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# On a differential operator for multivalent functions

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**Abstract.** In this article we define a differential operator for multivalent functions in the unit disk. Further, we introduce some classes of functions defined by this operator. Partial sums are also considered.

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## 1. Introduction

Let $T(p)$ denote the class of functions $f$ of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{n+p}, \quad (p \in \mathbb{N}, \;\; z \in U). \tag{1.1}$$

which are analytic and p-valent (multivalent) in the open unit disk

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Let be given two functions $f, g \in T(p)$,

$$f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{n+p}$$

and

$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n.$$

Then their *convolution* or *Hadamard product* $f(z) * g(z)$ is defined by

$$f(z) * g(z) = z^p + \sum_{n=1}^{\infty} a_n b_n z^{n+p}, \;\; (z \in U).$$

Define a function $\varphi_p(a,c;z)$ as follows

$$\varphi_p(a,c;z) := z^p + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} z^{n+p}, \ \ c \neq 0,-1,-2,...$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & n=0 \\ a(a+1)...(a+n-1), & n=\{1,2,...\}. \end{cases}$$

Assume that $a = k+p > 0$ and $c = 1$ where $k = 0,1,2,...$ in $\varphi_p(a,c;z)$ so we obtain the function

$$\varphi_p(k+p,1;z) = z^p + \sum_{n=1}^{\infty} \frac{(k+p)_n}{(1)_n} z^{n+p}. \tag{1.2}$$

Next we define the following differential operator $\mathcal{D}^k_{\lambda,p} : T(p) \to T(p)$ by

$$\begin{aligned} &D^0 f(z) = f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{n+p} \\ &D^1_{\lambda,p} f(z) = (1+\lambda p)f(z) - \lambda z f'(z) = z^p + \sum_{n=1}^{\infty} (1-\lambda n) a_n z^{n+p} \\ &\vdots \\ &D^k_{\lambda,p} f(z) = z^p + \sum_{n=1}^{\infty} (1-\lambda n)^k a_n z^{n+p}, \ \ (z \in U), \end{aligned} \tag{1.3}$$

where

$$\Big(p \in \mathbb{N}, \ \ k \in \mathbb{N}_0, \ 0 \leq \lambda < \frac{1}{n}, \quad n \in \mathbb{N}\Big).$$

Again by applying convolution product on (1.2) and (1.3) we have the following operator

$$\begin{aligned} \mathcal{D}^k_{\lambda,p} f(z) &= \frac{z^p}{(1-z)^{k+p}} * D^k_{\lambda,p} f(z) \\ &= z^p + \sum_{n=1}^{\infty} \frac{(k+p)_n}{(1)_n} (1-\lambda n)^k a_n z^{n+p} \\ &= z^p + \sum_{n=1}^{\infty} C(n,k)(1-\lambda n)^k a_n z^{n+p}, \ \ (z \in U), \end{aligned} \tag{1.4}$$

where $C(n,k) := \frac{(k+p)_n}{(1)_n}$.

**Remark 1.1.** The symbol $\mathcal{D}^k_{\lambda,p} f(z)$, when $\lambda = 0$, $p = 1$, was introduced by Ruscheweyh [1] and when $\lambda = 0$ by Goel and Sohi [2].

A function $f \in T(p)$ is said to be *p-valent starlike of order* $\mu, 0 \leq \mu < p$ if

$$\Re\Big\{\frac{zf'(z)}{f(z)}\Big\} > \mu, \ \ (z \in U).$$

The class of p-valent starlike functions of order $\mu$ is denoted by $S_p^*(\mu)$. A function $f \in T(p)$ is said to be *p-valent convex of order* $\mu, 0 \le \mu < p$ if

$$\Re\Big\{1 + \frac{zf''(z)}{f'(z)}\Big\} > \mu, \ \ (z \in U).$$

The class of p-valent convex functions of order $\mu$ is denoted by $C_p(\mu)$.
A function $f \in T(p)$ is said *to be in the class* $S_p^*(\mu, \lambda)$ *of order* $\mu$, where $0 \le \mu < p$ if

$$\Re\Big\{\frac{z[\mathcal{D}_{\lambda,p}^k f(z)]'}{\mathcal{D}_{\lambda,p}^k f(z)}\Big\} > \mu, \ \ (z \in U).$$

