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Behavior of a rational recursive sequences

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Abstract. We obtain in this paper the solutions of the difference equations

$$x_{n+1} = \frac{x_{n-7}}{\pm 1 \pm x_{n-1}x_{n-3}x_{n-5}x_{n-7}}, \quad n = 0, 1, ...,$$

where the initial conditions are arbitrary nonzero real numbers.

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1. Introduction

In this paper we obtain the solutions of the following recursive sequences

$$x_{n+1} = \frac{x_{n-7}}{\pm 1 \pm x_{n-1}x_{n-3}x_{n-5}x_{n-7}}, \quad n = 0, 1, ..., \tag{1.1}$$

where the initial conditions are arbitrary nonzero real numbers.

Recently there has been a great interest in studying the qualitative properties of rational difference equations. For the systematical studies of rational and nonrational difference equations, one can refer to the papers [1-41] and references therein.

The study of rational difference equations of order greater than one is quite challenging and rewarding because some prototypes for the development of the basic theory of the global behavior of nonlinear difference equations of order greater than one come from the results for rational difference equations. However, there have not been any effective general methods to deal with the global behavior of rational difference equations of order greater than one so far. Therefore, the study of rational difference equations of order greater than one is worth further consideration.

Aloqeili [5] has obtained the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}.$$

Cinar [7]-[9] investigated the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-1}}{1 + a x_n x_{n-1}}, \quad x_{n+1} = \frac{x_{n-1}}{-1 + a x_n x_{n-1}}, \quad x_{n+1} = \frac{a x_{n-1}}{1 + b x_n x_{n-1}}.$$

Elabbasy et al. [11]-[12] investigated the global stability, periodicity character and gave the solution of special case of the following recursive sequences

$$x_{n+1} = a x_n - \frac{b x_n}{c x_n - d x_{n-1}}, \quad x_{n+1} = \frac{d x_{n-l} x_{n-k}}{c x_{n-s} - b} + a.$$

Elabbasy et al. [15] gave the solution of the following difference equations

$$x_{n+1} = \frac{x_{n-7}}{\pm 1 \pm x_{n-3} x_{n-7}}.$$

Karatas et al. [26] get the form of the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2} x_{n-5}}.$$

Simsek et al. [33] obtained the solution of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}.$$

Here, we recall some notations and results which will be useful in our investigation.

Let I be some interval of real numbers and let

$$f : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{1.2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$ [29].

Definition 1.1. *(Equilibrium Point)*

A point $\overline{x} \in I$ is called an equilibrium point of Eq. (1.2) if

$$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq. (1.2), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 1.2. *(Stability)*

(i) The equilibrium point $\overline{x}$ of Eq. (1.2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq. (1.2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq. (1.2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ *of Eq. (1.2) is global attractor if for all* $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$, *we have*

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ *of Eq. (1.2) is globally asymptotically stable if* $\overline{x}$ *is locally stable, and* $\overline{x}$ *is also a global attractor of Eq. (1.2).*

(v) The equilibrium point $\overline{x}$ *of Eq. (1.2) is unstable if* $\overline{x}$ *is not locally stable.*

The linearized equation of Eq. (1.2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}.$$

Theorem 1.3. [28] *Assume that* $p, q \in \mathbb{R}$ *and* $k \in \{0, 1, 2, ...\}$. *Then*

$$|p| + |q| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n = 0, 1, ...$$

Remark 1.4. *Theorem 1.3 can be easily extended to a general linear equations of the form*

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \ \ n = 0, 1, ..., \tag{1.3}$$

where $p_1, p_2, ..., p_k \in \mathbb{R}$ *and* $k \in \{1, 2, ...\}$. *Then Eq. (1.3) is asymptotically stable provided that*

$$\sum_{i=1}^{k} |p_i| < 1.$$

Definition 1.5. *(Periodicity)*

A sequence $\{x_n\}_{n=-k}^{\infty}$ *is said to be periodic with period* p *if* $x_{n+p} = x_n$ *for all* $n \geq -k$.

