Stud. Univ. Babeş-Bolyai Math.
Volume LVI, Number 1
March 2011, pp. 109–115

The Sălăgean integral operator and strongly starlike functions

Mohamed K. Aouf

Abstract. Let A denote the class of analytic functions $f(z)$ defined in the unit disc $U = \{z : |z| < 1\}$ and satisfying the conditions $f(0) = f'(0) - 1 = 0$. We introduce some new subclasses of strongly starlike functions defined by the Sălăgean integral operator and study their properties.

Mathematics Subject Classification (2010): 30C45.

Keywords: Strongly starlike function, strongly convex function, Sălăgean integral operator.

1. Introduction

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$. A function $f(z) \in A$ is said to be starlike of order γ if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma \ \ (z \in U) \tag{1.2}$$

for some $\gamma\,(0 \le \gamma < 1)$. We denote by $S^*(\gamma)$ the subclass of A consisting of functions which are starlike of order γ in U. Also, a function $f(z) \in A$ is said to be convex of order γ if it satisfies

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \gamma \ \ (z \in U) \tag{1.3}$$

for some γ $(0 \le \gamma < 1)$. We denote by $C(\gamma)$ the subclass of A consisting of all functions which are convex of order γ in U.

It follows from (1.2) and (1.3) that

$$f(z) \in C(\gamma) \iff zf'(z) \in S^*(\gamma), \tag{1.4}$$

the classes $S^*(\gamma)$ and $C(\gamma)$ were introduced by Robertcen [8].
If $f(z) \in A$ satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - \gamma\right)\right| < \frac{\pi}{2}\beta \quad (z \in U) \tag{1.5}$$

for some $\gamma\,(0 \le \gamma < 1)$ and $\beta(0 < \beta \le 1)$, then $f(z)$ is said to be strongly starlike of order β and type γ in U. We denote this by $f(z) \in S^*(\beta, \gamma)$.

If $f(z) \in A$ satisfies

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)} - \gamma\right)\right| < \frac{\pi}{2}\beta \quad (z \in U) \tag{1.6}$$

for some $\gamma\,(0 \le \gamma < 1)$ and $\beta(0 < \beta \le 1)$, then we say that $f(z)$ is strongly convex of order β and type γ in U. We denote by $C(\beta, \gamma)$ the class of all such functions (see also Liu [3] and Nurokawa et al. [7]). In particular, the classes $S^*(\beta, 0)$ and $C(\beta, 0)$ have been extensively studied by Mocanu [5] and Nunokawa [6].

It follows from (1.5) and (1.6) that

$$f(z) \in C(\beta, \gamma) \iff zf'(z) \in S^*(\beta, \gamma). \tag{1.7}$$

Also, we note that $S^*(1, \gamma) = S^*(\gamma)$ and $C(1, \gamma) = C(\gamma)$.

For a function $f(z) \in A$, we define the integral operator $I^n f(z)$, $n \in N_0 = N \cup \{0\}$, where $N = \{1, 2, ...\}$, by

$$I^0 f(z) = f(z), \tag{1.8}$$

$$I^1 f(z) = I\,f(z) = \int_0^z f(t)\,t^{-1} dt, \tag{1.9}$$

and (in general)

$$I^n f(z) = I(I^{n-1} f(z)). \tag{1.10}$$

It is easy to see that:

$$(i) \qquad I^n f(z) = z + \sum_{k=2}^{\infty} \frac{a_k}{k^n} z^k \quad (n \in N_0), \tag{1.11}$$

and

$$(ii) \qquad z(I^n f(z))' = I^{n-1} f(z). \tag{1.12}$$

The integral operator $I^n f(z)\,(f \in A)$ was introduced by Sălăgean [9] and studied by Aouf et al. [1]. We call the operator I^n by Sălăgean integral operator. The relation (1.12) plays an important and significant role in obtaining our results.

Using the Sălăgean integral operator, we introduce and study the properties of some new classes of analytic functions, defined as follows:

$$S_n^*(\beta,\gamma) = \{ f(z) \in A : I^n f(z) \in S^*(\beta,\gamma), \frac{z(I^n f(z))'}{I^n f(z)} \neq \gamma \text{ for all } z \in U\}$$

and

$$C_n(\beta,\gamma) = \{f(z) \in A : I^n f(z) \in C(\beta,\gamma), 1 + \frac{z(I^n f(z))''}{(I^n f(z))'} \neq \gamma \text{ for all } z \in U\}.$$

Clearly,

$$f(z) \in C_n(\beta,\gamma) \Longleftrightarrow zf'(z) \in S_n^*(\beta,\gamma). \tag{1.13}$$

We note that:

(*i*) $S_n^*(\beta,\gamma) = S^*(\beta,\gamma)$ and $C_0^*(\beta,\gamma) = C(\beta,\gamma)$;

and

(*ii*) $S_0^*(1,\gamma) = S^*(\gamma)$ and $C_0^*(1,\gamma) = C(\gamma)$.

