Stud. Univ. Babeş-Bolyai Math.
Volume LVI, Number 1
March 2011, pp. 51–61

Asymptotic behavior of the solution of nonlinear parametric variational inequalities in notched beams

Iuliana Marchiş

Abstract. In this article we study the asymptotic behavior of the solution U_ϵ of a parametric variational inequality governed by a nonlinear differential operator posed in a notched beam (i.e. a thin cylinder with small part of it having a diameter much smaller than the rest) which depends on three positive parameters: ϵ, r_ϵ, and t_ϵ.

Mathematics Subject Classification (2010): 35B40, 35B45.

Keywords: homogenization, partial differential equations, notched beam.

1. Introduction

The aim of the paper is to study the asymptotic behavior of the solution of nonlinear variational inequalities in a notched beam (i.e. a thin cylinder with small part of it having a diameter much smaller than the rest). Mathematically, this notched beam is given by

$$\Omega_\epsilon = \{(x_1, x') \in \mathbb{R}^3 : -1 < x_1 < 1, |x'| < \epsilon \text{ if } |x_1| > t_\epsilon, |x'| < \epsilon r_\epsilon \text{ if } |x_1| \leq t_\epsilon\}, \tag{1.1}$$

where ϵ, r_ϵ, and t_ϵ are positive parameters.

Previous work on domains of this type was done by Hale & Vegas [6], Jimbo [7, 8], Cabib, Freddi, Morassi, & Percivale [2], Rubinstein, Schatzman & Sternberg [12], and Casado-Díaz, Luna-Laynez & Murat [3, 4], Kohn & Slastikov [9].

The most recent results are of Casado-Díaz, Luna-Laynez & Murat [4]. They studied the asymptotic behavior of the solution of a diffusion equation in the notched beam Ω_ϵ and obtained at the limit a one-dimensional model.

This research was partially supported by the research grant CNCSIS, PN II, ID 523/2007.

In the present article the geometrical setting is the same as in [4], but we consider nonlinear variational inequalities instead of linear variational equalities.

The paper is organized as follows. In Section 2 the geometrical setting is described, the studied problem is given, and the assumptions for our results are formulated. In Section 3 the asymptotic behavior of the solution is studied. The main results are Theorem 3.6 and Theorem 3.7.

2. Setting the problem

Let $\epsilon > 0$ be a parameter, r_ϵ ($r_\epsilon > 0$) and t_ϵ ($t_\epsilon > 0$) be two sequences of real numbers, with

$$r_\epsilon \to 0, \qquad t_\epsilon \to 0, \quad \text{when } \epsilon \to 0.$$

We assume that

$$\frac{t_\epsilon}{r_\epsilon^2} \to \mu, \quad \frac{\epsilon}{r_\epsilon} \to \nu, \qquad \text{with } 0 \leq \mu \leq +\infty,\ 0 \leq \nu \leq +\infty, \quad \text{when } \epsilon \to 0.$$

Let $S \subset \mathbb{R}^2$ be a bounded domain such that $0 \in S$, which is sufficiently smooth to apply the Poincaré-Wirtinger inequality.

Define the following subsets of $\mathbb{R}^3$:

$$\Omega_\epsilon^- = (-1, -t_\epsilon) \times (\epsilon S), \quad \Omega_\epsilon^0 = [-t_\epsilon, t_\epsilon] \times (\epsilon r_\epsilon S), \quad \Omega_\epsilon^+ = (t_\epsilon, 1) \times (\epsilon S),$$

$$\Omega_\epsilon = \Omega_\epsilon^- \cup \Omega_\epsilon^0 \cup \Omega_\epsilon^+, \quad \text{and} \quad \Omega_\epsilon = \Omega_\epsilon^- \cup \Omega_\epsilon^+.$$

Ω_ϵ is a notched beam, the main part of the beam is Ω_ϵ^1 and the notched part Ω_ϵ^0. The plane section of this domain is presented in Figure 1. A point of Ω^ϵ is denoted by $x = (x_1, x') = (x_1, x_2, x_3)$.

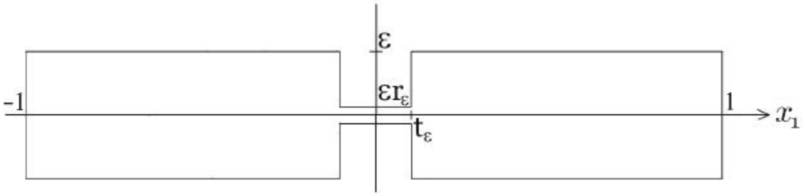

FIGURE 1. The plane section of the notched beam Ω_ϵ

Denote by

$$\Gamma_\epsilon^- = \{-1\} \times (\epsilon S) \ \text{ and } \ \Gamma_\epsilon^+ = \{1\} \times (\epsilon S)$$

the two bases of the beam, and let

$$\Gamma_\epsilon = \Gamma_\epsilon^- \cup \Gamma_\epsilon^+$$

be the union of the two bases.