2. On the Difference Equation $x_{n+1} = \frac{x_{n-7}}{1+x_{n-1}x_{n-3}x_{n-5}x_{n-7}}$

In this section we give a specific form of the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-7}}{1 + x_{n-1}x_{n-3}x_{n-5}x_{n-7}}, \quad n = 0, 1, ..., \tag{2.1}$$

where the initial conditions are arbitrary nonzero positive real numbers.

Theorem 2.1. *Let $\{x_n\}_{n=-7}^{\infty}$ be a solution of Eq. (2.1). Then for $n = 0, 1, ...$*

$$x_{8n-7} = \frac{h\prod_{i=0}^{n-1}(1+4ibdfh)}{\prod_{i=0}^{n-1}(1+(4i+1)bdfh)}, \quad x_{8n-3} = \frac{d\prod_{i=0}^{n-1}(1+(4i+2)bdfh)}{\prod_{i=0}^{n-1}(1+(4i+3)bdfh)},$$

$$x_{8n-6} = \frac{g\prod_{i=0}^{n-1}(1+4iaceg)}{\prod_{i=0}^{n-1}(1+(4i+1)aceg)}, \quad x_{8n-2} = \frac{c\prod_{i=0}^{n-1}(1+(4i+2)aceg)}{\prod_{i=0}^{n-1}(1+(4i+3)aceg)},$$

$$x_{8n-5} = \frac{f\prod_{i=0}^{n-1}(1+(4i+1)bdfh)}{\prod_{i=0}^{n-1}(1+(4i+2)bdfh)}, \quad x_{8n-1} = \frac{b\prod_{i=0}^{n-1}(1+(4i+3)bdfh)}{\prod_{i=0}^{n-1}(1+(4i+4)bdfh)},$$

$$x_{8n-4} = \frac{e\prod_{i=0}^{n-1}(1+(4i+1)aceg)}{\prod_{i=0}^{n-1}(1+(4i+2)aceg)}, \quad x_{8n} = \frac{a\prod_{i=0}^{n-1}(1+(4i+3)aceg)}{\prod_{i=0}^{n-1}(1+(4i+4)aceg)},$$

where $x_{-7} = h,\ x_{-6} = g,\ x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_{-0} = a,\ \prod_{i=0}^{-1} A_i = 1$.

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is;

$$x_{8n-15} = \frac{h\prod_{i=0}^{n-2}(1+4ibdfh)}{\prod_{i=0}^{n-2}(1+(4i+1)bdfh)}, \quad x_{8n-11} = \frac{d\prod_{i=0}^{n-2}(1+(4i+2)bdfh)}{\prod_{i=0}^{n-2}(1+(4i+3)bdfh)},$$

$$x_{8n-14} = \frac{g\prod_{i=0}^{n-2}(1+4iaceg)}{\prod_{i=0}^{n-2}(1+(4i+1)aceg)}, \quad x_{8n-10} = \frac{c\prod_{i=0}^{n-2}(1+(4i+2)aceg)}{\prod_{i=0}^{n-2}(1+(4i+3)aceg)},$$

$$x_{8n-13} = \frac{f\prod_{i=0}^{n-2}(1+(4i+1)bdfh)}{\prod_{i=0}^{n-2}(1+(4i+2)bdfh)}, \quad x_{8n-9} = \frac{b\prod_{i=0}^{n-2}(1+(4i+3)bdfh)}{\prod_{i=0}^{n-2}(1+(4i+4)bdfh)},$$

$$x_{8n-12} = \frac{e\prod_{i=0}^{n-2}(1+(4i+1)aceg)}{\prod_{i=0}^{n-2}(1+(4i+2)aceg)}, \quad x_{8n-8} = \frac{a\prod_{i=0}^{n-2}(1+(4i+3)aceg)}{\prod_{i=0}^{n-2}(1+(4i+4)aceg)}.$$