2. Main Results

In order to give our results, we need the following lemma, which is due to Nunokawa [6].

Lemma 2.1. *Let a function* $p(z) = 1 + c_1 z + c_2 z^2 + ...$ *be analytic in* U *and* $p(z) \neq 0$ $(z \in U)$. *If there exists a point* $z_0 \in U$ *such that*

$$|\arg f(z)| < \frac{\pi}{2}\beta, \quad (|z| < |z_0|) \text{ and } |\arg p(z_0)| = \frac{\pi}{2}\beta \ (0 < \beta \leq 1),$$

then we have $\dfrac{z p_0'(z)}{p(z_0)} = ik\beta$, *where*

$$k \geq \frac{1}{2}(a + \frac{1}{a}) \quad (\text{when} \quad \arg p(z_0) = \frac{\pi}{2}\beta),$$

$$k \leq \frac{-1}{2}(a + \frac{1}{a}) \quad (\text{when} \quad \arg p(z_0) = \frac{-\pi}{2}\beta),$$

and $(p(z_0))^{\frac{1}{\beta}} = \pm ia \ (a > 0)$.

Theorem 2.2. $S_n^*(\beta,\gamma) \subset S_{n+1}^*(\beta,\gamma)$ *for each* $n \in N_0$.

Proof. Let $f(z) \in S_n^*(\beta,\gamma)$. Then we put

$$\frac{z(I^{n+1}f(z))'}{I^{n+1}f(z)} = \gamma + (1-\gamma)\, p(z), \tag{2.1}$$

where $p(z) = 1 + c_1 z + c_2 z^2 + ...$ is analytic in U and $p(z) \neq 0$ for all $z \in U$. Using (1.12) and (2.1), we have

$$\frac{I^n f(z)}{I^{n+1}f(z)} = \gamma + (1-\gamma)\, p(z). \tag{2.2}$$

Differentiating (2.2) with respect to z logarithmically, we obtain

$$\begin{aligned}\frac{z(I^n f(z))'}{I^n f(z)} &= \frac{z(I^{n+1} f(z))'}{I^{n+1} f(z)} + \frac{(1-\gamma)zp'(z)}{\gamma+(1-\gamma)p(z)} \\ &= \gamma+(1-\gamma)p(z)+\frac{(1-\gamma)zp'(z)}{\gamma+(1-\gamma)p(z)}\ ,\end{aligned}$$

or

$$\frac{z(I^n\, f(z))^{'}}{I^n f(z)} - \gamma = (1-\gamma)\, p(z) + \frac{(1-\gamma)zp'(z)}{\gamma+(1-\gamma)p(z)}\,. \tag{2.3}$$

Suppose that there exists a point $z_0 \in U$ such that

$$|\arg\ f(z)| < \frac{\pi}{2}\beta\ (|z|<|z_0|)\ \text{ and }\ |\arg p(z_0)| = \frac{\pi}{2}\beta.$$

Then, applying Lemma 2.1, we can write that

$$\frac{z_0 p^{'}(z_0)}{p(z_0)} = ik\beta\ \text{ and }\ (p(z_0))^{\frac{1}{\beta}} = \pm\, ia\ (a>0).$$

Therefore, if $\arg p(z_0) = -\frac{\pi}{2}\beta$, then

$$\begin{aligned}\frac{z_0(I^n f(z_0))'}{I^n f(z_0)} - \gamma &= (1-\gamma)\, p(z_0)\left[1+\frac{\frac{zp'(z_0)}{p(z_0)}}{\gamma+(1-\gamma)p(z_0)}\right] \\ &= (1-\gamma)a^\beta e^{-\frac{i\Pi\beta}{2}}\left[1+\frac{ik\beta}{\gamma+(1-\gamma)a^\beta e^{-\frac{i\Pi\beta}{2}}}\right].\end{aligned}$$

This implies that

$$\arg\left\{\frac{z_0(I^n f(z_0))'}{I^n f(z_0)}-\gamma\right\} = -\frac{\pi}{2}\beta+\arg\left\{1+\frac{ik\beta}{\gamma+(1-\gamma)a^\beta e^{-\frac{i\Pi\beta}{2}}}\right\}$$

$$= \frac{-\pi}{2}\beta +$$

$$\tan^{-1}\left\{\frac{k\beta[\gamma+(1-\gamma)a^\beta\cos\,(\frac{\Pi}{2}\beta)]}{\gamma^2+2\gamma(1-\gamma)a^\beta\cos\,(\frac{\Pi}{2}\beta)+(1-\gamma)^2a^{2\beta}-k\beta(1-\gamma)a^\beta\sin(\frac{\Pi}{2}\beta)}\right\}$$

$$\le \frac{-\pi}{2}\beta\ \ (\text{where } k \le \frac{-1}{2}(a+\frac{1}{a}) \le -1),$$

which contradicts the condition $f(z) \in S^*_n(\beta,\gamma)$.