Denote

$$\mathcal{V}_\epsilon = \{V \in H^1(\Omega_\epsilon), \quad V = 0 \text{ on } \Gamma_\epsilon\}.$$

We consider the following problem:
find $U_\epsilon \in M_\epsilon$ such that, for all $V_\epsilon \in M_\epsilon$,

$$\int_{\Omega_\epsilon} [A_\epsilon \Phi_\epsilon(x, U_\epsilon, B_\epsilon \nabla U_\epsilon), \nabla(V_\epsilon - U_\epsilon)]\, dx \;+\; \int_{\Omega_\epsilon} \Psi_\epsilon(x, U_\epsilon, \nabla U_\epsilon)(V_\epsilon - U_\epsilon) dx \tag{2.1}$$

$$+ \int_{\Omega_\epsilon} [G_\epsilon, \nabla(V_\epsilon - U_\epsilon)]\, dx + \int_{\Omega_\epsilon} \Theta_\epsilon(x, U_\epsilon, V_\epsilon - U_\epsilon) \geq 0,$$

with A_ϵ, B_ϵ, Φ_ϵ, Ψ_ϵ, G_ϵ, and Θ_ϵ given functions, M_ϵ a closed, convex, nonempty subset of $\mathcal{V}_\epsilon$.

This problem has applications in Physics. Bruno [1] observed that when a ferromagnet has a thin neck, this will be preferred location for the domain wall. He also notice that if the geometry of the neck varies rapidly enough, it can influence and even dominate the structure of the wall.

We impose the following assumptions:

(B1) The matrix A_ϵ has the following form

$$A_\epsilon(x) = \chi_{\Omega^1_\epsilon}(x) A^1\left(x_1, \frac{x'}{\epsilon}\right) + \chi_{\Omega^0_\epsilon}(x) A^0\left(\frac{x_1}{t_\epsilon}, \frac{x'}{\epsilon r_\epsilon}\right),$$

where $A^1, A^0 \in L^\infty((-1,1) \times S)^{3\times 3}$.

(B2) The matrix B_ϵ has the following form

$$B_\epsilon(x) = \chi_{\Omega^1_\epsilon}(x) B^1\left(x_1, \frac{x'}{\epsilon}\right) + \chi_{\Omega^0_\epsilon}(x) B^0\left(\frac{x_1}{t_\epsilon}, \frac{x'}{\epsilon r_\epsilon}\right),$$

where $B^1, B^0 \in L^\infty((-1,1) \times S)^{3\times 3}$.

(B3) The functions $\Phi_\epsilon : \Omega_\epsilon \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ and $\Psi_\epsilon : \Omega_\epsilon \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ are Carathédory mappings having the following form

$$\Phi_\epsilon(x, \eta, \xi) = \chi_{\Omega^1_\epsilon}(x) \Phi^1_\epsilon\left(x_1, \frac{x'}{\epsilon}, \eta, B^1\left(x_1, \frac{x'}{\epsilon}\right)\xi\right)$$
$$+ \chi_{\Omega^0_\epsilon}(x) \Phi^0_\epsilon\left(\frac{x_1}{t_\epsilon}, \frac{x'}{\epsilon r_\epsilon}, \eta, B^0\left(\frac{x_1}{t_\epsilon}, \frac{x'}{\epsilon r_\epsilon}\xi\right)\xi\right);$$

$$\Psi_\epsilon(x, \eta, \xi) = \chi_{\Omega^1_\epsilon}(x) \Psi^1_\epsilon\left(x_1, \frac{x'}{\epsilon}, \eta, \xi\right) + \chi_{\Omega^0_\epsilon}(x) \Psi^0_\epsilon\left(\frac{x_1}{t_\epsilon}, \frac{x'}{\epsilon r_\epsilon}, \eta, \xi\right);$$

for a.e. $x \in \Omega_\epsilon$, for all $\eta \in \mathbb{R}$, and $\xi \in \mathbb{R}^3$;
for all $U_\epsilon \in H^1(\Omega_\epsilon)$, $\Phi^1_\epsilon(\cdot, U_\epsilon(\cdot), B^1_\epsilon(\cdot)\nabla U_\epsilon(\cdot)), \Phi^0_\epsilon(\cdot, U_\epsilon(\cdot), B^0_\epsilon(\cdot)\nabla U_\epsilon(\cdot)) \in L^2((-1,1) \times S)^3$; $\Psi^1_\epsilon(\cdot, U_\epsilon(\cdot), \nabla U_\epsilon(\cdot)), \Psi^0_\epsilon(\cdot, U_\epsilon(\cdot), \nabla U_\epsilon(\cdot)) \in L^2((-1,1) \times S)$.

(B4) *Coercivity conditions*
There exist $C_1, C_2 > 0$ and $k_1 \in L^\infty(\Omega_\epsilon)$ such that for all $\xi \in \mathbb{R}^3$, $\eta \in \mathbb{R}$

$$[A_\epsilon(x)\Phi_\epsilon(x, \eta, B_\epsilon(x)\xi), \xi] + \Psi_\epsilon(x, \eta, \xi)\eta \geq C_1\|\xi\|^2 + C_2|\eta|^{q_1} - k_1(x) \text{ a.e. } x \in \Omega_\epsilon, \tag{2.2}$$

for some $1 < q_1 < 2$, for each $\epsilon > 0$.

(B5) *Growth conditions*
There exist $C > 0$ and $\alpha \in L^\infty(\Omega_\epsilon)$ such that for all $\xi \in \mathbb{R}^3$, $\eta \in \mathbb{R}$

$$\|A_\epsilon(x)\Phi_\epsilon(x,\eta,\xi)\| \le C\|\xi\| + C|\eta| + \alpha(x) \quad \text{a.e. } x \in \Omega_\epsilon, \tag{2.3}$$

for each $\epsilon > 0$.

There exist $C > 0$ and $\beta \in L^\infty(\Omega_\epsilon)$ such that for all $\xi \in \mathbb{R}^3$, $\eta \in \mathbb{R}$

$$|\Psi_\epsilon(x,\eta,\xi)| \le C\|\xi\| + C|\eta| + \beta(x) \quad \text{a.e. } x \in \Omega_\epsilon, \tag{2.4}$$

for each $\epsilon > 0$.