Now, it follows from Eq. (2.1) that

$$x_{8n-7} = \frac{x_{8n-15}}{1 + x_{8n-9}x_{8n-11}x_{8n-13}x_{8n-15}}$$

$$= \frac{\frac{h\prod_{i=0}^{n-2}(1+4ibdfh)}{\prod_{i=0}^{n-2}(1+(4i+1)bdfh)}}{1+\frac{b\prod_{i=0}^{n-2}(1+(4i+3)bdfh)}{\prod_{i=0}^{n-2}(1+(4i+4)bdfh)}\frac{d\prod_{i=0}^{n-2}(1+(4i+2)bdfh)}{\prod_{i=0}^{n-2}(1+(4i+3)bdfh)}\frac{f\prod_{i=0}^{n-2}(1+(4i+1)bdfh)}{\prod_{i=0}^{n-2}(1+(4i+2)bdfh)}\frac{h\prod_{i=0}^{n-2}(1+4ibdfh)}{\prod_{i=0}^{n-2}(1+(4i+1)bdfh)}}$$

$$= \frac{h\prod_{i=0}^{n-2}(1+4ibdfh)}{\prod_{i=0}^{n-2}(1+(4i+1)bdfh)}\left(\frac{1}{1+\frac{bdfh}{\prod_{i=0}^{n-2}(1+(4i+4)bdfh)}\prod_{i=0}^{n-2}(1+4ibdfh)}\right)$$

$$= \frac{h\prod_{i=0}^{n-2}(1+4ibdfh)}{\prod_{i=0}^{n-2}(1+(4i+1)bdfh)}\left(\frac{1}{1+\frac{bdfh}{(1+(4n-4)bdfh)}}\left\{\frac{(1+(4n-4)bdfh)}{(1+(4n-4)bdfh)}\right\}\right)$$

$$= \frac{h\prod_{i=0}^{n-2}(1+4ibdfh)}{\prod_{i=0}^{n-2}(1+(4i+1)bdfh)}\left(\frac{1+(4n-4)bdfh}{1+(4n-4)bdfh+bdfh}\right)$$

$$= \frac{h\prod_{i=0}^{n-2}(1+4ibdfh)}{\prod_{i=0}^{n-2}(1+(4i+1)bdfh)}\left(\frac{1+(4n-4)bdfh}{1+(4n-3)bdfh}\right).$$

Hence, we have

$$x_{8n-7} = \frac{h\prod_{i=0}^{n-1}(1+4ibdfh)}{\prod_{i=0}^{n-1}(1+(4i+1)bdfh)}.$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed.

Theorem 2.2. *Eq. (2.1) has a unique equilibrium point which is the number zero and this equilibrium point is not locally asymptotically stable.*

Proof. For the equilibrium points of Eq. (2.1), we can write

$$\overline{x} = \frac{\overline{x}}{1+\overline{x}^4}.$$

Then

$$\overline{x}+\overline{x}^5=\overline{x},$$

or

$$\overline{x}^5=0.$$

Thus the equilibrium point of Eq. (2.1) is $\overline{x}=0$. Let $f:(0,\infty)^4\longrightarrow(0,\infty)$ be a function defined by

$$f(u,v,w,t)=\frac{u}{1+uvwt}.$$

Therefore it follows that

$$f_u(u,v,w,t)=\frac{1}{(1+uvwt)^2},\quad f_v(u,v,w,t)=\frac{-u^2wt}{(1+uvwt)^2},$$
$$f_w(u,v,w,t)=\frac{-u^2vt}{(1+uvwt)^2},\quad f_t(u,v,w,t)=\frac{-u^2vw}{(1+uvwt)^2},$$

we see that

$$f_u(\overline{x},\overline{x},\overline{x},\overline{x})=1,\quad f_v(\overline{x},\overline{x},\overline{x},\overline{x}x)=0,\quad f_w(\overline{x},\overline{x},\overline{x},\overline{x})=0,\quad f_t(\overline{x},\overline{x},\overline{x},\overline{x})=0.$$

The proof follows by using Theorem 1.3.

Theorem 2.3. *Every positive solution of Eq. (2.1) is bounded and* $\lim_{n\to\infty}x_n=0$.

Proof. It follows from Eq. (2.1) that

$$x_{n+1}=\frac{x_{n-7}}{1+x_{n-1}x_{n-3}x_{n-5}x_{n-7}}\leq x_{n-7}.$$

Then the subsequences $\{x_{8n-7}\}_{n=0}^{\infty}$, $\{x_{8n-6}\}_{n=0}^{\infty}$, $\{x_{8n-5}\}_{n=0}^{\infty}$, $\{x_{8n-4}\}_{n=0}^{\infty}$, $\{x_{8n-3}\}_{n=0}^{\infty}$, $\{x_{8n-2}\}_{n=0}^{\infty}$, $\{x_{8n-1}\}_{n=0}^{\infty}$, $\{x_{8n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by $M=\max\{x_{-7},x_{-6},x_{-5},x_{-4},x_{-3},x_{-2},x_{-1},x_0\}$.

