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On $A_{p,q}^{lip}(G)$ spaces

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Abstract. In this paper, the space $A_{p,q}^{lip}(G)$ consisting of all complex valued functions $f \in lip(\alpha, 1)$ whose Fourier transform $\widehat{f}$ belongs to $L(p,q)\left(\widehat{G}\right)$ is investigated.

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1. Introduction

Let G denote a locally compact Abelian group, with dual group $\widehat{G}$ and Haar measure μ and $\widehat{\mu}$, respectively. The Fourier transform of a function $f \in L^1(G)$ will be denoted by $\widehat{f}$ which is continuous on $\widehat{G}$, vanishes at infinity and satisfies the inequality $\left\|\widehat{f}\right\|_\infty \leq \|f\|_1$. It is known that the space

$$A_p(G) = \left\{ f \in L^1(G) : \widehat{f} \in L^p\left(\widehat{G}\right)\right\}$$

is a Banach algebra for $1 \leq p < \infty$ and for $1 < p < \infty,\ 1 \leq q < \infty$, the space

$$A(p,q)(G) = \left\{ f \in L^1(G) : \widehat{f} \in L(p,q)\left(\widehat{G}\right)\ \right\}$$

is a Segal Algebra with respect to the usual convolution product and the norms defined by $\|f\| = \|f\|_1 + \left\|\widehat{f}\right\|_p$, $\|f\| = \|f\|_1 + \left\|\widehat{f}\right\|_{p,q}$ respectively. These spaces are examined by Larsen-Liu-Wang [15], Lai [11-13], Martin-Yap [16], Yap [23,24] and others.

For the convenience of the reader, we briefly review what we need from the theory of $L(p,q)(G)$ spaces. Let (G, Σ, μ) be a positive measure space and let f be a complex-valued, measurable function on G. For each $y \geq 0$ let

$$\lambda_f(y) = \mu\{x \in G : \ |f(x)| > y\}.$$

The function λ_f is called the distribution function of f. The rearrangement of f on $(0,\infty)$ is defined by

$$f^*(t)=\inf\{y>0:\lambda_f(y)\le t\}=\sup\{y>0:\lambda_f(y)>t\},\ t>0,$$

where $\inf\phi=\infty$. Also, the average function of f is defined by

$$f^{**}(t)=\frac{1}{t}\int_0^t f^*(s)\,ds\ ,\ t>0.$$

For $p,q\in(0,\infty)$ we define

$$\|f\|^*_{p,q} = \|f\|^*_{p,q,\mu}=\left(\frac{q}{p}\int_0^\infty [f^*(t)]^q\, t^{\frac{q}{p}-1}dt\right)^{\frac{1}{q}}$$

$$\|f\|_{p,q} = \|f\|_{p,q,\mu}=\left(\frac{q}{p}\int_0^\infty [f^{**}(t)]^q\, t^{\frac{q}{p}-1}dt\right)^{\frac{1}{q}}.$$

Also, if $0<p, q=\infty$ we define

$$\|f\|^*_{p,\infty}=\sup_{t>0} t^{\frac{1}{p}}f^*(t)\ \text{ and }\ \|f\|_{p,\infty}=\sup_{t>0} t^{\frac{1}{p}}f^{**}(t).$$

For $0<p<\infty$ and $0<q\le\infty$, Lorentz spaces are denoted by $L(p,q)(G,\Sigma,\mu)$ (or shortly $L(p,q)(G)$) is defined to be the vector space of all (equivalence classes of) measurable functions f on G such that $\|f\|^*_{p,q}<\infty$. We know that, for $1\le p\le\infty$, $\|f\|^*_{p,p}=\|f\|_p$ and so $L_p(G)=L(p,p)(G)$ where $L_p(G)$ is the usual Lebesgue space. It is also known that if $1<p<\infty$ and $1\le q\le\infty$ then

$$\|f\|^*_{p,q}\le\|f\|_{p,q}\le\frac{p}{p-1}\|f\|^*_{p,q}$$

for each $f\in L(p,q)(G)$ and $\left(L(p,q)(G),\|\cdot\|_{pq}\right)$ is a Banach space [10].