(B6) *Monotonicity condition* For all $\xi, \tau \in \mathbb{R}^n$, $\eta \in \mathbb{R}$,

$$[A_\epsilon(x)\phi_\epsilon(x,\eta,B_\epsilon(x)\xi) - A_\epsilon(x)\phi_\epsilon(x,\eta,B_\epsilon(x)\tau), \xi - \tau] \ge 0, \quad \text{a. e. } x \in \Omega_\epsilon,$$

for each $\epsilon > 0$.

(B7) The function $G_\epsilon \in L^2((-1,1)\times S)^3$ has the following form

$$G_\epsilon(x) = \chi_{\Omega_\epsilon^1}(x) G_\epsilon^1\left(x_1, \frac{x'}{\epsilon}\right) + \chi_{\Omega_\epsilon^0}(x) G_\epsilon^0\left(\frac{x_1}{t_\epsilon}, \frac{x'}{\epsilon r_\epsilon}\right) \quad \text{a.e. } x \in \Omega_\epsilon,$$

where $G_\epsilon^1, G_\epsilon^0 \in L^2((-1,1)\times S)^3$.

(B8) There exists $C > 0$ such that

$$\frac{1}{\epsilon^2}\int_{\Omega_\epsilon} \|G_\epsilon(x)\|^2 \, \mathrm{d}x < C, \tag{2.5}$$

for each $\epsilon > 0$.

(B9) $\Theta_\epsilon : \Omega_\epsilon \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\Theta_\epsilon(x,\cdot,\cdot)$ is upper semi-continuous for almost all $x \in \Omega_\epsilon$; $\Theta_\epsilon(\cdot, y, z)$ is measurable for all $y, z \in \mathbb{R}$; Θ_ϵ is sublinear in its second variable, for each ϵ.

(B10) There exists $g_1, g_2 \in L^\infty(\Omega_\epsilon)$ nonnegative functions such that

$$|\Theta_\epsilon(x,y,z)| \le g_1(x) + g_2(x)|z| \tag{2.6}$$

for almost all $x \in \Omega_\epsilon$, for all $z \in \mathbb{R}$, for each $\epsilon > 0$.

Remark 2.1. *From Theorem 3.4 in* [10] *it follows that, for all $\epsilon > 0$, the variational inequality (*2.1*) has at least one solution.*

3. Asymptotic behavior of the solution

To study the asymptotic behavior we use the change of variables $y = y_\epsilon(x)$ given by

$$y_1 = x_1 \quad y' = \frac{x'}{\epsilon} \tag{3.1}$$

which transforms the beam (except the notch) in a cylinder of fixed diameter. This change of variable is classical in the study of asymptotic behavior of variational equalities in thin cylinders or beams (see [5], [11], [13]). We denote by Y_ϵ^-, Y_ϵ^0, Y_ϵ^+, Y_ϵ, and Y_ϵ^1 the images of Ω_ϵ^-, Ω_ϵ^0, Ω_ϵ^+, Ω_ϵ, and Ω_ϵ^1 by the change of variables $y = y_\epsilon(x)$, i.e.

$$Y_\epsilon^- = (-1, -t_\epsilon) \times S, \quad Y_\epsilon^0 = [-t_\epsilon, t_\epsilon] \times (r_\epsilon S), \quad Y_\epsilon^+ = (t_\epsilon, 1) \times S,$$
$$Y_\epsilon = Y_\epsilon^- \cup Y_\epsilon^0 \cup Y_\epsilon^+, \quad Y_\epsilon^1 = Y_\epsilon^- \cup Y_\epsilon^+.$$

Denote by Y^-, Y^+, and Y^1 the "limits" of Y_ϵ^-, Y_ϵ^+, and Y_ϵ^1, i.e.

$$Y^- = (-1, 0) \times S, \quad Y^+ = (0, 1) \times S, \quad Y^1 = Y^- \cup Y^+.$$

Note that Y_ϵ^1 is contained in its limit Y^1.

The two bases of the beam Γ_ϵ^- and Γ_ϵ^+ are transformed to Λ^- and Λ^+, respectively, where

$$\Lambda^- = \{-1\} \times S \quad \text{and} \quad \Lambda^+ = \{1\} \times S.$$

Γ_ϵ transforms to $\Lambda = \Lambda^- \cup \Lambda^+$, which doesn't depend on ϵ.

Let $U_\epsilon \in M_\epsilon$ be the solution of the variational inequality (2.1). Define $u_\epsilon \in K_\epsilon$ by

$$u_\epsilon(y) = U_\epsilon(y_\epsilon^{-1}(y)) \quad \text{a.e. } y \in Y_\epsilon, \tag{3.2}$$

K_ϵ being the image of M_ϵ. K_ϵ is a closed, convex, nonempty cone in $\mathcal{D}_\epsilon$, with $\mathcal{D}_\epsilon = \{v \in H^1(Y_\epsilon) \mid v = 0 \text{ on } \Lambda\}$. We need the following two assumptions

(B11) There exists a nonempty, convex cone K in $H^1(Y^1)$ such that
(i) $K \cap H^1((-1,0) \cup (0,1)) \neq \emptyset$;
(ii) $\epsilon_i \to 0$, $u_{\epsilon_i} \in K_{\epsilon_i}$, $u \in H^1((-1,0) \cup (0,1))$, $u_{\epsilon_i} \rightharpoonup u$ (weakly) in $H^1(Y^1)$
imply $u \in K$.