A function $f \in T(p)$ is said *to be in the class* $C_p(\mu, \lambda)$ *of order* $\mu$, where $0 \le \mu < p$ if

$$\Re\Big\{1 + \frac{z[\mathcal{D}_{\lambda,p}^k f(z)]''}{[\mathcal{D}_{\lambda,p}^k f(z)]'}\Big\} > \mu, \ \ (z \in U).$$

For $0 \le \alpha < p$ and $\beta \ge 0$, let $S_p^*(\alpha, \beta, \lambda)$ be the subclass of $T(p)$ consisting of functions of the form (1.1) satisfying the analytic criterion

$$\Re\Big\{\frac{z[\mathcal{D}_{\lambda,p}^k f(z)]'}{\mathcal{D}_{\lambda,p}^k f(z)} - \alpha\Big\} > \beta\Big|\frac{z[\mathcal{D}_{\lambda,p}^k f(z)]'}{\mathcal{D}_{\lambda,p}^k f(z)} - p\Big|, \ \ (z \in U). \tag{1.5}$$

Also, for $0 \le \alpha < p$ and $\beta \ge 0$, let $C_p(\alpha, \beta, \lambda)$ be the subclass of $T(p)$ satisfying the analytic criterion

$$\Re\Big\{1 + \frac{z[\mathcal{D}_{\lambda,p}^k f(z)]''}{[\mathcal{D}_{\lambda,p}^k f(z)]'} - \alpha\Big\} > \beta\Big|\frac{z[\mathcal{D}_{\lambda,p}^k f(z)]''}{[\mathcal{D}_{\lambda,p}^k f(z)]'} - (p-1)\Big|, \ \ (z \in U). \tag{1.6}$$

The main goal of this work is to determine sufficient conditions for the analytic functions to belong to these general classes. Sharp results involving partial sums $f_{m+p}(z)$ of functions $f(z)$ in the classes $S_p^*(\alpha, \beta, \lambda)$ and $C_p(\alpha, \beta, \lambda)$ are obtained.

## 2. The classes $S_p^*(\alpha, \beta, \lambda)$ and $C_p(\alpha, \beta, \lambda)$

In this section we obtain sufficient conditions for functions $f(z)$ to be in the classes $S_p^*(\alpha, \beta, \lambda)$ and $C_p(\alpha, \beta, \lambda)$.
**Theorem 2.1.** A sufficient condition for a function $f(z)$ of the form (1.1) to be in $S_p^*(\alpha, \beta, \lambda)$ is

$$\sum_{n=1}^{\infty}[(1+\beta)n + (p-\alpha)]C(n,k)(1-\lambda n)^k|a_n| < p - \alpha, \ \ (z \in U), \tag{2.1}$$

for $0 \le \alpha < p$, $\beta \ge 0$ and $0 \le \lambda < \frac{1}{n}, \quad n \in \mathbb{N}$.
*Proof.* It suffices to show that

$$\beta\Big|\frac{z[\mathcal{D}_{\lambda,p}^k f(z)]'}{\mathcal{D}_{\lambda,p}^k f(z)} - p\Big| - \Re\Big\{\frac{z[\mathcal{D}_{\lambda,p}^k f(z)]'}{\mathcal{D}_{\lambda,p}^k f(z)} - p\Big\} \le p - \alpha, \ \ (z \in U).$$

We have

$$\beta\Big|\frac{z[\mathcal{D}^k_{\lambda,p}f(z)]'}{\mathcal{D}^k_{\lambda,p}f(z)}-p\Big|-\Re\Big\{\frac{z[\mathcal{D}^k_{\lambda,p}f(z)]'}{\mathcal{D}^k_{\lambda,p}f(z)}-p\Big\}\le(1+\beta)\Big|\frac{z[\mathcal{D}^k_{\lambda,p}f(z)]'}{\mathcal{D}^k_{\lambda,p}f(z)}-p\Big|$$

$$\le\frac{(1+\beta)\sum_{n=1}^{\infty}nC(n,k)(1-\lambda n)^k|a_n||z|^{n+p}}{1-\sum_{n=1}^{\infty}C(n,k)(1-\lambda n)^k|a_n||z|^{n+p}}$$

$$\le\frac{(1+\beta)\sum_{n=1}^{\infty}nC(n,k)(1-\lambda n)^k|a_n|}{1-\sum_{n=p+1}^{\infty}C(n,k)(1-\lambda n)^k|a_n|}.$$

This last expression is bounded above by $(p-\alpha)$ if

$$\sum_{n=1}^{\infty}[(1+\beta)n+(p-\alpha)]C(n,k)(1-\lambda n)^k|a_n|<p-\alpha,$$

and the proof is complete.