In [4], Chen and Lai showed that there is an approximate identity $\{a_\alpha\}_{\alpha\in I}$ of $L_1(G)$ such that $\|a_\alpha\|_1=1$ for each $\alpha\in I$ and $f*a_\alpha\to f$ for every $f\in L(p,q)(G)$, whenever $1<p<\infty$, $1\le q<\infty$. It can be derived from [2],[3] and [20] that $L(p,q)(G)$ is an essential Banach $L_1(G)-$module with the usual convolution and the norm $\|\cdot\|_{p,q}$. Also, in [4], Chen and Lai showed that $(L_1(G),L(p,q)(G))$ is isometrically isomorphic to $L(p,q)(G)$ for $1<p,q<\infty$. One can also review [2-5,10,17,20,22] for more properties of $L(p,q)(G)$ Lorentz spaces.

Throughout the paper G will denote a metrizable locally compact Abelian group with a translation invariant metric d such that for any $y\in G$, $|y|=d(0,y)$ and Haar measure μ.We assume that there is a decreasing countable (open) basis $\{V_n\}_{n\in\mathbb{N}}$ of the identity e of G such that

$$\mu((y+V_n)\,\triangle\,V_n)/|y|^\alpha\to 0\ \text{ as } y\to 0,$$

where $\triangle$ denotes the symmetric difference, $\alpha \in (0,1)$ and $y \in G$. Quek and Yap showed that the above condition is not unduly restrictive. Example of groups that have these properties are $R^k, T^k\,(k \geq 1)$, the 0-dimensional groups, etc.[19]. While χ denotes the characteristic function, it is easy to see that $\{e_n\}_{n\in\mathbb{N}}$ is an approximate identity for $L^1(G)$ which is defined by $e_n = \mu(V_n)^{-1}\chi_{V_n}$. For any $f \in L^1(G)$ and $\delta > 0$, define

$$\omega_1(f;\delta) = \sup\left\{\|\tau_y f - f\|_1 : |y| \leq \delta\right\},$$

where $\tau_y f(x) = f(x-y)$. Following Zygmund [25],Bloom [1] and Quek-Yap [19], we define

$$\begin{aligned} Lip(\alpha,1) &= \left\{f \in L^1(G) : \omega_1(f;\delta) = O(\delta^\alpha)\ \right\} \\ lip(\alpha,1) &= \{f \in Lip(\alpha,1) : \omega_1(f;\delta) = o(\delta^\alpha)\ \}. \end{aligned}$$

These spaces are called as Lipschitz spaces and the function $\|\cdot\|_{(\alpha,1)}$ defined by

$$\|f\|_{(\alpha,1)} = \|f\|_1 + \sup_{y\neq 0}\frac{\|\tau_y f - f\|_1}{|y|^\alpha}$$

is a norm in both Lipschitz spaces. Quek and Yap in [19],Feichtinger in [6,7] proved a series of results concerning Lipschitz spaces.

2. The space $A_{p,q}^{lip}(G)$

Let G be a metrizable locally compact Abelian group, $\alpha \in (0,1)$ and $1 < p < \infty$, $1 \leq q < \infty$. We define the vector space $A_{p,q}^{lip}(G)$ by

$$A_{p,q}^{lip}(G) = \left\{f \in lip(\alpha,1)(G) :\ \widehat{f} \in L(p,q)\left(\widehat{G}\right)\right\}.$$

If one endows it with the norm

$$\|f\|_{p,q}^{lip} = \|f\|_{(\alpha,1)} + \left\|\widehat{f}\right\|_{p,q}$$

where $f \in A_{p,q}^{lip}(G)$, then it is easy to see that $A_{p,q}^{lip}(G) = lip(\alpha,1)(G) \cap A(p,q)(G)$ becomes a normed space.

Theorem 2.1. *The space $\left(A_{p,q}^{lip}(G), \|\cdot\|_{p,q}^{lip}\right)$ is a Banach space for $p = q = 1$, $p = q = \infty$ or $1 < p \leq \infty, 1 \leq q \leq \infty$.*

Proof. Assume that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $A_{p,q}^{lip}(G)$. Clearly, $\{f_n\}_{n\in\mathbb{N}}$ and $\left\{\widehat{f_n}\right\}_{n\in\mathbb{N}}$ are also Cauchy sequences in $lip(\alpha,1)(G)$ and $L(p,q)\left(\widehat{G}\right)$, respectively. Since $lip(\alpha,1)(G)$ and $L(p,q)\left(\widehat{G}\right)$ are Banach spaces, there exist $f \in lip(\alpha,1)(G)$ and $g \in L(p,q)\left(\widehat{G}\right)$ such that $\|f_n - f\|_{(\alpha,1)} \to 0$, $\|f_n - f\|_1 \to 0$ and $\left\|\widehat{f_n} - g\right\|_{p,q} \to 0$. Using Lemma 2.2 in