(B12) There exists a nonempty, convex cone L in $L^2((-1,1); H^1(S))$ such that $\epsilon_i \to 0$, $w_{\epsilon_i} \in K_{\epsilon_i}$, $w \in L^2((-1,1); H^1(S))$, $w_{\epsilon_i} \rightharpoonup w$ (weakly) in $L^2((-1,1); H^1(S))$ imply $w \in L$.

By change of variables $y = y_\epsilon(x)$ the operator ∇ transforms to

$$\nabla^\epsilon \cdot = \left(\frac{\partial \cdot}{\partial y_1}, \frac{1}{\epsilon} \frac{\partial \cdot}{\partial y_2}, \frac{1}{\epsilon} \frac{\partial \cdot}{\partial y_3} \right). \tag{3.3}$$

Using the change of variables $y = y_\epsilon(x)$, given by (3.1), the inequality (2.1) transforms to

$$\begin{aligned}
&\int_{Y_\epsilon} \left[A_\epsilon(y_\epsilon^{-1}(y)) \Phi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), B_\epsilon(y_\epsilon^{-1}(y)) \nabla^\epsilon u_\epsilon(y)), \nabla^\epsilon (v_\epsilon(y) - u_\epsilon(y)) \right] \, dy \\
&+ \int_{Y_\epsilon} \Psi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), \nabla^\epsilon u_\epsilon(y)) (v_\epsilon(y) - u_\epsilon(y)) \, dy \\
&+ \int_{Y_\epsilon} \left[G_\epsilon(y_\epsilon^{-1}(y)), \nabla^\epsilon (v_\epsilon(y) - u_\epsilon(y)) \right] \, dy \\
&+ \int_{Y_\epsilon} \Theta_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), v_\epsilon(y) - u_\epsilon(y)) \, dy \geq 0,
\end{aligned} \tag{3.4}$$

for all $v_\epsilon \in K_\epsilon$, where $v_\epsilon(y) = V_\epsilon(y_\epsilon^{-1}(y))$ a. e. $y \in Y_\epsilon$.

Lemma 3.1. *Assume that (B4) holds, $U_\epsilon \in M_\epsilon$, and $u_\epsilon \in K_\epsilon$ is given by (3.2). Then there exist $C_1, C_2 > 0$ and $C_3 \in \mathbb{R}$ such that*

$$\begin{aligned}&\int_{Y_\epsilon} \left[A_\epsilon(y_\epsilon^{-1}(y))\Phi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), B_\epsilon(y_\epsilon^{-1}(y))\nabla^\epsilon u_\epsilon(y)), \nabla^\epsilon u_\epsilon(y)\right] \, \mathrm{d}y \\ &+ \int_{Y_\epsilon} \Psi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), \nabla^\epsilon u_\epsilon(y))u_\epsilon(y) \, \mathrm{d}y \\ &\geq C_1\|\nabla^\epsilon u_\epsilon\|^2_{L^2(Y_\epsilon)} - C_2\|u_\epsilon\|^{q_1}_{L^2(Y_\epsilon)} - C_3\end{aligned} \tag{3.5}$$

Proof. Putting $\eta = U_\epsilon(x)$ and $\xi = \nabla U_\epsilon(x)$ in coercivity condition (2.2), integrating on Ω_ϵ we get

$$\begin{aligned}&\int_{\Omega_\epsilon} [A_\epsilon(x)\Phi_\epsilon(x, U_\epsilon(x), B_\epsilon(x)\nabla U_\epsilon(x)), \nabla U_\epsilon(x)] \, \mathrm{d}x \\ &+ \int_{\Omega_\epsilon} \Psi_\epsilon(x, U_\epsilon(x), \nabla U_\epsilon(x))U_\epsilon(x) \, \mathrm{d}x \\ &\geq C_1 \int_{\Omega_\epsilon} \|\nabla U_\epsilon(x)\|^2 \, \mathrm{d}x - C_2 \int_{\Omega_\epsilon} |U_\epsilon(x)|^{q_1} \, \mathrm{d}x - |\Omega_\epsilon|\|k_1\|_\infty.\end{aligned}$$

Multiplying by $\frac{1}{\epsilon^2}$ and using the change of variables $y = y_\epsilon(x)$, given by (3.1), we obtain

$$\begin{aligned}&\int_{Y_\epsilon} [A_\epsilon(y_\epsilon^{-1}(y))\Phi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), B_\epsilon(y_\epsilon^{-1}(y))\nabla^\epsilon u_\epsilon(y)), \nabla^\epsilon u_\epsilon(y)] \, \mathrm{d}y \\ &+ \int_{Y_\epsilon} \Psi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), \nabla^\epsilon u_\epsilon(y))u_\epsilon(y) \, \mathrm{d}y \\ &\geq C_1 \int_{Y_\epsilon} \|\nabla^\epsilon u_\epsilon(y)\|^2 \, \mathrm{d}y - C_2 \int_{Y_\epsilon} |u_\epsilon(y)|^{q_1} \, \mathrm{d}y - \overline{k}_1 \\ &\geq C_1\|\nabla^\epsilon u_\epsilon\|^2_{L^2(Y_\epsilon)} - C_2\|u_\epsilon\|^{q_1}_{L^{q_1}(Y_\epsilon)} - \overline{k}_1,\end{aligned}$$