By setting $\beta=\lambda=0$ (Goel-Sohi operator [2]) in Theorem 2.1, we obtain the following result:

**Corollary 2.2.** Let $f$ be given by (1.1) and satisfying

$$\sum_{n=2}^{\infty}(n+p-\alpha)C(n,k)|a_n|\le p-\alpha,\quad(0\le\alpha<p,z\in U)$$

then $f\in S_p^*(\alpha)$ (p-valent starlike).

By letting $\beta=\lambda=0$ and $p=1$ (Ruscheweyh operator [1]) in Theorem 2.1, we obtain the following result:

**Corollary 2.3.** Let $f$ be given by (1.1) and satisfying

$$\sum_{n=2}^{\infty}(n+1-\alpha)C(n,k)|a_n|\le 1-\alpha,\quad(0\le\alpha<1,z\in U)$$

then $f\in S^*(\alpha)$ (starlike).

In the same manner we can obtain the next result.

**Theorem 2.4.** A sufficient condition for a function $f$ of the form (1.1) to be in $C_p(\alpha,\beta,\lambda)$ is

$$\sum_{n=1}^{\infty}(n+p)[n(1+\beta)+(p-\alpha)]C(n,k)(1-\lambda n)^k|a_n|<p(p-\alpha),\ \ (p\in\mathbb{N},\ z\in U),\tag{2.2}$$

for $0\le\alpha<p$ and $\beta\ge0$.

*Proof.* It suffices to show that

$$\beta\Big|\frac{z[\mathcal{D}^k_{\lambda,p}f(z)]''}{[\mathcal{D}^k_{\lambda,p}f(z)]'}-(p-1)\Big|-\Re\Big\{\frac{z[\mathcal{D}^k_{\lambda,p}f(z)]''}{[\mathcal{D}^k_{\lambda,p}f(z)]'}-(p-1)\Big\}\le p-\alpha,$$

$$(p\in\mathbb{N},\ 0\le\alpha<p,\ \beta\ge0,\ z\in U).$$

Then we have

$$\beta\Big|\frac{z[\mathcal{D}^k_{\lambda,p}f(z)]''}{[\mathcal{D}^k_{\lambda,p}f(z)]'}-(p-1)\Big|-\Re\Big\{\frac{z[\mathcal{D}^k_{\lambda,p}f(z)]''}{[\mathcal{D}^k_{\lambda,p}f(z)]'}-(p-1)\Big\}$$

$$\leq (1+\beta)\Big|\frac{z[\mathcal{D}^k_{\lambda,p}f(z)]''}{[\mathcal{D}^k_{\lambda,p}f(z)]'}-(p-1)\Big|$$

$$\leq \frac{(1+\beta)\sum_{n=1}^{\infty} n(n+p)C(n,k)(1-\lambda n)^k|a_n||z|^{n+p-1}}{p|z|^{p-1}-\sum_{n=1}^{\infty}(n+p)C(n,k)(1+\lambda n)^k|a_n||z|^{n+p-1}}$$

$$\leq \frac{(1+\beta)\sum_{n=1}^{\infty} n(n+p)C(n,k)(1+\lambda n)^k|a_n|}{p-\sum_{n=1}^{\infty}(n+p)C(n,k)(1-\lambda n)^k|a_n|}.$$

This last expression is bounded above by $(p-\alpha)$ if

$$\sum_{n=1}^{\infty}(n+p)[n(1+\beta)+(p-\alpha)]C(n,k)(1-\lambda n)^k|a_n| < p(p-\alpha),\ \ (p\in\mathbb{N}).$$

This completes the proof.