[24], there exists a subsequence $\left\{\widehat{f_{n_k}}\right\}_{n\in\mathbb{N}}$ of $\left\{\widehat{f_n}\right\}_{n\in\mathbb{N}}$ which converges to g almost everywhere. It follows from the inequality

$$\left\|\widehat{f_n}-\widehat{f}\right\|_\infty \leq \|f_n - f\|_1 \leq \|f_n - f\|_{(\alpha,1)}$$

that $\left\|\widehat{f_n}-\widehat{f}\right\|_\infty \to 0$. Hence it is easly showed that $\left\|\widehat{f_{n_k}}-\widehat{f}\right\|_\infty \to 0$. Therefore $\widehat{f}=g$, $\|f_n - f\|_{p,q}^{lip} \to 0$ and $f \in A_{p,q}^{lip}(G)$. Thus $\left(A_{p,q}^{lip}(G), \|\cdot\|_{p,q}^{lip}\right)$ is a Banach space. □

By using the propositions and lemmas proved in [19], one can easly prove the following propositions.

Proposition 2.2. *The space $lip(\alpha,1)(G)$ is a Banach algebra with usual convolution product.*

Proposition 2.3. *The space $lip(\alpha,1)(G)$ is strongly translation and character invariant.*

Proof. It is known that $L^1(G)$ is strongly translation invariant, i.e., $\tau_x f \in L^1(G)$ and $\|\tau_x f\|_1 = \|f\|_1$ for all $x \in G$, $f \in L^1(G)$. Let us take any $f \in lip(\alpha,1)(G)$ and $x \in G$. Then for any $\varepsilon > 0$, there exits a $\delta_\varepsilon > 0$ such that

$$\sup_{|y|\leq\delta} \frac{\|\tau_y f - f\|_1}{\delta^\alpha} < \varepsilon$$

whenever $0 < \delta < \delta_\varepsilon$. For the same $\varepsilon > 0$, we have

$$\sup_{|y|\leq\delta} \frac{\|\tau_y(\tau_x f) - (\tau_x f)\|_1}{\delta^\alpha} = \sup_{|y|\leq\delta} \frac{\|\tau_y f - f\|_1}{\delta^\alpha} < \varepsilon$$

whenever $0 < \delta < \delta_\varepsilon$. Therefore $\omega_1(\tau_x f; \delta) = o(\delta^\alpha)$, $\tau_x f \in lip(\alpha,1)(G)$ and $\|\tau_x f\|_{(\alpha,1)} = \|f\|_{(\alpha,1)}$.

Strongly character invariance of $lip(\alpha,1)(G)$ can be seen in a similar way. □

Proposition 2.4. *The function $x \to \tau_x f$ is continuous from G into $lip(\alpha,1)(G)$ for every $f \in lip(\alpha,1)(G)$.*

Proposition 2.5. *The space $lip(\alpha,1)(G)$ has an approximate identity $\{e_n\}_{n\in\mathbb{N}}$ defined by $e_n = \mu(V_n)^{-1}\chi_{V_n}$.*

Proposition 2.6. *The space $lip(\alpha,1)(G)$ is a homogeneous Banach space.*

Proposition 2.7. *The space $lip(\alpha,1)(G)$ is an essential $L^1(G)-$module.*

Theorem 2.8. *The space $\left(A_{p,q}^{lip}(G), \|\cdot\|_{p,q}^{lip}\right)$ is a Banach module over $L^1(G)$ and $lip(\alpha,1)(G)$. Hence it is a Banach algebra with respect to the usual convolution.*

Proof. Let $f,g \in A_{p,q}^{lip}(G)$ be given. Since the space $lip(\alpha,1)(G)$ is a Banach algebra under convolution, then $f*g \in lip(\alpha,1)(G)$. Since $\widehat{f},\widehat{g} \in L(p,q)\left(\widehat{G}\right)$, we have

$$\begin{aligned} \lambda_{\widehat{f}\widehat{g}}(y) &= \mu\left\{x\in\widehat{G}:\ \left|\widehat{f}(x)\,\widehat{g}(x)\right|>y\right\} \\ &\le \mu\left\{x\in\widehat{G}:\ \left(\sup\widehat{f}(x)\right)|\widehat{g}(x)|>y\right\} \\ &= \mu\left\{x\in\widehat{G}:\ K|\widehat{g}(x)|>y\right\}=\lambda_{K\widehat{g}}(y), \end{aligned}$$