as $q_1 < 2$. □

Lemma 3.2. *Assume that (B5) holds and let $v_\epsilon \in K_\epsilon$, $(v_\epsilon)_\epsilon$ bounded in $H^1(Y_\epsilon)$. Then the following properties hold*

a) There exist k_1, k_2, and k_3 constants such that

$$\begin{aligned}&\int_{Y_\epsilon} \left[A_\epsilon(y_\epsilon^{-1}(y))\Phi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), B_\epsilon(y_\epsilon^{-1}(y))\nabla^\epsilon u_\epsilon(y)), \nabla^\epsilon v_\epsilon(y)\right] \, \mathrm{d}y \\ &\quad \leq k_1\|\nabla^\epsilon u_\epsilon\|_{L^2(Y_\epsilon)} + k_2\|u_\epsilon\|_{L^2(Y_\epsilon)} + k_3.\end{aligned} \tag{3.6}$$

b) There exists k_4, k_5, and k_6 such that

$$\int_{Y_\epsilon} \Psi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), \nabla^\epsilon u_\epsilon(y))v_\epsilon(y) \, \mathrm{d}y \leq k_4\|\nabla^\epsilon u_\epsilon\|_{L^2(Y_\epsilon)} + k_5\|u_\epsilon\|_{L^2(Y_\epsilon)} + k_6. \tag{3.7}$$

Proof. a) Applying the Cauchy-Schwarz inequality and then the growth condition (2.3) for $x = y_\epsilon^{-1}(y)$ we get

$$\int_{Y_\epsilon} \left[A_\epsilon(y_\epsilon^{-1}(y))\Phi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), B_\epsilon(y_\epsilon^{-1}(y))\nabla^\epsilon u_\epsilon(y)), \nabla^\epsilon v_\epsilon(y)\right] \, \mathrm{d}y$$
$$\leq \int_{Y_\epsilon} \|A_\epsilon(y_\epsilon^{-1}(y))\Phi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), B_\epsilon(y_\epsilon^{-1}(y))\nabla^\epsilon u_\epsilon(y))\|\|\nabla^\epsilon v_\epsilon(y)\| \, \mathrm{d}y$$
$$\leq \int_{Y_\epsilon} \left(C\|\nabla^\epsilon u_\epsilon(y)\| + C|u_\epsilon(y)| + \overline{\alpha}(y_\epsilon^{-1}(y))\right) \|\nabla^\epsilon v_\epsilon(y)\| \, \mathrm{d}y$$

(by Cauchy-Schwarz inequality)

$$\leq \left(C\|\nabla^\epsilon u_\epsilon\|_{L^2(Y_\epsilon)} + C\|u_\epsilon\|_{L^2(Y_\epsilon)} + \overline{\alpha}\right) \|\nabla^\epsilon v_\epsilon\|_{L^2(Y_\epsilon)},$$

as $(v_\epsilon)_\epsilon$ is bounded.

b) Using the growth condition (2.4) for $x = y_\epsilon^{-1}(y)$ and the Cauchy-Schwarz inequality, we get

$$\int_{Y_\epsilon} \Psi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), \nabla^\epsilon u_\epsilon(y)) v_\epsilon(y) \, \mathrm{d}y$$
$$\leq \int_{Y_\epsilon} \left(C\|\nabla^\epsilon u_\epsilon(y)\| + C|u_\epsilon(y)| + \beta(y_\epsilon^{-1}(y))\right) |v_\epsilon(y)| \, \mathrm{d}y$$
$$\leq \left(C\|\nabla^\epsilon u_\epsilon\|_{L^2(Y_\epsilon)} + C\|u_\epsilon\|_{L^2(Y_\epsilon)} + \overline{\beta}\right) \|v_\epsilon\|_{L^2(Y_\epsilon)},$$

as $(v_\epsilon)_\epsilon$ is bounded. □

Lemma 3.3. *If assumption (B10) is satisfied, $U_\epsilon, V_\epsilon \in M_\epsilon$, u_ϵ and v_ϵ are given by (3.2), then there exist $\bar{g}_1, \bar{g}_2 \in \mathbb{R}$ such that*

$$\int_{Y_\epsilon} \Theta_\epsilon(u_\epsilon(y), v_\epsilon(y) - u_\epsilon(y)) \, \mathrm{d}y \leq \bar{g}_1 + \bar{g}_2\|v_\epsilon - u_\epsilon\|_{L^2(Y_\epsilon)}.$$

Proof. Putting $y = U_\epsilon(x)$ and $z = V_\epsilon(x) - U_\epsilon(x)$ in (2.6), multiplying by $\frac{1}{\epsilon^2}$, then integrating over Ω_ϵ, we obtain

$$\frac{1}{\epsilon^2}\int_{\Omega_\epsilon} \Theta_\epsilon(U_\epsilon(x), V_\epsilon(x) - U_\epsilon(x)) \, \mathrm{d}x \leq \frac{1}{\epsilon^2}\int_{\Omega_\epsilon} |\Theta_\epsilon(U_\epsilon(x), V_\epsilon(x) - U_\epsilon(x))| \, \mathrm{d}x$$
$$\leq \frac{1}{\epsilon^2}\int_{\Omega_\epsilon} (g_1(x) + g_2(x)|V_\epsilon(x) - U_\epsilon(x)|) \, \mathrm{d}x$$
$$\leq \bar{g}_1\frac{|\Omega_\epsilon|}{\epsilon^2} + \frac{1}{\epsilon^2}\bar{g}_2\int_{\Omega_\epsilon} |V_\epsilon(x) - U_\epsilon(x)| \, \mathrm{d}x,$$

where $\bar{g}_1 = \|g_1\|_\infty$ and $\bar{g}_2 = \|g_2\|_\infty$. Using the change of variable y_ϵ, the result follows. □