Similarly, if $\arg\ p(z_0) = \frac{\Pi}{2}\beta$, then we obtain that

$$\left|\arg\left\{\frac{z_0(I^n f(z_0))'}{I^n f(z_0)}-\gamma\right\}\right| \ge \frac{\pi}{2}\beta,$$

which also contradicts the hypothesis that $f(z) \in S^*_n(\beta,\gamma)$.

Thus the function $p(z)$ has to satisfy $|\arg p(z)| < \frac{\Pi}{2}\,\beta\ (z\in U)$. This shows that

$$\left|\arg\left\{\frac{z\,(I^{n+1} f(z))'}{I^{n+1}\, f(z)}-\gamma\right\}\right| < \frac{\pi}{2}\beta \quad (z\in U),$$

or $f(z) \in S^*_{n+1}(\beta,\gamma)$. This completes the proof of Theorem 2.2.

Theorem 2.3. *$C_n(\beta,\gamma) \subset C_{n+1}(\beta,\gamma)$ for each $n \in N_0$.*

Proof. $f(z) \in C_n(\beta,\gamma) \Longleftrightarrow I^n f(z) \in C(\beta,\gamma) \Longleftrightarrow z(I^n f(z))' \in S^*(\beta,\gamma)$
$\Longleftrightarrow I^n(zf'(z)) \in S^*(p,\gamma) \Longleftrightarrow zf'(z) \in S_n^*(\beta,\gamma)$
$\Longrightarrow zf'(z) \in S_{n+1}^*(\beta,\gamma) \Longleftrightarrow I^{n+1}(zf'(z)) \in S^*(\beta,\gamma)$
$\Longleftrightarrow z(I^{n+1}f(z))' \in S^*(\beta,\gamma) \Longleftrightarrow I^{n+1}f(z) \in C(\beta,\gamma)$
$\Longleftrightarrow f(z) \in C_{n+1}(\beta,\gamma)$.
This completes the proof of Theorem 2.3.

For $c > -1$ and $f(z) \in A$, we define the integral operator $L_c(f)$ as

$$L_c f(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt. \tag{2.4}$$

The operator $L_c(f)$ when $c \in N$ was studied by Bernardi [2]. For $c = 1$, $L_1(f)$ was introduced by Libera [4].

Theorem 2.4. *Let $c > -\gamma$ and $0 \le \gamma < 1$. If $f(z) \in S_n^*(\beta,\gamma)$ with $\dfrac{z(I^n L_c f(z))'}{I^n L_c f(z)} \neq \gamma$ for all $z \in U$, then we have $L_c(f) \in S_n^*(\beta,\gamma)$.*

Proof. Set

$$\frac{z(I^n L_c\ f(z))'}{I^n L_c f(z)} = \gamma + (1-\gamma)\,p(z), \tag{2.5}$$

where $p(z)$ is analytic in U, $p(0) = 1$, and $p(z) \neq 0\,(z \in U)$. From (2.4), we have

$$z(I^n L_c f(z))' = (c+1)\,I^n f(z) - cI^n L_c f(z). \tag{2.6}$$

Using (2.5) and (2.6), we have

$$(c+1)\frac{I^n f(z)}{I^n L_c f(z)} = c + \gamma + (1-\gamma)p(z). \tag{2.7}$$

Differentiating both sides of (2.7) with respect to z logarithmically, we obtain

$$\frac{z(I^n f(z))'}{I^n f(z)} - \gamma = (1-\gamma)p(z) + \frac{(1-\gamma)zp'(z)}{c+\gamma+(1-\gamma)p(z)}.$$

Suppose that there exists a point $z_0 \in U$ such that

$$|\arg\, p(z)| < \frac{\pi}{2}\beta\ \ (|z| < |z_0|) \text{ and } |\arg\ p(z_0)| = \frac{\pi}{2}\beta.$$

Then, applying Lemma 2.1, we can write that

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta \text{ and } (p(z_0))^{\frac{1}{\beta}} = \pm ia\ (a > 0).$$