Numerical examples

For confirming the results of this section, we consider numerical examples which represent different types of solutions to Eq. (2.1).

Example 2.4. *Consider* $x_{-7}=2,\ x_{-6}=7,\ x_{-5}=3,\ x_{-4}=2,\ x_{-3}=6,\ x_{-2}=9,\ x_{-1}=5,\ x_0=14$. *See Fig. 1.*

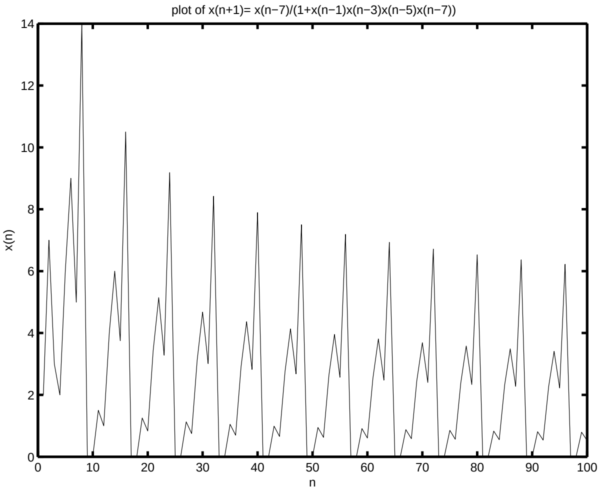

Figure 1.

Example 2.5. *See Fig. 2, since* $x_{-7} = 7,\ x_{-6} = 5,\ x_{-5} = 0.3,\ x_{-4} = 0.2,\ x_{-3} = 4,\ x_{-2} = 1,\ x_{-1} = 1.5,\ x_0 = 2.$

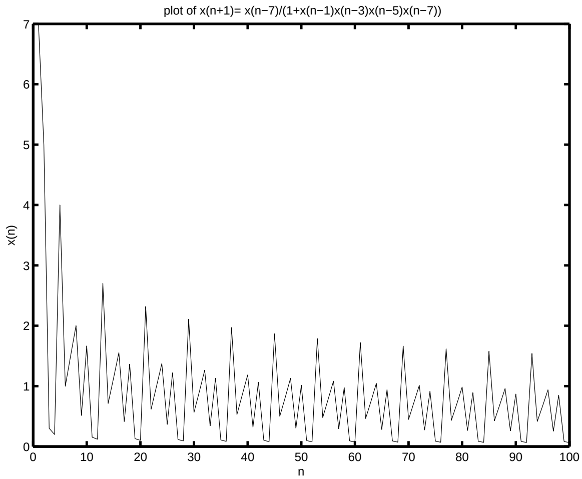

Figure 2.

3. On the Difference Equation $x_{n+1} = \frac{x_{n-7}}{1-x_{n-1}x_{n-3}x_{n-5}x_{n-7}}$

In this section we give a specific form of the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-7}}{1 - x_{n-1}x_{n-3}x_{n-5}x_{n-7}}, \quad n = 0, 1, ..., \tag{3.1}$$

where the initial conditions are arbitrary nonzero real numbers.

Theorem 3.1. *Let $\{x_n\}_{n=-7}^{\infty}$ be a solution of Eq. (3.1). Then for* $n = 0, 1, ...$

$$x_{8n-7} = \frac{h \prod_{i=0}^{n-1} (1 - 4ibdfh)}{\prod_{i=0}^{n-1} (1 - (4i+1)bdfh)}, \quad x_{8n-3} = \frac{d \prod_{i=0}^{n-1} (1 - (4i+2)bdfh)}{\prod_{i=0}^{n-1} (1 - (4i+3)bdfh)},$$

