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# The extensions for the univalence conditions of certain general integral operators

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**Abstract.** In this paper, we generalize certain integral operators given by Pescar [8] and determine conditions for univalence of these general integral operators.

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## 1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$$

and satisfy the following usual normalization condition

$$f(0) = f'(0) - 1 = 0.$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{U}$.

In [6] and [7], Pescar gave the following univalence conditions for the functions $f \in \mathcal{A}$.

**Theorem 1.1.** [6] *Let $\alpha$ be a complex number, $\Re(\alpha) > 0$, and $c$ be a complex number, $|c| \leq 1$, $c \neq -1$ and $f(z) = z + \cdots$ a regular function in $\mathbb{U}$. If*

$$\left| c\,|z|^{2\alpha} + \left(1 - |z|^{2\alpha}\right) \frac{zf''(z)}{\alpha f'(z)} \right| \leq 1,$$

*for all* $z \in \mathbb{U}$*, then the function*

$$F_\alpha(z) = \left(\alpha \int_0^z t^{\alpha-1} f'(t)dt\right)^{\frac{1}{\alpha}} = z + \cdots$$

*is regular and univalent in* $\mathbb{U}$*.*

**Theorem 1.2.** [7] *Let* $\alpha$ *be a complex number,* $\Re(\alpha) > 0$*, and* $c$ *be a complex number,* $|c| \leq 1$*,* $c \neq -1$ *and* $f \in \mathcal{A}$*. If*

$$\frac{1 - |z|^{2\Re(\alpha)}}{\Re(\alpha)} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 - |c|,$$

*for all* $z \in \mathbb{U}$*, then for any complex number* $\beta$*,* $\Re(\beta) \geq \Re(\alpha)$*, the function*

$$F_\beta(z) = \left(\beta \int_0^z t^{\beta-1} f'(t)dt\right)^{\frac{1}{\beta}}$$

*is in the class* $\mathcal{S}$*.*

On the other hand, for the functions $f \in \mathcal{A}$, Ozaki and Nunokawa [5] proved another univalence condition asserted by Theorem 1.3.

**Theorem 1.3.** [5] *Let* $f \in \mathcal{A}$ *satisfy the condition*

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| \leq 1 \quad (z \in \mathbb{U}). \tag{1.1}$$

*Then* $f$ *is univalent in* $\mathbb{U}$*.*

Furthermore in [8], Pescar determined necessary conditions for univalence of some integral operators.

**Theorem 1.4.** [8] *Let the function* $g \in \mathcal{A}$ *satisfy (*1.1*),* $M$ *be a positive real number fixed and* $c$ *be a complex number. If*

$$\alpha \in \left[\frac{2M+1}{2M+2}, \frac{2M+1}{2M}\right],$$

$$|c| \leq 1 - \left|\frac{\alpha - 1}{\alpha}\right| (2M+1), \ c \neq -1$$

*and*

$$|g(z)| \leq M$$

*for all* $z \in \mathbb{U}$*, then the function*

$$G_\alpha(z) = \left(\alpha \int_0^z (g(t))^{\alpha-1} dt\right)^{\frac{1}{\alpha}} \tag{1.2}$$

*is in the class* $\mathcal{S}$*.*

**Theorem 1.5.** [8] *Let $g \in \mathcal{A}$, $\alpha$ be a real number, $\alpha \geq 1$, and $c$ be a complex number, $|c| \leq \frac{1}{\alpha}$, $c \neq -1$. If*

$$\left| \frac{g''(z)}{g'(z)} \right| \leq 1 \quad (z \in \mathbb{U}),$$

*then the function*

$$H_\alpha(z) = \left( \alpha \int_0^z (tg'(t))^{\alpha-1} \, dt \right)^{\frac{1}{\alpha}} \tag{1.3}$$

*is in the class $\mathcal{S}$.*

**Theorem 1.6.** [8] *Let $g \in \mathcal{A}$ satisfies (1.1), $\alpha$ be a complex number, $M > 1$ fixed, $\Re(\alpha) > 0$ and $c$ be a complex number, $|c| < 1$. If*

$$|g(z)| \leq M$$

*for all $z \in \mathbb{U}$, then for any complex number $\beta$*

$$\Re(\beta) \geq \Re(\alpha) \geq \frac{2M+1}{|\alpha|(1-|c|)},$$

*the function*

$$H_\beta(z) = \left( \beta \int_0^z t^{\beta-1} \left( \frac{g(t)}{t} \right)^{\frac{1}{\alpha}} dt \right)^{\frac{1}{\beta}} \tag{1.4}$$

*is in the class $\mathcal{S}$.*

Finally, Breaz and Breaz [1] considered the following family of integral operators and proved that the function $G_{n,\alpha}$ defined by

$$G_{n,\alpha}(z) = \left( [n(\alpha-1)+1] \int_0^z \prod_{j=1}^n (g_j(t))^{\alpha-1} \, dt \right)^{\frac{1}{n(\alpha-1)+1}} \quad (g_1, \ldots, g_n \in \mathcal{A}) \tag{1.5}$$

is univalent in $\mathbb{U}$. For some recent investigations of the integral operator $G_{n,\alpha}$, see the works by Breaz et al. [2] and [3].