## 3. Partial sums

In this section, applying methods used by Silverman [3] and Silvia [4], we will investigate the ratio of a function $f(z)$ of the form (1.1) to its sequence of partial sums

$$f_{m+p}(z) = z^p + \sum_{n=1}^{m} a_n z^{n+p},\quad (z\in U) \tag{3.1}$$

when the coefficients are small enough in order to satisfy either condition (2.1) or (2.2). More precisely, we will determine sharp lower bounds for

$$\Re\Big\{\frac{f(z)}{f_{m+p}(z)}\Big\}, \Re\Big\{\frac{f_{m+p}(z)}{f(z)}\Big\}, \Re\Big\{\frac{f'(z)}{f'_{m+p}(z)}\Big\} \text{ and } \Re\Big\{\frac{f'_{m+p}(z)}{f'(z)}\Big\}.$$

In the sequel, we will make use of the fact that

$$\Re\Big\{\frac{(1+w(z))}{(1-w(z))}\Big\} > 0,\ \ (z\in U)$$

if and only if $w(z)=\sum_{n=1}^{\infty} c_n z^n$ satisfies the inequality $|w(z)|<|z|$.

**Theorem 3.1.** Let $f$ given by (1.1) and satisfies (2.1). Then

$$\Re\Big\{\frac{f(z)}{f_{m+p}(z)}\Big\} > 1-\frac{p-\alpha}{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}, \tag{3.2}$$

$$\Big(z\in U,\ p>\alpha, m=0,1,2,...,\quad 0\leq\lambda<\frac{1}{m+p+1}\Big).$$

The result is sharp for every $m$ with the extremal function

$$f(z) = z^p+\frac{p-\alpha}{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}z^{m+p+1}, \tag{3.3}$$

$$(z\in U,\ \ m\geq 0,\ \ p>\alpha).$$

*Proof.* Assume that $f \in T(p)$ satisfies (2.1). By setting

$$w(z) = \tfrac{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}{p-\alpha}\Big\{\tfrac{f(z)}{f_{m+p}(z)}$$

$$-\Big(1-\tfrac{p-\alpha}{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}\Big)\Big\}$$

$$:= 1 + \frac{H_{m+p+1}\sum_{n=m+1}^{\infty} a_n z^n}{1+\sum_{n=1}^{m} a_n z^n},$$

where

$$H_{m+p+1} := \tfrac{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}{p-\alpha}.$$

Thus we find that

$$\Big|\frac{w(z)-1}{w(z)+1}\Big| \leq \frac{H_{m+p+1}\sum_{n=m+1}^{\infty}|a_n|}{2-2\sum_{n=1}^{m}|a_n| - H_{m+p+1}\sum_{n=m+1}^{\infty}|a_n|}$$

$$\leq 1, \quad (z \in U)$$

if and only if

$$2H_{m+p+1}\sum_{n=m+1}^{\infty}|a_n| \leq 2-2\sum_{n=1}^{m}|a_n|$$

which is equivalent to

$$\sum_{n=1}^{m}|a_n| + H_{m+p+1}\sum_{n=m+1}^{\infty}|a_n| \leq 1. \tag{3.4}$$

In order to see that

$$f(z) = z^p + \frac{z^{m+p+1}}{H_{m+p+1}}, \quad (z \in U)$$

gives a sharp result, we observe that for

$$z = re^{\frac{\pi i}{m+p}}, \quad (z \in U)$$

that

$$\frac{f(z)}{f_{m+p}(z)} = 1 + \frac{z^{m+p}}{H_{m+p+1}} \to 1 - \frac{1}{H_{m+p+1}} \quad as \quad z \to 1^-.$$

This completes the proof.

**Theorem 3.2.** Let $f$ given by (1.1) satisfying (2.1). Then

$$\Re\Big\{\tfrac{f_{m+p}(z)}{f(z)}\Big\} > \tfrac{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}{(p-\alpha)+[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}, \tag{3.5}$$

$$\Big(z \in U,\ p > \alpha,\ \ m = 0,1,2,...,\quad 0 \leq \lambda < \frac{1}{m+p+1}\Big).$$

The result is sharp for every $m$ with an extremal function given by (3.3).