$$x_{8n-6} = \frac{g \prod_{i=0}^{n-1} (1 - 4iaceg)}{\prod_{i=0}^{n-1} (1 - (4i+1)aceg)}, \quad x_{8n-2} = \frac{c \prod_{i=0}^{n-1} (1 - (4i+2)aceg)}{\prod_{i=0}^{n-1} (1 - (4i+3)aceg)},$$

$$x_{8n-5} = \frac{f \prod_{i=0}^{n-1} (1 - (4i+1)bdfh)}{\prod_{i=0}^{n-1} (1 - (4i+2)bdfh)}, \quad x_{8n-1} = \frac{b \prod_{i=0}^{n-1} (1 - (4i+3)bdfh)}{\prod_{i=0}^{n-1} (1 - (4i+4)bdfh)},$$

$$x_{8n-4} = \frac{e \prod_{i=0}^{n-1} (1 - (4i+1)aceg)}{\prod_{i=0}^{n-1} (1 - (4i+2)aceg)}, \quad x_{8n} = \frac{a \prod_{i=0}^{n-1} (1 - (4i+3)aceg)}{\prod_{i=0}^{n-1} (1 - (4i+4)aceg)},$$

where $jbdfh \neq 1, jaceg \neq 1$ *for* $j = 1, 2, 3, ...$.

Proof. It is similar to the proof of Theorem 2.1 and will be omitted.

Theorem 3.2. *Eq. (3.1) has a unique equilibrium point which is the number zero and this equilibrium point is not locally asymptotically stable.*

Numerical examples

Example 3.3. *Consider* $x_{-7} = 7$, $x_{-6} = 5$, $x_{-5} = 3$, $x_{-4} = 2$, $x_{-3} = 4$, $x_{-2} = 1$, $x_{-1} = 11$, $x_0 = 2$. *See Fig. 3.*

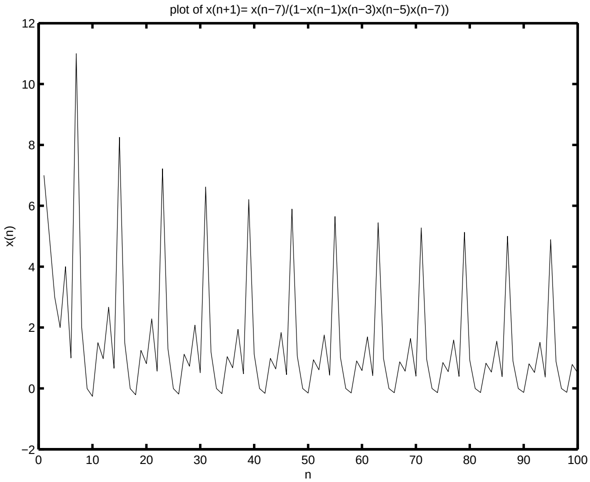

Figure 3.

Example 3.4. *See Fig. 4, since* $x_{-7} = 0.7,\ x_{-6} = 0.5,\ x_{-5} = 0.3,\ x_{-4} = 0.2,\ x_{-3} = 0.4,\ x_{-2} = 0.5,\ x_{-1} = 0.1,\ x_0 = 1.2.$

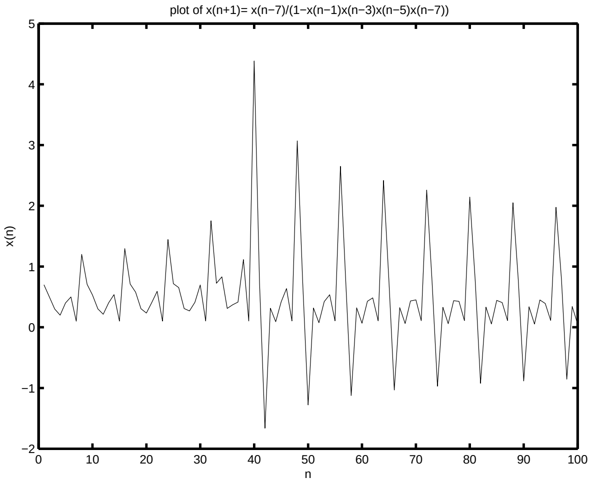

Figure 4.