Now we introduce two new general integral operators as follows:

$$H_{n,\alpha}(z) := \left( [n(\alpha-1)+1] \int_0^z \prod_{j=1}^n \left( tg_j'(t) \right)^{\alpha-1} dt \right)^{\frac{1}{n(\alpha-1)+1}} \quad (g_1, \ldots, g_n \in \mathcal{A}), \tag{1.6}$$

$$H_{n,\beta}(z) := \left( [n(\beta-1)+1] \int_0^z t^{n(\beta-1)} \prod_{j=1}^n \left( \frac{g_j(t)}{t} \right)^{\frac{1}{\alpha}} dt \right)^{\frac{1}{n(\beta-1)+1}} \quad (g_1, \ldots, g_n \in \mathcal{A}). \tag{1.7}$$

**Remark 1.7.** *For $n = 1$, the integral operators in (1.5), (1.6) and (1.7) would reduce to the integral operators in (1.2), (1.3) and (1.4), respectively.*

In this paper, we investigate univalence conditions involving the general family of integral operators defined by (1.5), (1.6) and (1.7). For this purpose, we need the following result.

**General Schwarz Lemma.** [4] *Let the function $f$ be regular in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed $M$. If $f$ has one zero with multiplicity order bigger than $m$ for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m \quad (z \in \mathbb{U}_R).$$

*The equality can hold only if*

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

*where $\theta$ is constant.*

## 2. Main Results

**Theorem 2.1.** *Let $M > 0$ and the functions $g_j \in \mathcal{A}$ $(j \in \{1, \ldots, n\})$ satisfies the inequality (1.1). Also let*

$$\alpha \in \mathbb{R} \quad \left(\alpha \in \left[\frac{(2M+1)\,n}{(2M+1)\,n+1}, \frac{(2M+1)\,n}{(2M+1)\,n-1}\right]\right) \quad \text{and} \quad c \in \mathbb{C}.$$

*If*

$$|c| \leq 1 - \left|\frac{\alpha - 1}{n(\alpha-1)+1}\right| (2M+1)\,n, \quad c \neq -1 \tag{2.1}$$

*and*

$$|g_j(z)| \leq M \quad (z \in \mathbb{U};\ j \in \{1, \ldots, n\}),$$

*then the function $G_{n,\alpha}$ defined by (1.5) is in the class $\mathcal{S}$.*

*Proof.* Define a function

$$h(z) = \int_0^z \prod_{j=1}^{n} \left(\frac{g_j(t)}{t}\right)^{\alpha-1} dt.$$

Then we obtain

$$h'(z) = \prod_{j=1}^{n} \left(\frac{g_j(z)}{z}\right)^{\alpha-1}.$$

Also, a simple computation yields

$$\frac{zh''(z)}{h'(z)} = (\alpha - 1) \sum_{j=1}^{n} \left(\frac{zg_j'(z)}{g_j(z)} - 1\right),$$

which readily shows that

$$\begin{aligned}
&\left| c\,|z|^{2[n(\alpha-1)+1]} + \left(1-|z|^{2[n(\alpha-1)+1]}\right) \frac{zh''(z)}{[n(\alpha-1)+1]\,h'(z)} \right| \\
\leq\;& |c| + \frac{1}{|n(\alpha-1)+1|} \left| \frac{zh''(z)}{h'(z)} \right| \\
\leq\;& |c| + \left| \frac{\alpha-1}{n(\alpha-1)+1} \right| \sum_{j=1}^{n} \left( \left| \frac{z^2 g_j'(z)}{(g_j(z))^2} \right| \left| \frac{g_j(z)}{z} \right| + 1 \right).
\end{aligned}$$

Since

$$|g_j(z)| \leq M \;\; (z \in \mathbb{U};\; j \in \{1,\ldots,n\}),$$

by using the inequality (1.1) and the general Schwarz lemma, we obtain

$$\begin{aligned}
&\left| c\,|z|^{2[n(\alpha-1)+1]} + \left(1-|z|^{2[n(\alpha-1)+1]}\right) \frac{zh''(z)}{[n(\alpha-1)+1]\,h'(z)} \right| \\
\leq\;& |c| + \left| \frac{\alpha-1}{n(\alpha-1)+1} \right| (2M+1)\,n,
\end{aligned}$$

which, by (2.1), yields

$$\left| c\,|z|^{2[n(\alpha-1)+1]} + \left(1-|z|^{2[n(\alpha-1)+1]}\right) \frac{zh''(z)}{[n(\alpha-1)+1]\,h'(z)} \right| \leq 1 \quad (z \in \mathbb{U}).$$

Applying Theorem 1.1, we conclude that the function $G_{n,\alpha}$ defined by (1.5) is in the class $\mathcal{S}$. □