Lemma 3.4. *Let $U_\epsilon \in M_\epsilon$ be the solution of the variational inequality (2.1) and $u_\epsilon \in K_\epsilon$ defined by*

$$u_\epsilon(y) = U_\epsilon(y_\epsilon^{-1}(y)) \quad a.e.\ y \in Y_\epsilon.$$

If assumptions (B1)-(B10) are verified then the following statements hold

2) $(u_\epsilon)_\epsilon$ is bounded in $H^1(Y_\epsilon)$;

1) $\left(\frac{1}{\epsilon}\frac{\partial u_\epsilon}{\partial y_2}\right)_\epsilon$ *and* $\left(\frac{1}{\epsilon}\frac{\partial u_\epsilon}{\partial y_3}\right)_\epsilon$ *are bounded in* $L^2(Y_\epsilon)$*;*
3) $(\sigma_\epsilon)_\epsilon$ *is bounded in* $L^2(Y_\epsilon)^3$*, where*

$$\sigma_\epsilon(y) = A_\epsilon(y_\epsilon^{-1}(y))\Phi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), B_\epsilon(y_\epsilon^{-1}(y))\nabla^\epsilon u_\epsilon(y)) \quad a.e. \quad y \in Y_\epsilon.$$

Proof. Suppose that $(v_\epsilon)_\epsilon$ is bounded in $H^1(Y_\epsilon)$. From coercivity condition (B4) by Lemma 3.1, then inequality (3.4), we obtain

$$\begin{aligned}
&C_1\|\nabla^\epsilon u_\epsilon\|^2_{L^2(Y_\epsilon)} - C_2\|u_\epsilon\|^{q_1}_{L^2(Y_\epsilon)} - C_3\\
&\leq \int_{Y_\epsilon} \left[A_\epsilon(y_\epsilon^{-1}(y))\Phi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), B_\epsilon(y_\epsilon^{-1}(y))\nabla^\epsilon u_\epsilon(y)), \nabla^\epsilon u_\epsilon(y)\right] \,\mathrm{d}y\\
&+ \int_{Y_\epsilon} \Psi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), \nabla^\epsilon u_\epsilon(y))u_\epsilon(y) \,\mathrm{d}y\\
&\leq \int_{Y_\epsilon} \left[A_\epsilon(y_\epsilon^{-1}(y))\Phi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), B_\epsilon(y_\epsilon^{-1}(y))\nabla^\epsilon u_\epsilon(y)), \nabla^\epsilon v_\epsilon(y)\right] \,\mathrm{d}y\\
&+ \int_{Y_\epsilon} \Psi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), \nabla^\epsilon u_\epsilon(y))v_\epsilon(y) \,\mathrm{d}y\\
&+ \int_{Y_\epsilon} \left[G_\epsilon(y_\epsilon^{-1}(y)), \nabla^\epsilon v_\epsilon(y) - \nabla^\epsilon u_\epsilon(y)\right] \,\mathrm{d}y\\
&+ \int_{Y_\epsilon} \Theta_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), v_\epsilon(y) - u_\epsilon(y)) \,\mathrm{d}y \leq
\end{aligned}$$

(using Lemma 3.2 for the first two terms, the Cauchy-Schwarz inequality and then assumption (2.5) for the third term, assumption (2.6) for the fourth term)

$$\begin{aligned}
&\leq k_1\|\nabla^\epsilon u_\epsilon\|_{L^2(Y_\epsilon)} + k_2\|u_\epsilon\|_{L^2(Y_\epsilon)}\\
&+ C\|\nabla^\epsilon v_\epsilon - \nabla^\epsilon u_\epsilon\|_{L^2(Y_\epsilon)} + c_1'\|v_\epsilon - u_\epsilon\|_{L^2(Y_\epsilon)} + k\\
&\leq c_1\|\nabla^\epsilon u_\epsilon\|_{L^2(Y_\epsilon)} + c_2,
\end{aligned}$$

using the Poincaré inequality , where c_1 and c_2 are constants. On the other hand

$$\begin{aligned}
&C_1\|\nabla^\epsilon u_\epsilon\|^2_{L^2(Y_\epsilon)} - C_2\|u_\epsilon\|^{q_1}_{L^2(Y_\epsilon)} - C_3\\
&\geq c_3\|\nabla^\epsilon u_\epsilon\|^2_{L^2(Y_\epsilon)} - c_4\|\nabla^\epsilon u_\epsilon\|^{q_1}_{L^2(Y_\epsilon)} - c_5,
\end{aligned}$$

by the Poincaré inequality , where c_3, c_4, and c_5 are constants. Thus

$$c_3\|\nabla^\epsilon u_\epsilon\|^2_{L^2(Y_\epsilon)} \leq c_1\|\nabla^\epsilon u_\epsilon\|_{L^2(Y_\epsilon)} + c_4\|\nabla^\epsilon u_\epsilon\|^{q_1}_{L^2(Y_\epsilon)} + c_6,$$

where c_6 is a constant, $q_1 < 2$, and $c_3 > 0$.

It follows that, for $\epsilon \leq 1$, $\|\nabla^\epsilon u_\epsilon\|_{L^2(Y_\epsilon)}$ is bounded.