4. On the Difference Equation $x_{n+1} = \frac{x_{n-7}}{-1+x_{n-1}x_{n-3}x_{n-5}x_{n-7}}$

In this section we investigate the solutions of the following difference equation

$$x_{n+1} = \frac{x_{n-7}}{-1 + x_{n-1}x_{n-3}x_{n-5}x_{n-7}}, \quad n = 0, 1, ..., \tag{4.1}$$

where the initial conditions are arbitrary nonzero real numbers with

$$x_{-7}x_{-5}x_{-3}x_{-1} \neq 1,\ x_{-6}x_{-4}x_{-2}x_0 \neq 1.$$

Theorem 4.1. *Let $\{x_n\}_{n=-7}^{\infty}$ be a solution of Eq. (4.1). Then Eq. (4.1) has unbounded solutions and for $n = 0, 1, ...$*

$$x_{8n-7} = \frac{h}{(-1+bdfh)^n}, \qquad x_{8n-3} = \frac{d}{(-1+bdfh)^n},$$
$$x_{8n-6} = \frac{g}{(-1+aceg)^n}, \qquad x_{8n-2} = \frac{c}{(-1+aceg)^n},$$

$$x_{8n-5} = f\left(-1+bdfh\right)^n, \qquad x_{8n-1} = b\left(-1+bdfh\right)^n,$$
$$x_{8n-4} = e\left(-1+aceg\right)^n, \qquad x_{8n} = a\left(-1+aceg\right)^n.$$

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1$. That is;

$$x_{8n-15} = \frac{h}{(-1+bdfh)^{n-1}}, \qquad x_{8n-11} = \frac{d}{(-1+bdfh)^{n-1}},$$
$$x_{8n-14} = \frac{g}{(-1+aceg)^{n-1}}, \qquad x_{8n-10} = \frac{c}{(-1+aceg)^{n-1}},$$

$$x_{8n-13} = f\left(-1+bdfh\right)^{n-1}, \qquad x_{8n-9} = b\left(-1+bdfh\right)^{n-1},$$
$$x_{8n-12} = e\left(-1+aceg\right)^{n-1}, \qquad x_{8n-8} = a\left(-1+aceg\right)^{n-1}.$$

Now, it follows from Eq. (4.1) that

$$x_{8n-7} = \frac{x_{8n-15}}{1+x_{8n-9}x_{8n-11}x_{8n-13}x_{8n-15}}$$

$$= \frac{\frac{h}{(-1+bdfh)^{n-1}}}{-1+b\left(-1+bdfh\right)^{n-1}\frac{d}{(-1+bdfh)^{n-1}}f\left(-1+bdfh\right)^{n-1}\frac{h}{(-1+bdfh)^{n-1}}}$$

$$= \frac{\frac{h}{(-1+bdfh)^{n-1}}}{-1+bdfh}.$$

Hence, we have

$$x_{8n-7} = \frac{h}{(-1+bdfh)^{n-1}}.$$

Similarly

$$x_{8n-4} = \frac{x_{8n-12}}{1+x_{8n-6}x_{8n-8}x_{8n-10}x_{8n-12}}$$

$$= \frac{e\left(-1+aceg\right)^{n-1}}{-1+\frac{g}{(-1+aceg)^n}a\left(-1+aceg\right)^{n-1}\frac{c}{(-1+aceg)^{n-1}}e\left(-1+aceg\right)^{n-1}}$$

$$= \frac{e\left(-1+aceg\right)^{n-1}}{-1+\dfrac{aceg}{(-1+aceg)}}\left(\frac{-1+aceg}{-1+aceg}\right).$$

Hence, we have

$$x_{8n-4} = e\left(-1+aceg\right)^{n}.$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed.

Theorem 4.2. *Eq. (4.1) has three equilibrium points which are* $0, \pm\sqrt[4]{2}$ *and these equilibrium points are not locally asymptotically stable.*

Proof. The proof as in Theorem 2.2.

Theorem 4.3. *Eq. (4.1) has a periodic solutions of period eight iff* $aceg = bdfh = 2$ *and will be take the form* $\{h, g, f, e, d, c, b, a, h, g, f, e, d, c, b, a, ...\}$.