**Remark 2.2.** *Setting* $n = 1$ *in Theorem* 2.1*, we have Theorem 1.4.*

**Theorem 2.3.** *Let* $g_j \in \mathcal{A}$ $(j \in \{1,\ldots,n\})$*,* $\alpha$ *be a real number,* $\alpha \geq 1$*, and* $c$ *be a complex number with*

$$|c| \leq \frac{1}{n(\alpha-1)+1}, \quad c \neq -1. \tag{2.2}$$

*If*

$$\left| \frac{g_j''(z)}{g_j'(z)} \right| \leq 1 \quad (z \in \mathbb{U};\; j \in \{1,\ldots,n\}), \tag{2.3}$$

*then the function* $H_{n,\alpha}$ *defined by* (1.6) *is in the class* $\mathcal{S}$*.*

*Proof.* Define a function

$$h(z) = \int_0^z \prod_{j=1}^{n} \left(g_j'(t)\right)^{\alpha-1} dt.$$

Then we obtain

$$h'(z) = \prod_{j=1}^{n} \left(g_j'(z)\right)^{\alpha-1}.$$

Also, a simple computation yields

$$\frac{zh''(z)}{h'(z)} = (\alpha-1) \sum_{j=1}^{n} \frac{zg_j''(z)}{g_j'(z)},$$

which readily shows that

$$\begin{aligned} & \left| c\,|z|^{2[n(\alpha-1)+1]} + \left(1-|z|^{2[n(\alpha-1)+1]}\right) \frac{zh''(z)}{[n(\alpha-1)+1]\,h'(z)} \right| \\ \leq & \quad |c| + \frac{1}{n(\alpha-1)+1} \left| \frac{zh''(z)}{h'(z)} \right| \\ \leq & \quad |c| + \left( \frac{\alpha-1}{n(\alpha-1)+1} \right) \sum_{j=1}^{n} \left| \frac{zg_j''(z)}{g_j'(z)} \right| . \end{aligned}$$

By (2.2) and (2.3), we obtain

$$\left| c\,|z|^{2[n(\alpha-1)+1]} + \left(1-|z|^{2[n(\alpha-1)+1]}\right) \frac{zh''(z)}{[n(\alpha-1)+1]\,h'(z)} \right| \leq 1 \quad (z \in \mathbb{U}).$$

Applying Theorem 1.1, we conclude that the function $H_{n,\alpha}$ defined by (1.6) is in the class $\mathcal{S}$. □

**Remark 2.4.** *Setting $n = 1$ in Theorem 2.3, we have Theorem 1.5.*

**Theorem 2.5.** *Let $M > 0$ and the functions $g_j \in \mathcal{A}$ $(j \in \{1, \ldots, n\})$ satisfies the inequality (1.1). Also let $\alpha$ be a complex number, $\Re(\alpha) > 0$, and $c$ be a complex number, $|c| < 1$. If*

$$|g_j(z)| \leq M \,(z \in \mathbb{U};\; j \in \{1, \ldots, n\}),$$

*then for any complex number $\beta$ with*

$$\Re\,(n(\beta-1)+1) \geq \Re\,(\alpha) \geq \frac{(2M+1)\,n}{|\alpha|\,(1-|c|)}, \tag{2.4}$$

*the function $H_{n,\beta}$ defined by (1.7) is in the class $\mathcal{S}$.*

*Proof.* Define a function

$$h(z) = \int_0^z \prod_{j=1}^{n} \left( \frac{g_j(t)}{t} \right)^{\frac{1}{\alpha}} dt.$$

Then we obtain

$$h'(z) = \prod_{j=1}^{n} \left( \frac{g_j(z)}{z} \right)^{\frac{1}{\alpha}} .$$

Also, a simple computation yields

$$\frac{zh''(z)}{h'(z)} = \frac{1}{\alpha} \sum_{j=1}^{n} \left( \frac{zg_j'(z)}{g_j(z)} - 1 \right),$$

which readily shows that

$$\frac{1-|z|^{2\Re(\alpha)}}{\Re\,(\alpha)} \left| \frac{zh''(z)}{h'(z)} \right| \leq \frac{1}{|\alpha|\,\Re\,(\alpha)} \sum_{j=1}^{n} \left( \left| \frac{z^2 g_j'(z)}{(g_j(z))^2} \right| \left| \frac{g_j(z)}{z} \right| + 1 \right).$$

Since

$$|g_j(z)| \leq M \;\; (z \in \mathbb{U};\; j \in \{1, \ldots, n\}),$$

by using the inequality (1.1) and the general Schwarz lemma, we obtain

$$\frac{1-|z|^{2\Re(\alpha)}}{\Re(\alpha)}\left|\frac{zh''(z)}{h'(z)}\right| \leq \frac{1}{|\alpha|\,\Re(\alpha)}\,(2M+1)\,n,$$

which, by (2.4), yields

$$\frac{1-|z|^{2\Re(\alpha)}}{\Re(\alpha)}\left|\frac{zh''(z)}{h'(z)}\right| \leq 1-|c| \quad (z\in\mathbb{U}).$$

Applying Theorem 1.2, we conclude that the function $H_{n,\beta}$ defined by (1.7) is in the class $\mathcal{S}$. □

**Remark 2.6.** *Setting* $n=1$ *in Theorem* 2.5*, we have Theorem 1.6.*

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