if $\sup\limits_{x\in\widehat{G}}\widehat{f}(x)=K$. Therefore we get

$$\left(\widehat{f}\widehat{g}\right)^{*}(t)=\inf\left\{y>0:\lambda_{\widehat{f}\widehat{g}}(y)\le t\ \right\}\le K(\widehat{g})^{*}(t),$$

$$\left(\widehat{f}\widehat{g}\right)^{**}(t)=\frac{1}{t}\int\limits_{0}^{t}\left(\widehat{f}\widehat{g}\right)^{*}(s)\,ds\le K(\widehat{g})^{**}(t)$$

and so

$$\begin{aligned} \left\|\widehat{f}\cdot\widehat{g}\right\|_{p,q} &\le K\|\widehat{g}\|_{p,q}\le\sup\widehat{f}(x)\|\widehat{g}\|_{p,q}\le\left\|\widehat{f}\right\|_{\infty}\|\widehat{g}\|_{p,q} \\ &\le \|f\|_{1}\|\widehat{g}\|_{p,q}. \end{aligned}$$

Thus, we obtain $\widehat{f*g}\in L(p,q)\left(\widehat{G}\right)$ and $f*g\in A_{p,q}^{lip}(G)$. Also, we have

$$\begin{aligned} \|f*g\|_{p,q}^{lip} &= \|f*g\|_{(\alpha,1)}+\left\|\widehat{f*g}\right\|_{p,q} \\ &\le \|f\|_{(\alpha,1)}\|g\|_{(\alpha,1)}+\left\|\widehat{f}\cdot\widehat{g}\right\|_{p,q} \\ &\le \|f\|_{(\alpha,1)}\|g\|_{(\alpha,1)}+\|f\|_{1}\|\widehat{g}\|_{p,q} \\ &\le \|f\|_{(\alpha,1)}\|g\|_{p,q}^{lip}\le\|f\|_{p,q}^{lip}\|g\|_{p,q}^{lip}, \end{aligned}$$

for any $f,g\in A_{p,q}^{lip}(G)$. □

By Proposition 2.3 in [19], Proposition 2.2 and Proposition 2.3, the following can be easily proved.

Proposition 2.9. *The space* $\left(A_{p,q}^{lip}(G),\|\cdot\|_{p,q}^{lip}\right)$ *is strongly translation invariant and the function* $x\to\tau_x f$ *is continuous from* G *into* $A_{p,q}^{lip}(G)$ *for every* $f\in A_{p,q}^{lip}(G)$.

Proposition 2.10. *The space* $\left(A_{p,q}^{lip}(G),\|\cdot\|_{p,q}^{lip}\right)$ *is a homogeneous Banach space.*

Proposition 2.11. *The space* $\left(A_{p,q}^{lip}(G),\|\cdot\|_{p,q}^{lip}\right)$ *is strongly character invariant.*

Proposition 2.12. *The space* $\left(A_{p,q}^{lip}(G), \|\cdot\|_{p,q}^{lip}\right)$ *is a semi-simple Banach algebra.*

Proof. Let $f \in A_{p,q}^{lip}(G)$ be given. It will be sufficient to show that $f = 0$ whenever $\left\|\widehat{f}\right\|_{\infty} = 0$. Since $A_{p,q}^{lip}(G)$ is a commutative Banach algebra by Theorem 2.8, it is known that

$$\lim_n \left(\|f^n\|_{p,q}^{lip}\right)^{\frac{1}{n}} = \left\|\widehat{f}\right\|_{\infty}.$$

Moreover, we have

$$\|f^n\|_1^{\frac{1}{n}} \leq \|f^n\|_{(\alpha,1)}^{\frac{1}{n}} \leq \left(\|f^n\|_{p,q}^{lip}\right)^{\frac{1}{n}}$$

and

$$\lim_n \|f^n\|_1^{\frac{1}{n}} \leq \lim_n \left(\|f^n\|_{p,q}^{lip}\right)^{\frac{1}{n}}.$$

If we set

$$\lim_n \|f^n\|_1^{\frac{1}{n}} = \left\|\widehat{f}\right\|_{\infty}',$$

then we have the inequality

$$\left\|\widehat{f}\right\|_{\infty}' \leq \left\|\widehat{f}\right\|_{\infty}.$$

