{C\epsilon}{1-q}=C_1\epsilon, \quad (2.14)$$

where C_1 is a constant of ϵ. The proof Theorem 2.1 is completed.

Let us denote the characteristic function corresponding to $\overline{G_p}(x,y)$ by $\psi_p(t_1,t_2)$. Now, we consider the second stability theorem.

Theorem 2.2. *If both X_j and Y_j have ϵ-exponential distribution $(j=1,2,...,n)$ as described in Theorem 2.1, then we have the inequality*

$$\lambda(\psi_p,\varphi_0)=\lambda[\psi_p(t_1,t_2);\varphi_0(t_1,t_2)]\le max\{C_2\epsilon;\frac{1}{T^*(\epsilon)}\} \quad (2.15)$$

Proof of Theorem 2.2. At first, denoting by $\psi(t_1,t_2)$ the characteristic function of (Z_1,Z_2), then

$$\psi_p(t_1,t_2)=\psi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2].$$

But, in the proof of Theorem 1 in [3], we have

$$\psi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2]$$
$$=\frac{\varphi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2][p_{12}+p_1\psi(0,t_2)+p_2\psi(t_1,0)]}{1-\varphi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2]}; \quad (2.16)$$

in [2], we have already proved that if X_j is ϵ-exponentially distributed then

$$|\psi(t_1,0)-\frac{1}{1-it_1}|=|r_1(t_1)|\le max_{|t_1|\le T_1(\epsilon)}\{\frac{\epsilon}{2};\frac{1}{T_1(\epsilon)}\}\quad\forall t_1,\quad |t_1|\le T_1(\epsilon) \quad (2.17)$$

and, more, if Y_j is ϵ-exponentially distributed then

$$|\psi(0,t_2)-\frac{1}{1-it_2}| = |r_2(t_2)| \leq max_{|t_2|\leq T_1(\epsilon)}\{\frac{\epsilon}{2};\frac{1}{T_2(\epsilon)}\} \quad \forall t_2, \quad |t_2| \leq T_2(\epsilon), \tag{2.18}$$

and from (2.16), (2.17) and (2.18) it follows that

$$\psi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2]$$
$$= \frac{\varphi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2][p_{12}+\frac{p_1}{1-it_1}+\frac{p_2}{1-it_2}]}{1-q\varphi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2]}$$
$$+\frac{\varphi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2][p_1r_1(t_1)+p_2r_2(t_2)]}{1-q\varphi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2]} \tag{2.19}$$

Therefore

$$|\psi_p(t_1,t_2)-\varphi_0(t_1,t_2)|$$
$$\leq ||\frac{\varphi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2][p_{12}+\frac{p_1}{1-it_1}+\frac{p_2}{1-it_2}]}{1-q\varphi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2]}-\varphi_0(t_1,t_2)|$$
$$+|\frac{\varphi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2]}{1-q\varphi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2]}||p_1r_1(t_1)+p_2r_2(t_2)| = J_1+J_2. \tag{2.20}$$

Taking into account (2.9), (2.10) and (2.13), we get

$$J_2 \leq max\{C_2\epsilon;\frac{1}{T^*(\epsilon)}\} \tag{2.21}$$

where $T^*(\epsilon) = min\{T_1(\epsilon);T_2(\epsilon)\}$ and C_2 is a constant of ϵ.

According to the proof of Theorem 2 in [3], we have

$$\varphi_0(t_1,t_2) = \frac{\varphi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2][p_{12}+\frac{p_1}{1-it_1}+\frac{p_2}{1-it_2}]}{1-q\varphi[(p_{12}+p_1)t_1,(p_{12}+p_2)t_2]}. \tag{2.22}$$

Thus, $J_1 = 0$ and we have:

$$J_1+J_2 \leq max\{C_2\epsilon;\frac{1}{T^*(\epsilon)}\}. \tag{2.23}$$

where C_2 is a constant independent of ϵ. Therefore it follows that

$$\lambda(\psi_P;\varphi_0) \leq max\{C_2\epsilon;\frac{1}{T*(\epsilon)}\} \tag{2.24}$$

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