Then $\left(\frac{1}{\epsilon}\frac{\partial u_\epsilon}{\partial y_2}\right)_\epsilon$ and $\left(\frac{1}{\epsilon}\frac{\partial u_\epsilon}{\partial y_3}\right)_\epsilon$ are bounded in $L^2(Y_\epsilon)$. Using

$$\|\nabla u_\epsilon\|_{L^2(Y_\epsilon)} \leq \|\nabla^\epsilon u_\epsilon\|_{L^2(Y_\epsilon)},$$

we get that $(u_\epsilon)_\epsilon$ is bounded in $H^1(Y_\epsilon)$, so 2) is true.

To prove 3), we take the square of the first inequality of (B5) and we obtain

$$\|A_\epsilon(x)\Phi_\epsilon(x,U_\epsilon(x),B_\epsilon(x)\nabla U_\epsilon(x))\|^2 \le C\|\nabla U_\epsilon(x)\|^2 + C|U_\epsilon(x)|^2 + |\alpha(x)|^2$$

for a.e. $x \in \Omega_\epsilon$.

Multiplying by $\frac{1}{\epsilon^2}$ and integrating on Ω_ϵ we get

$$\begin{aligned} &\frac{1}{\epsilon^2}\int_{\Omega_\epsilon} \|A_\epsilon(x)\Phi_\epsilon(x,U_\epsilon(x),B_\epsilon(x)\nabla U_\epsilon(x))\|^2 \,\mathrm{d}x \\ &\le \frac{C}{\epsilon^2}\int_{\Omega_\epsilon}\|\nabla U_\epsilon(x)\|^2 \,\mathrm{d}x + \frac{C}{\epsilon^2}\int_{\Omega_\epsilon}|U_\epsilon(x)|^2 \,\mathrm{d}x + \frac{1}{\epsilon^2}\int_{\Omega_\epsilon}|\alpha|^2 \,\mathrm{d}x \\ &\le \frac{C}{\epsilon^2}\int_{\Omega_\epsilon}\|\nabla U_\epsilon(x)\|^2 \,\mathrm{d}x + \frac{C}{\epsilon^2}\int_{\Omega_\epsilon}|U_\epsilon(x)|^2 \,\mathrm{d}x + \frac{|\Omega_\epsilon|}{\epsilon^2}\bar{\alpha}, \end{aligned}$$

where $\bar{\alpha}$ is a constant. Using the change of variables $y = y_\epsilon(x)$, we get

$$\begin{aligned} &\int_{Y_\epsilon} \|A_\epsilon(y_\epsilon^{-1}(y))\Phi_\epsilon(y_\epsilon^{-1}(y),u_\epsilon(y),B_\epsilon(y_\epsilon^{-1}(y))\nabla^\epsilon u_\epsilon(y))\|^2 \,\mathrm{d}y \\ &\le C\int_{Y_\epsilon}\|\nabla^\epsilon u_\epsilon(y)\|^2 \,\mathrm{d}y + C\int_{Y_\epsilon}|u_\epsilon(y)|^2 \,\mathrm{d}y + \bar{\alpha}, \end{aligned}$$

which can be written as

$$\begin{aligned} &\|A_\epsilon(y_\epsilon^{-1}(\cdot))\Phi_\epsilon(y_\epsilon^{-1}(\cdot),u_\epsilon,B_\epsilon(y_\epsilon^{-1}(\cdot))\nabla^\epsilon u_\epsilon)\|^2_{L^2(Y_\epsilon)} \\ &\le C\|\nabla^\epsilon u_\epsilon\|^2_{L^2(Y_\epsilon)} + C\|u_\epsilon(y)\|^2_{L^2(Y_\epsilon)} + \bar{\alpha} \le \bar{C}, \end{aligned}$$

as $\|\nabla^\epsilon u_\epsilon\|_{L^2(Y_\epsilon)}$ and $\|u_\epsilon\|_{L^2(Y_\epsilon)}$ are bounded. It follows that $(\sigma_\epsilon)_\epsilon$ is bounded in $L^2(Y_\epsilon)$. □

Corollary 3.5. *Let $U_\epsilon \in M_\epsilon$ be the solution of the inequality (2.1) and $u_\epsilon \in K_\epsilon$ given by (3.2). If assumptions (B1) - (B10) are verified then the sequence U_ϵ satisfies*

$$U_\epsilon \in M_\epsilon, \quad \frac{1}{|\Omega_\epsilon|}\int_{\Omega_\epsilon}|\nabla U_\epsilon|^2 dx \le C. \tag{3.8}$$

Proof. By Lemma 3.4 we get that $(\nabla^\epsilon u_\epsilon)_\epsilon$ is bounded in $L^2(Y_\epsilon)$, i.e. there exists $C > 0$ such that

$$\int_{Y_\epsilon}\|\nabla^\epsilon u_\epsilon(y)\|^2 \,\mathrm{d}y \le C.$$

Using the change of variables $x = y_\epsilon^{-1}(y)$, we get

$$\frac{1}{\epsilon^2}\int_{\Omega_\epsilon}\|\nabla^\epsilon U_\epsilon(x)\|^2 \,\mathrm{d}x < C,$$

from where the statement of the corollary follows, as

$$|\Omega_\epsilon| = 2\pi|S|^2\epsilon^2(1 - t_\epsilon + t_\epsilon r_\epsilon^2).$$

