Stud. Univ. Babeş-Bolyai Math.
Volume LVI, Number 1
March 2011, pp. 81–84

Metric relations on mountain slopes

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Abstract. It is well-known that the Ceva and Menelaus theorems are deducible from each other in the Euclidean case. In this paper we show that Ceva's theorem holds whereas Menelaus' theorem fails on Matsumoto's mountain slope geometry.

Mathematics Subject Classification (2010): 58B20, 53A35, 53C22.

Keywords: Non-Euclidean geometry, Finsler surface, geodesics.

1. Introduction

It is well-known that in Euclidean geometry, Ceva's and Menelaus' theorems are dual results, i.e., they are deducible from each other. In the Euclidean context, several extensions of these theorems can be found, see Green [2], Landy [4], Lipman [5], Wernicke [8]. Moreover, Masal'tsev [6] generalized Ceva's theorem to geodesic triangles on Riemannian surfaces of constant curvature (hyperbolic plane, sphere).

A natural question arises in the validity of these two theorems on non-Riemannian surfaces, even with constant curvature. Our aim is to prove that on the *Matsumoto's mountain slope* - which is one of the simplest *non*-Riemannian Finsler surface whose flag curvature is identically 0 - Ceva's theorem holds whereas Menelaus' theorem fails except the case when the slope becomes the horizontal plane.

2. Results

First, we recall the *Matsumoto's mountain slope metric*, see Matsumoto [7] or Kozma-Tamássy [3]. Let us consider an inclined plane (slope) with an angle $\alpha \in [0, \pi/2)$ to the horizontal plane, denoted by (S_α). If a man moves with a constant speed v $[m/s]$ on a horizontal plane, he goes $l_t = vt + \frac{g}{2}t^2 \sin\alpha\cos\theta$

Both authors were supported by Grant PN II IDEI_527 by CNCSIS.

meters in t seconds on (S_α), where θ is the angle between the straight road and the direct downhill road (θ is measured in clockwise direction). The point here is that the travel speed depends heavily on both the slope of the terrain and the direction of travel, due to the presence of the gravity. The precise law of the above phenomenon - by using the so-called Okubo's technique - can be described relatively to the horizontal plane by the parameterized function

$$F_\alpha(y_1,y_2)=\frac{y_1^2+y_2^2}{v\sqrt{y_1^2+y_2^2}+\frac{g}{2}y_1\sin\alpha},\qquad (y_1,y_2)\in\mathbb{R}^2\setminus\{(0,0)\}.$$

Here, $g\approx 9.81$ $[m/s^2]$ and we assume $g\sin\alpha\le v$.

For every $\alpha\in(0,\pi/2)$, $(\mathbb{R}^2,F_\alpha)$ is a typical non-Riemannian, Finsler surface. A classification of Finsler manifolds shows that $(\mathbb{R}^2,F_\alpha)$ is a locally Minkowski space with the following additional properties:

(a) its *flag curvature* is identically 0, see Bao-Chern-Shen [1, p. 384];
(b) its geodesics are *straight lines,* see also Bao-Chern-Shen [1, p. 384];
(c) every two points in $(\mathbb{R}^2,F_\alpha)$ determine a unique geodesic which lies them, due to Cartan-Hadamard's and Hopf-Rinow's theorems.

On account of (a)-(c), there is a strong similarity between $(\mathbb{R}^2,F_\alpha)$ and the standard two-dimensional Euclidean space. However, differences appear once we start to measure distances on these spaces. Exploiting the shape of geodesics on $(\mathbb{R}^2,F_\alpha)$, the *distance* (measuring actually the *physical time* to arrive) from $P=(P^1,P^2)$ to $Q=(Q^1,Q^2)$ on $(\mathbb{R}^2,F_\alpha)$ is

$$d_\alpha(P,Q)=F_\alpha(Q^1-P^1,Q^2-P^2).$$

Note that usually $d_\alpha(P,Q)\neq d_\alpha(Q,P)$.

Since geodesics are straight lines on $(\mathbb{R}^2,F_\alpha)$, see (b) from above, we may introduce the following two notions:

- $[PQ]=\{t(Q-P)+P : t\in[0,1]\}$ is the *geodesic segment* lying the points $P,Q\in\mathbb{R}^2$, and
- $[PQ[=\{t(Q-P)+P : t\ge 1\}$ is the *geodesic semi-line* defined by $P,Q\in\mathbb{R}^2$.

Let A,B,C be three arbitrarily fixed points in $(\mathbb{R}^2,F_\alpha)$, and let M,N,P points on the geodesic segments $[BC],[CA],[AB]$, respectively. We consider the following two statements:

(C_1^α): $\frac{d_\alpha(A,P)}{d_\alpha(P,B)}\cdot\frac{d_\alpha(B,M)}{d_\alpha(M,C)}\cdot\frac{d_\alpha(C,N)}{d_\alpha(N,A)}=1$;
(C_2): The geodesic segments $[AM],[BN],[CP]$ are concurrent.

Theorem 2.1. *For every $\alpha\in[0,\pi/2)$, we have $(C_1^\alpha)\Leftrightarrow(C_2)$.*

Thus, Ceva's theorem holds on the mountain slope $(\mathbb{R}^2,F_\alpha)$ for every $\alpha\in[0,\pi/2)$.

Now, let A,B,C be fixed points in $(\mathbb{R}^2,F_\alpha)$, and fix the points N,P on the geodesic segments $[CA],[AB]$, while M on the geodesic semi-line $[BC[$. We formulate the following two statements (the first being formally the same as (C_1^α)):

(M_1^α): $\frac{d_\alpha(A,P)}{d_\alpha(P,B)} \cdot \frac{d_\alpha(B,M)}{d_\alpha(M,C)} \cdot \frac{d_\alpha(C,N)}{d_\alpha(N,A)} = 1$;
(M_2): The points M, N, P are on the same geodesic (straight line).