Since $\left\|\widehat{f}\right\|_{\infty} = 0$, then $\left\|\widehat{f}\right\|_{\infty}' = 0$. Also, since $L^1(G)$ is semi-simple [14], then $f = 0$. □

Theorem 2.13. *The space* $A_{p,q}^{lip}(G)$ *is an essential Banach* $L^1(G)-$*module.*

Proof. In view of Lemma 4.1 in [8], it will be sufficient to show that any bounded approximate identity $\{e_\alpha\}_{\alpha\in I}$ of $L^1(G)$ which belongs to

$$\Lambda^K = \left\{f \in L^1(G) : \text{ supp } \widehat{f} \text{ compact}\right\}$$

is also an approximate identity for $A_{p,q}^{lip}(G)$. Let $f \in A_{p,q}^{lip}(G) \subset lip(\alpha, 1) \subset L^1(G)$. By the same Lemma, the bounded approximate identity $\{e_\alpha\}_{\alpha\in I} \subset \Lambda^K$ is also an approximate identity for $L^1(G)$, and so, for any given $\varepsilon > 0$, we have $\|e_\alpha * f - f\|_1 < \varepsilon$ for sufficiently large α. For each $\alpha \in I, \|e_\alpha\|_1 = 1$ implies that $\sup_\alpha \|\widehat{e_\alpha}\|_\infty \leq 1$. Hence, for any $g \in L^1(G)$, the inequality

$$|\widehat{g}|\,|1 - \widehat{e_\alpha}| \leq \|\widehat{g} - \widehat{g}\widehat{e_\alpha}\|_\infty \leq \|g - g * e_\alpha\|_1 \to 0$$

implies uniformly convergence of $\{\widehat{e_\alpha}\}_{\alpha\in I}$ to 1 over compact sets. Since $\widehat{f} \in L(p,q)\left(\widehat{G}\right)$, we can choose a compact set $\widehat{K} \subset \widehat{G}$ such that

$$\left\|\widehat{f} - \widehat{f}\chi_{\widehat{K}}\right\|_{p,q} < \frac{\varepsilon}{8}$$

and the local convergence to 1 implies that one can find an α_0 with

$$\left\|\widehat{e_\alpha}\chi_{\widehat{K}} - \chi_{\widehat{K}}\right\|_\infty < \frac{\varepsilon}{4\left\|\widehat{f}\right\|_{p,q}} \text{ for all } \alpha > \alpha_0.$$

Altogether,

$$\begin{aligned}
\left\|f\widehat{-e_\alpha * f}\right\|_{p,q} &= \left\|\widehat{f} - \widehat{e_\alpha}\cdot\widehat{f}\right\|_{p,q} \qquad (2.1)\\
&\le \left\|\widehat{f} - \widehat{f}\chi_{\widehat{K}}\right\|_{p,q} + \left\|\widehat{f}\chi_{\widehat{K}} - \widehat{f}\chi_{\widehat{K}}\widehat{e_\alpha}\right\|_{p,q} + \left\|\widehat{f}\chi_{\widehat{K}}\widehat{e_\alpha} - \widehat{f}\widehat{e_\alpha}\right\|_{p,q}\\
&\le (1 + \|\widehat{e_\alpha}\|_\infty)\left\|\widehat{f} - \widehat{f}\chi_{\widehat{K}}\right\|_{p,q} + \left\|\widehat{f}\right\|_{p,q}\left\|\widehat{e_\alpha}\chi_{\widehat{K}} - \chi_{\widehat{K}}\right\|_\infty\\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}
\end{aligned}$$

for all $\alpha > \alpha_0$. Also it is known that

$$\|f - f * e_\alpha\|_{(\alpha,1)} < \frac{\varepsilon}{2} \qquad (2.2)$$

for any $f \in lip\,(\alpha, 1)$ by Proposition 2.4. Finally, by using (2.1) and (2.2), we obtain

$$\begin{aligned}
\|f - f * e_\alpha\|_{p,q}^{lip} &= \|f - f * e_\alpha\|_{(\alpha,1)} + \left\|f\widehat{-e_\alpha * f}\right\|_{p,q}\\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

for all $\alpha > \alpha_0$. Therefore it follows that $\|f - f * e_\alpha\|_{p,q}^{lip} \to 0$ for any $f \in A_{p,q}^{lip}(G)$. Consequently, $A_{p,q}^{lip}(G)$ is an essential Banach module by Module Factorization Theorem in [9]. This means $\left(A_{p,q}^{lip}(G)\right)_e = A_{p,q}^{lip}(G)$. □

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