□

Theorem 3.6. *Let U_ϵ be the solution of the variational inequality (2.1) and $u_\epsilon \in K_\epsilon$ defined by*

$$u_\epsilon(y) = U_\epsilon(y_\epsilon^{-1}(y)) \quad a.e.\ y \in Y_\epsilon.$$

If assumptions (B1)-(B12) are verified, then there exist three functions u, w, and σ^1 with

$$u \in H^1((-1,0) \cup (0,1)) \cap K, \quad u(-1) = u(1) = 0,$$
$$w \in L, \quad \sigma^1 \in L^2(Y^1)^3,$$

such that up to extraction of a subsequence

$$\chi_{Y_\epsilon^1} u_\epsilon \to u \quad in \quad L^2(Y^1); \tag{3.9}$$
$$\chi_{Y_\epsilon^-} \frac{\partial u_\epsilon}{\partial y_1} \rightharpoonup \frac{\partial u}{\partial y_1} \quad in \quad L^2(Y^-);$$
$$\chi_{Y_\epsilon^+} \frac{\partial u_\epsilon}{\partial y_1} \rightharpoonup \frac{\partial u}{\partial y_1} \quad in \quad L^2(Y^+);$$
$$\chi_{Y_\epsilon^1} \frac{1}{\epsilon} \nabla_{y'} u_\epsilon \rightharpoonup \nabla_{y'} w \quad in \ \ L^2(Y^1)^2;$$

and

$$\chi_{Y_\epsilon^1} \sigma_\epsilon \rightharpoonup \sigma^1 \quad in \ \ L^2(Y^1)^3.$$

Proof. From Lemma 3.4 it follows that there exist three functions $u \in H^1((-1,0) \cup (0,1))$, $w \in L^2((-1,1); H^1(S))$, and $\sigma^1 \in L^2(Y^1)^3$, which satisfy the statement of the lemma. From assumption (B11) we get that $u \in H^1((-1,0) \cup (0,1)) \cap K$, and from (B12) we obtain that $w \in L$. □

Theorem 3.7. *Let U_ϵ be the solution of the variational inequality (2.1) and $u \in H^1((-1,0) \cup (0,1)) \cap K$ given in Theorem 3.6. If assumptions (B1)-(B11) are verified, then there exists a subsequence of solutions U_ϵ, also denoted by U_ϵ, such that*

$$\lim_{\epsilon \to 0} \frac{1}{|\Omega_\epsilon|} \int_{\Omega_\epsilon} |U_\epsilon(x) - u(x_1)|^2 \, dx = 0. \tag{3.10}$$

Proof. Let $u_\epsilon \in K_\epsilon$ given by (3.2). From Theorem 3.6 follows that there exists u with

$$u \in H^1((-1,0) \cup (0,1)) \cap K, \quad u(-1) = u(1) = 0,$$

such that up to extraction of a subsequence

$$\chi_{Y_\epsilon^1} u_\epsilon \to u \quad \text{in} \quad L^2(Y^1),$$

which is equivalent with

$$\int_{Y_\epsilon} |u_\epsilon(y) - u(y_1)|^2 \, dy = 0.$$

Using the change of variables $x = y_\epsilon^{-1}(y)$, we get (3.10). □

References

[1] Bruno, P., *Geometrically constrained magnetic wall*, Phys. Rev. Lett., **83**(1999), 2425-2428.

[2] Cabib, E., Freddi, L., Morassi, A., Percivale, D., *Thin notched beams*, J. Elasticity, **64**(2002), 157-178.

[3] Casado-Díaz, J., Luna-Laynez, M., Murat, F., *Asymptotic behavior of diffusion problems in a domain made of two cylinders of different diameters and lengths*, C. R. Acad. Sci. Paris, Sér. I, **338**(2004), 133-138.

[4] Casado-Díaz, J., Luna-Laynez, M., Murat, F., *The diffusion equation in a notched beam*, Calculus of Variations, **31**(2008), 297-323.

[5] Gustafsson, B., Mossino, J., *Non-periodic explicit homogenization and reduction of dimension: the linear case*, IMA Journal of Applied Mathematics, **68**(2003), 269-298.

[6] Hale, J. K., Vegas, J., *A nonlinear parabolic equation with varying domain*, Arch. Rat. Mech. Anal., **86**(1984), 99-123.

[7] Jimbo, S., *Singular perturbation of domains and semilinear elliptic equation*, J. Fac. Sci. Univ. Tokyo, **35**(1988), 27-76.

[8] Jimbo, S., *Singular perturbation of domains and semilinear elliptic equation 2*, J. Diff. Eq., **75**(1988), 264-289.

[9] Kohn, R. V., Slastikov, V. V., *Geometrically Constrained Walls*, Calculus of Variations and Partial Differential Equations, **28**(2007), no. 1, 33-57.

[10] Kolumbán, J., Marchis, I., Szász, T., *Homogenization and reduction of dimension for nonlinear parametric variational inequalities*, Nonlinear Analysis-Theory Methods & Applications, **71**(2009), no. 3-4, 819-828.

[11] Murat, F., Sili, A., *Problèmes monotones dans des cylindres de faible diamètre formés de matériaux hétérogènes*, C. R. Acad. Sci. de Paris, Série 1, **320**(1995), 1199-1204.

[12] Rubinstein, J., Schatzman, M., Sternberg, P., *Ginzburg-Landau model in thin loops with narrow contrictions*, SIAM J. Appl. Math., **64**(2004), 2186-2204.

[13] Sili, A., *Asymptotic behavior of the solutions of monotone problems in flat cylinders*, Asymptotic Analysis, **19**(1999), 19-33.

Iuliana Marchiş
Psychology and Educational Sciences Faculty
Babeş-Bolyai University, Cluj-Napoca, Romania
e-mail: krokusz@gmail.com