Theorem 2.2. *The equivalence* $(M_1^\alpha) \Leftrightarrow (M_2)$ *holds if and only if* $\alpha = 0$.

Consequently, Menelaus' theorem holds on mountain slopes for every geodesic triangle if and only if the 'slope' becomes the horizontal plane (i.e., $\alpha = 0$), which corresponds exactly to the Euclidean case.

3. Proofs

In the sequel, we denote by d_E the usual two-dimensional Euclidean metric.
Proof of Theorem 2.1. Since $P = \frac{d_E(P,B)}{d_E(A,B)} A + \frac{d_E(A,P)}{d_E(A,B)} B$, we have

$$P - A = \frac{d_E(A,P)}{d_E(A,B)}(B - A).$$

Since F_α is positively homogeneous of degree 1, one has

$$d_\alpha(A,P) = F_\alpha(P^1 - A^1, P^2 - A^2) = \frac{d_E(A,P)}{d_E(A,B)} F_\alpha(B^1 - A^1, B^2 - A^2).$$

A similar calculation for $d_\alpha(P,B)$ implies that

$$\frac{d_\alpha(A,P)}{d_\alpha(P,B)} = \frac{d_E(A,P)}{d_E(P,B)}.$$

Repeating this argument for the other two sides of the triangle, (C_1^α) is equivalent to

$$\frac{d_E(A,P)}{d_E(P,B)} \cdot \frac{d_E(B,M)}{d_E(M,C)} \cdot \frac{d_E(C,N)}{d_E(N,A)} = 1.$$

But, in the Euclidean case, the latter relation is equivalent to the fact that the segments $[AM], [BN], [CP]$ are concurrent, thus the proof is done.
Proof of Theorem 2.2. If $\alpha = 0$, the equivalence $(M_1^\alpha) \Leftrightarrow (M_2)$ is just the well-known Menelaus' theorem in the Euclidean case.

Now, we assume the equivalence $(M_1^\alpha) \Leftrightarrow (M_2)$ holds for every points A, B, C as well as M, N, P in $(\mathbb{R}^2, F_\alpha)$ specified above. We prove that $\alpha = 0$. To see this, we consider the following specific constellation of points: $B = (0,0)$, $C = (1,0)$, $M = (2,0)$, A is arbitrary, while P and N are situated on $[AB]$ and $[AC]$ such that M belongs to the unique geodesic lying them, see (c) from above. Thus, (M_2) holds. Since $(M_2) \Leftrightarrow (M_1^\alpha)$, we have

$$\frac{d_\alpha(A,P)}{d_\alpha(P,B)} \cdot \frac{d_\alpha(B,M)}{d_\alpha(M,C)} \cdot \frac{d_\alpha(C,N)}{d_\alpha(N,A)} = 1. \tag{3.1}$$

On the other hand, (M_2) also implies for the Euclidean metric that

$$\frac{d_E(A,P)}{d_E(P,B)} \cdot \frac{d_E(B,M)}{d_E(M,C)} \cdot \frac{d_E(C,N)}{d_E(N,A)} = 1. \tag{3.2}$$

As in the proof of Theorem 2.1, by using the positive homogeneity of F_α, we deduce

$$\frac{d_\alpha(A,P)}{d_\alpha(P,B)} = \frac{d_E(A,P)}{d_E(P,B)} \quad \text{and} \quad \frac{d_\alpha(C,N)}{d_\alpha(N,A)} = \frac{d_E(C,N)}{d_E(N,A)}.$$

Combining (3.1) and (3.2) with the above relations, we obtain

$$\frac{d_\alpha(B,M)}{d_\alpha(M,C)} = \frac{d_E(B,M)}{d_E(M,C)}.$$

After substitutions, we obtain $\dfrac{F_\alpha(2,0)}{F_\alpha(-1,0)} = 2$. An elementary calculation shows that the latter equation holds only in the case when $g \sin\alpha = 0$, i.e., $\alpha = 0$. This concludes our proof.

Acknowledgment. Research for this article was supported by the Grant PN II, ID_527/2007.

References

[1] Bao, D., Chern, S. S., Shen, Z., *Introduction to Riemann-Finsler Geometry,* Graduate Texts in Mathematics, 200, Springer Verlag, 2000.

[2] Green, H. G., *On the theorems of Ceva and Menelaus,* Amer. Math. Monthly, **64** (1957), 354-357.

[3] Kozma, L., Tamássy, L., *Finsler geometry inspired. Finslerian geometries* (Edmonton, AB, 1998), 9-14, Fund. Theories Phys., 109, Kluwer Acad. Publ., Dordrecht, 2000.

[4] Landy, S., *A generalization of Ceva's theorem to higher dimensions*, Amer. Math. Monthly, **95** (1988), no. 10, 936-939.

[5] Lipman, J., *A generalization of Ceva's theorem,* Amer. Math. Monthly, **67**(1960), 162-163.

[6] Masal'tsev, L. A., *Incidence theorems in spaces of constant curvature,* (Russian) Ukrain. Geom. Sb., **35**(1992), 67-74, 163; translation in J. Math. Sci., **72**(1994), no. 4, 3201-3206.

[7] Matsumoto, M., *A slope of a mountain is a Finsler surface with respect to a time measure,* J. Math. Kyoto Univ., **29**(1989), no. 1, 17-25.

[8] Wernicke, P., *The theorems of Ceva and Menelaus and their extension,* Amer. Math. Monthly, **34**(1927), no. 9, 468-472.

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