*Proof.* Assume that $f \in T(p)$ and satisfies (2.1). Write

$$w(z) = \left(1 + \frac{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}{p-\alpha}\right)\left\{\frac{f_{m+p}(z)}{f(z)}\right.$$

$$\left. - \frac{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}{(p-\alpha)+[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}\right\}$$

$$= 1 - \frac{\left(1 + H_{m+p+1}\right)\sum_{n=m+1}^{\infty} a_n z^n}{1 + \sum_{n=1}^{m} a_n z^n}$$

where $H_{m+p+1}$ is defined in Theorem 3.1. This yields that

$$\left|\frac{w(z)-1}{w(z)+1}\right| \leq \frac{(1 + H_{m+p+1})\sum_{n=m+1}^{\infty} |a_n|}{2 - 2\sum_{n=p+1}^{m} |a_n| - (1 + H_{m+p+1})\sum_{n=m+1}^{\infty} |a_n|}$$

$$\leq 1, \quad (z \in U)$$

if and only if

$$2[(1 + H_{m+p+1})\sum_{n=m+1}^{\infty} |a_n|] \leq 2 - 2\sum_{n=2}^{m} |a_n|$$

or

$$\sum_{n=p+1}^{m} |a_n| + (1 + H_{m+p+1})\sum_{n=m+1}^{\infty} |a_n| \leq 1, \tag{3.6}$$

which gives (3.5). The bound in (3.5) is sharp for all $m \in \mathbb{N}$ with the extremal function given by (3.3). This completes the proof.

**Theorem 3.3.** Let $f$ given by (1.1) satisfies (2.1). Then

$$\Re\left\{\frac{f'(z)}{f'_{m+p}(z)}\right\} \geq 1 - \frac{(m+p+1)(p-\alpha)}{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}, \tag{3.7}$$

$$\left(z \in U, \;\; p > \alpha, \;\; m = 0, 1, 2, ..., 0 \leq \lambda < \frac{1}{m+p+1}\right).$$

*Proof.* Assume that $f \in T(p)$ satisfies (2.1). Write

$$w(z) = \frac{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}{p-\alpha}\left\{\frac{f'(z)}{f'_{m+p}(z)}\right.$$

$$\left. -\left(1 - \frac{(m+p+1)(p-\alpha)}{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}\right)\right\}$$

$$= \frac{1 + \frac{H_{m+p+1}}{(m+p+1)}\sum_{n=m+1}^{\infty} \frac{n+p}{p} a_n z^n + \sum_{n=1}^{\infty} \frac{n+p}{p} a_n z^n}{1 + \sum_{n=1}^{m} \frac{n+p}{p} a_n z^n}$$

$$= 1 + \frac{\frac{H_{m+p+1}}{(m+p+1)}\sum_{n=m+1}^{\infty} \frac{n+p}{p} a_n z^n}{1 + \sum_{n=1}^{m} \frac{n+p}{p} a_n z^n},$$

where $H_{m+p+1}$ is defined in Theorem 3.1. This implies

$$\Big|\frac{w(z)-1}{w(z)+1}\Big| \le \frac{\frac{H_{m+p+1}}{m+p+1}\sum_{n=m+1}^{\infty}\frac{n+p}{p}|a_n|}{2-2\sum_{n=1}^{m}\frac{n+p}{p}|a_n|-\frac{H_{m+p+1}}{m+p+1}\sum_{n=m+1}^{\infty}\frac{n+p}{p}|a_n|}$$

$$\le 1, \quad (z\in U)$$

if and only if

$$2[\frac{H_{m+p+1}}{m+p+1}\sum_{n=m+1}^{\infty}\frac{n}{p}|a_n|] \le 2-2\sum_{n=p+1}^{m}\frac{n}{p}|a_n|,$$

i.e.

$$\sum_{n=1}^{m+p}\frac{n}{p}|a_n| + \frac{H_{m+p+1}}{m+p+1}\sum_{n=m+1}^{\infty}\frac{n}{p}|a_n| \le 1.$$

We therefore obtain (3.7). The result is sharp with functions given by (3.3). The proof of the Theorem 3.3 is completed.