If $\arg\, p(z_0) = \frac{\Pi}{2}\beta$, then

$$\frac{z_0(I^n f(z_0))'}{I^n f(z_0)} - \gamma = (1-\gamma)p(z_0)\left[1 + \frac{\dfrac{z_0 p'(z_0)}{p(z_0)}}{c+\gamma+(1-\gamma)p(z_0)}\right]$$

$$= (1-\gamma)a^{\beta}e^{i\frac{\Pi\beta}{2}}\left[1+\frac{ik\beta}{c+\gamma+(1-\gamma)a^{\beta}e^{i\frac{\Pi\beta}{2}}}\right].$$

This shows that

$$\arg\left\{\frac{z_0(I^n f(z_0))'}{I^n f(z_0)}-\gamma\right\} = \frac{\pi}{2}\beta + \arg\left[1+\frac{ik\beta}{c+\gamma+(1-\gamma)a^{\beta}e^{i\frac{\Pi\beta}{2}}}\right] = \frac{\Pi}{2}\beta$$

$$+\tan^{-1}\left\{\frac{k\beta[c+\gamma+(1-\gamma)a^{\beta}\cos(\frac{\Pi\beta}{2})]}{(c+\gamma)^2+2(c+\gamma)(1-\gamma)a^{\beta}\cos(\frac{\Pi\beta}{2})+(1-\gamma)^2a^{2\beta}+k\beta(1-\gamma)a^{\beta}\sin(\frac{\Pi\beta}{2})}\right\}$$

$$\geq \frac{\pi}{2}\beta \quad (\text{where } k \geq \frac{1}{2}(a+\frac{1}{a}) \geq 1),$$

which contradicts the condition $f(z) \in S_n^*(\beta,\gamma)$.

Similarly, we can prove the case $\arg\ p(z_0) = -\frac{\Pi}{2}\beta$. Thus we conclude that the function $p(z)$ has to satisfy $|\arg\ p(z)| < \frac{\Pi}{2}\beta$ for all $z \in U$. This gives that

$$\left|\arg\left\{\frac{z(I^n L_c f(z))'}{I^n L_c f(z)}-\gamma\right\}\right| < \frac{\pi}{2}\beta \ \ (z \in U),$$

or $L_c f(z) \in S_n^*(\beta,\gamma)$. This completes the proof of Theorem 2.4.

Theorem 2.5. *Let $c > -\gamma$ and $0 \leq \gamma < 1$. If $f(z) \in C_n(\beta,\gamma)$ and*

$$1+\frac{z(I^n L_c f(z))''}{(I^n L_c f(z))'} \neq \gamma$$

for all $z \in U$, then we have $L_c f(z) \in C_n(\beta,\gamma)$.

Proof. $f(z) \in C_n(\beta,\gamma) \iff zf'(z) \in S_n^*(\beta,\gamma) \implies L_c(zf'(z)) \in S_n^*(\beta,\gamma) \iff z(L_c f(z))' \in S_n^*(\beta,\gamma) \iff L_c f(z) \in C_n(\beta,\gamma)$.
This completes the proof of Theorem 2.5.

Acknowledgements. The author is thankful to the referee for his comments and suggestions.

References

[1] Aouf, M. K., Al-Oboudi, F. M., Haidan, M. M., *An application of certain integral operator*, Math. (Cluj), **47(70)**(2005), no. 2, 121-124.

[2] Bernardi, S. D., *Convex and starlike univalent functions*, Trans. Amer. Math. Soc., **135**(1969), 429-446.

[3] Liu, J. L., *The Noor integral and strongly starlike functions*, J. Math. Anal. Appl., **261**(2001), 441-447.

[4] Libera, R. J., *Some classes of regular univalent functions*, Proc. Amer. Math Soc., **16**(1965), 755-758.

[5] Mocanu, P. T., *Alpha-convex integral operators and strongly starlike functions*, Stud. Univ. Babes-Bolyai Math., **34**(1989), 18-24.

[6] Nunokawa, M., *On the order of strongly starlikeness of strongly convex functions*, Proc. Japan Acad. Ser. A Math. Sci., **69**(1993), 234-237.

[7] Nunokawa, M., Owa, S., Saitoh, H., Ikeda, A., Koike, N., *Some results for strongly starlike functions*, J. Math. Anal. Appl., **212**(1997), 98-106.

[8] Robertson, M. S., *On the theory of univalent functions*, Ann. Math., **37**(1936), 374-408.

[9] Sălăgean, G. S., *Subclasses of univalent functions*, Lecture Notes in Math. 1013, Springer-Verlag, Berlin - Heidelberg - New York, 1983, 362-372.

Mohamed K. Aouf
Department of Mathematics
Faculty of Science, Mansoura University
Mansoura 35516, Egypt
e-mail: mkaouf127@yahoo.com