Proof. First suppose that there exists a prime period eight solution

$$h, g, f, e, d, c, b, a, h, g, f, e, d, c, b, a, ...,$$

of Eq. (4.1), we see from Eq. (4.1) that

$$h = \frac{h}{\left(-1+bdfh\right)^{n}}, \qquad d = \frac{d}{\left(-1+bdfh\right)^{n}},$$
$$g = \frac{g}{(-1+aceg)^{n}}, \qquad c = \frac{c}{(-1+aceg)^{n}},$$

$$f = f\left(-1+bdfh\right)^{n}, \qquad b = b\left(-1+bdfh\right)^{n},$$
$$e = e\left(-1+aceg\right)^{n}, \qquad a = a\left(-1+aceg\right)^{n}.$$

or

$$\left(-1+bdfh\right)^{n} = 1, \quad \left(-1+aceg\right)^{n} = 1.$$

Then

$$bdfh = 2, \quad aceg = 2.$$

Second suppose $aceg = 2,\ bdfh = 2$. Then we see from Eq. (4.1) that

$$x_{8n-7} = h,\ x_{8n-6} = g,\ x_{8n-5} = f,\quad x_{8n-4} = e,\quad x_{8n-3} = d,$$
$$x_{8n-2} = c,\quad x_{8n-1} = b,\quad x_{8n} = a.$$

Thus we have a period eight solution and the proof is complete.

Numerical examples

Example 4.4. *We consider* $x_{-7} = 7,\ x_{-6} = 8,\ x_{-5} = 11,\ x_{-4} = 2,\ x_{-3} = 4,\ x_{-2} = 1,\ x_{-1} = 3,\ x_{0} = 9$. *See Fig. 5.*

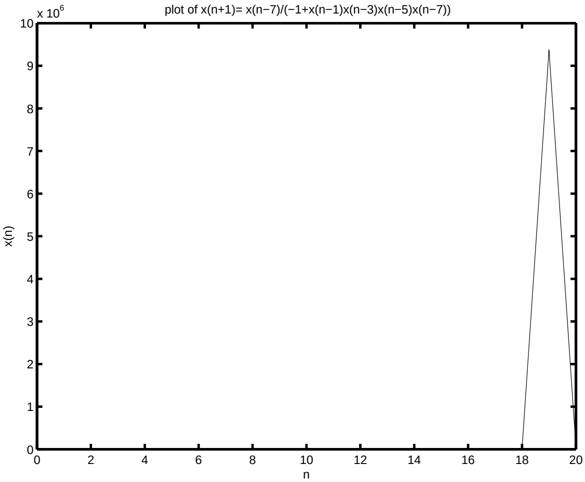

Figure 5.

Example 4.5. *See Fig. 6, since* $x_{-7} = 7,\ x_{-6} = 0.5,\ x_{-5} = 10,\ x_{-4} = 12,\ x_{-3} = 0.4,\ x_{-2} = 1/12,\ x_{-1} = 1/14,\ x_0 = 4.$

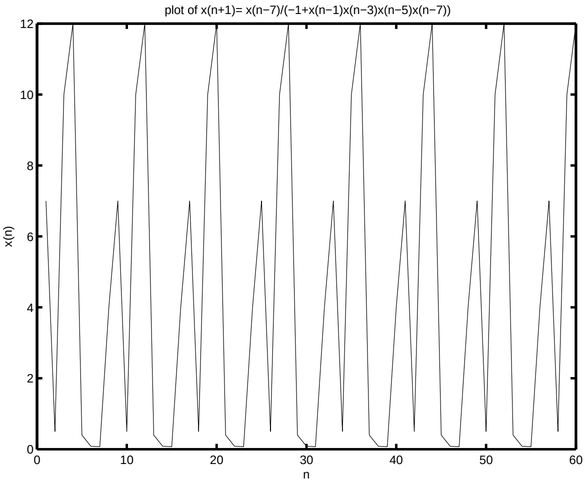

Figure 6.