**Theorem 3.4.** Let $f$ given by (1.1) satisfying (2.1). Then

$$\Re\Big\{\frac{f_m'(z)}{f'(z)}\Big\} \ge \frac{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}{(m+p+1)(p-\alpha)+[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}, \tag{3.8}$$

$$\Big(z\in U,\ \ p>\alpha,\ \ m=0,1,2,...,\quad 0\le\lambda<\frac{1}{m+p+1}\Big).$$

*Proof.* Assume that $f\in T(p)$ satisfies (2.1). Consider

$$w(z)=\Big((m+p+1)+\frac{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}{p-\alpha}\Big)\Big\{\frac{f_m'(z)}{f'(z)}$$

$$-\frac{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}{(m+p+1)(p-\alpha)+[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}\Big\}$$

$$=1-\frac{(1+\frac{H_{m+p+1}}{m+p+1})\sum_{n=m+1}^{\infty}\frac{n+p}{p}a_nz^n}{1+\sum_{n=2}^{m}\frac{n+p}{p}a_nz^n}.$$

This implies that

$$\Big|\frac{w(z)-1}{w(z)+1}\Big| \le \frac{(1+\frac{H_{m+p+1}}{m+p+1})\sum_{n=m+1}^{\infty}\frac{n+p}{p}|a_n|}{2-2\sum_{n=1}^{m}\frac{n+p}{p}|a_n|-(1+\frac{H_{m+p+1}}{m+p+1})\sum_{n=m+1}^{\infty}\frac{n+p}{p}|a_n|}$$

$$\le 1, \quad (z\in U)$$

if and only if

$$2[(1+\frac{H_{m+p+1}}{m+p+1})\sum_{n=m+1}^{\infty}\frac{n+p}{p}|a_n|] \le 2-2\sum_{n=1}^{m}\frac{n+p}{p}|a_n|,$$

i.e.

$$\sum_{n=1}^{m} \frac{n+p}{p}|a_n| + (1 + \frac{H_{m+p+1}}{m+p+1}) \sum_{n=m+1}^{\infty} \frac{n+p}{p}|a_n| \leq 1.$$

We therefore obtain (3.8). The result is sharp with functions given by (3.3). The proof of Theorem 3.4 is complete.

In the same manner as the proof of Theorems 3.1-3.4, we can show the following results:

**Theorem 3.5.** Let $f$ given by (1.1) satisfying (2.2). Then

$$\Re\left\{\frac{f(z)}{f_{m+p}(z)}\right\} > 1 - \frac{p(p-\alpha)}{(m+p+1)[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}. \tag{3.9}$$

The result is sharp for every $m$ with the extremal function

$$f(z) = z^p + \frac{p(p-\alpha)}{(m+p+1)[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k} z^{m+p+1}, \tag{3.10}$$

$$\left(z \in U,\ \ p > \alpha,\ \ m = 0, 1, 2, ...,\quad 0 \leq \lambda < \frac{1}{m+p+1}\right).$$

**Theorem 3.6.** Let $f$ given by (1.1) satisfies (2.2). Then

$$\Re\left\{\frac{f_{m+p}(z)}{f(z)}\right\} > \frac{(m+p+1)[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}{p(p-\alpha)+(m+p+1)[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}, \tag{3.11}$$

$$\left(z \in U,\ \ p > \alpha,\ \ m = 0, 1, 2, ...,\quad 0 \leq \lambda < \frac{1}{m+p+1}\right).$$

The result is sharp for every $m$ with the extremal function given by (3.10).

**Theorem 3.7.** Let $f$ given by (1.1) satisfies (2.2). Then

$$\Re\left\{\frac{f'(z)}{f'_{m+p}(z)}\right\} \geq 1 - \frac{p(m+p+1)(p-\alpha)}{(m+p+1)[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}, \tag{3.12}$$

$$\left(z \in U,\ \ p > \alpha,\ \ m = 0, 1, 2, ...,\quad 0 \leq \lambda < \frac{1}{m+p+1}\right).$$

**Theorem 3.8.** Let $f$ given by (1.1) satisfies (2.2). Then

$$\Re\left\{\frac{f'_m(z)}{f'(z)}\right\}$$

$$\geq \frac{(m+p+1)[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}{p(m+p+1)(p-\alpha)+(m+p+1)[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k} \tag{3.13}$$

where

$$\left(z \in U,\ \ p > \alpha,\ \ m = 0, 1, 2, ...,\quad 0 \leq \lambda < \frac{1}{m+p+1}\right).$$

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