5. On the Difference Equation $x_{n+1} = \frac{x_{n-7}}{-1-x_{n-1}x_{n-3}x_{n-5}x_{n-7}}$

In this section we investigate the solutions of the following difference equation

$$x_{n+1} = \frac{x_{n-7}}{-1 - x_{n-1}x_{n-3}x_{n-5}x_{n-7}}, \quad n = 0, 1, ..., \tag{5.1}$$

where the initial conditions are arbitrary nonzero real numbers with

$$x_{-5}x_{-3}x_{-1} \neq -1, \; x_{-4}x_{-2}x_0 \neq -1.$$

Theorem 5.1. *Let $\{x_n\}_{n=-7}^{\infty}$ be a solution of Eq. (5.1). Then Eq. (5.1) has unbounded solutions and for $n = 0, 1, ...$*

$$x_{8n-7} = \frac{(-1)^n h}{(1+bdfh)^n}, \qquad x_{8n-3} = \frac{(-1)^n d}{(1+bdfh)^n},$$
$$x_{8n-6} = \frac{(-1)^n g}{(1+aceg)^n}, \qquad x_{8n-2} = \frac{(-1)^n c}{(1+aceg)^n},$$

$$x_{8n-5} = f(-1)^n (1+bdfh)^n, \qquad x_{8n-1} = b(-1)^n (1+bdfh)^n,$$
$$x_{8n-4} = e(-1)^n (1+aceg)^n, \qquad x_{8n} = a(-1)^n (1+aceg)^n.$$

Theorem 5.2. *Eq. (5.1) has one equilibrium point which is number zero and this equilibrium point is not locally asymptotically stable.*

Theorem 5.3. *Eq. (5.1) has a periodic solutions of period eight iff $aceg = bdfh = -2$ and will be take the form $\{h, g, f, e, d, c, b, a, h, g, f, e, d, c, b, a, ...\}$.*

Numerical examples

Example 5.4. *Fig. 7 shows the solution when* $x_{-7} = -7, \; x_{-6} = 8, \; x_{-5} = 11, \; x_{-4} = 2, \; x_{-3} = -4, \; x_{-2} = 1, \; x_{-1} = 3, \; x_0 = -9.$

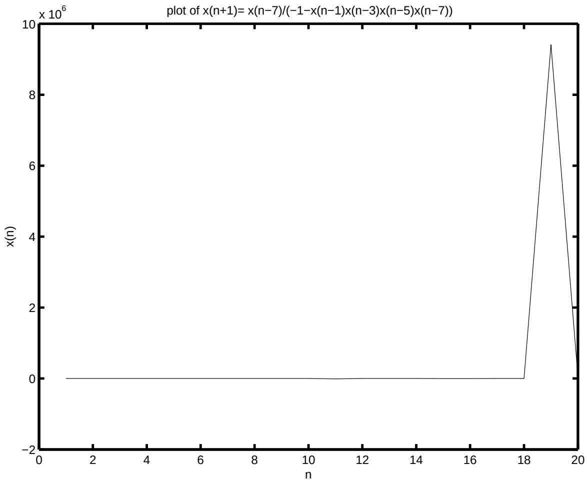

Figure 7.

Example 5.5. *See Fig. 8, since* $x_{-7} = -7, \; x_{-6} = 10, \; x_{-5} = 30, \; x_{-4} = 2, \; x_{-3} = -0.4, \; x_{-2} = 0.6, \; x_{-1} = -1/42, \; x_0 = -1/6$

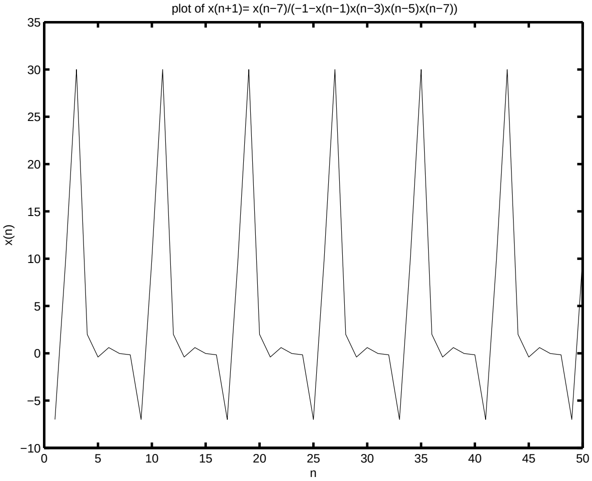

Figure 8.

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