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On the extension and torsion functors of local cohomology of weakly Laskerian and Matlis reflexive modules

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Abstract. Let R be a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ an ideal of R and M, N two R-modules. The main purpose of this paper is to study the circumstances under which, for fixed integers $j \in \mathbb{N}_0$ and $n \in \mathbb{N}$, the R-modules $\mathrm{Ext}^j_R(N, H^n_{\mathfrak{a}}(M))$ and $\mathrm{Tor}^R_j(N, H^n_{\mathfrak{a}}(M))$ are weakly Laskerian or Matlis reflexive. In this way, we also get to some results about the associated primes, coassociated primes and Bass numbers of $H^n_{\mathfrak{a}}(M)$.

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1. Introduction

Throughout this paper, we will generally assume that R is a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ be an ideal of R and M, N be two R-modules. We shall use $V(\mathfrak{a})$ to denote the set of all prime ideals containing $\mathfrak{a}$. Also, we shall use $\mathbb{N}_0$ (respectively $\mathbb{N}$) to denote the set of non-negative (respectively positive) integers.

For a non-negative integer i, the i-th local cohomology module of M with respect to $\mathfrak{a}$ is defined as:

$$H^i_{\mathfrak{a}}(M) = \varinjlim_{n \in \mathbb{N}_0} \mathrm{Ext}^i_R(R/\mathfrak{a}^n, M).$$

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The reader can refer to [6], for the basic properties of local cohomology.

This paper studies the circumstances under which the R-modules $\mathrm{Ext}^j_R(N, H^n_{\mathfrak{a}}(M))$ and $\mathrm{Tor}^R_j(N, H^n_{\mathfrak{a}}(M))$ are weakly Laskerian or Matlis reflexive, for fixed integers j and n when M, N are certain R-modules. One motivation for our work comes from the concept of cofiniteness for local cohomology modules introduced by Hartshorne in [15]. The local cohomology module $H^n_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite if $\mathrm{Ext}^i_R(R/\mathfrak{a}, H^n_{\mathfrak{a}}(M))$ is finitely generated for all $i \in \mathbb{N}_0$. It is a question of Huneke in [16] that when the local cohomology module $H^n_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite. In this regard, there has been a great deal of work. For instance, we refer the reader to the papers of Huneke and Koh [17], Delfino [8], Delfino and Marley [9], Yoshida [26] and Chiriacescu [7]. A question here arises that for fixed integers j and n, if M, N are certain R-modules, when the R-modules $\mathrm{Ext}^j_R(N, H^n_{\mathfrak{a}}(M))$ and $\mathrm{Tor}^R_j(N, H^n_{\mathfrak{a}}(M))$ are finitely generated. There has also been a couple of work regarding to this question when M is a finitely generated R-module and $N = R/\mathfrak{b}$ for some ideal $\mathfrak{b}$ of R containing $\mathfrak{a}$ (cf. [11] and [18]). The goal of the present paper is to obtain similar results as above, but for a larger class of modules.

Let E be the minimal injective cogenerator of the category of R-modules and $D(M) = \mathrm{Hom}_R(M, E)$. Recall that, an R-module M is called Matlis reflexive if the canonical map $M \to D(D(M))$ is an isomorphism. Moreover, Divaani-Aazar and Mafi, in [12], introduced and studied another type of modules called weakly Laskerian. A module M is called weakly Laskerian if the set of associated primes of any quotient module of M is finite. Note that, the class of weakly Laskerian modules includes all finitely generated, Artinian, linearly compact and Matlis reflexive modules. Also, the class of Matlis reflexive modules over a complete local ring contains all finitly generated and Artinian modules. Therefore, for fixed integers j and n, it is desirable to ask that when the R-modules $\mathrm{Ext}^j_R(N, H^n_{\mathfrak{a}}(M))$ and $\mathrm{Tor}^R_j(N, H^n_{\mathfrak{a}}(M))$ are weakly Laskerian or Matlis reflexive which is a generalization of mentioned question "*in some sense*".

In the second section of this paper we list some facts about the weakly Laskerian modules which will be useful in later sections. In the third section, at first, we investigate the above mentioned question for the R-module $\mathrm{Ext}^j_R(N, H^n_{\mathfrak{a}}(M))$. In fact, we show that for fixed integers $j \in \mathbb{N}_0$ and $n \in \mathbb{N}$, if N is a finitely generated R-module with $\mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$ and M is a weakly Laskerian R-module such that

(i) $\mathrm{Ext}^{j+t+1}_R(N, H^{n-t}_{\mathfrak{a}}(M))$ is weakly Laskerian for all $t = 1, \dots, n$, and
(ii) $\mathrm{Ext}^{j-s-1}_R(N, H^{n+s}_{\mathfrak{a}}(M))$ is weakly Laskerian for all $s = 1, \dots, \dim M - n$,

then $\mathrm{Ext}^j_R(N, H^n_{\mathfrak{a}}(M))$ is also weakly Laskerian. Next, we present some generalizations of [13, Theorem 3.1], [1, Theorem 1.2], [11, Theorem B], [5, Theorem 2.2] and [19, Theorem B(β)], to some extent.

In the forth section, we use an analogue of the above results for $\mathrm{Tor}^R_j(N, H^n_{\mathfrak{a}}(M))$. At last, in the final section, when $(R, \mathfrak{m})$ is a complete local ring with respect to $\mathfrak{m}$-adic topology, in a similar way, we study the

Matlis reflexivity of the R-modules $\mathrm{Ext}^j_R(N, H^n_{\mathfrak{a}}(M))$ and $\mathrm{Tor}^R_j(N, H^n_{\mathfrak{a}}(M))$ and get to some interesting results.

2. Preliminary results

First of all, we recall the definition of weakly Laskerian modules.

Definition 2.1. (i) (*See* [*12, Definition 2.1*].) *An R-module M is called weakly Laskerian if the set of associated primes of any quotient module of M is finite.*

(ii) (*See* [*13, Definition 2.4*].) *An R-module M is called $\mathfrak{a}$-weakly cofinite if* $\mathrm{Supp}_R(M) \subseteq V(\mathfrak{a})$ *and* $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ *is weakly Laskerian for all* $i \in \mathbb{N}_0$.

In the following lemma, we gather together some basic properties of weakly Laskerian modules.

Lemma 2.2. (See [12, Lemma 2.3] and [13, Remark 2.7].)

(i) *Let* $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ *be an exact sequence of R-modules and R-homomorphisms. Then M is weakly Laskerian if and only if L and N are weakly Laskerian. Hence, if* $L \longrightarrow M \longrightarrow N$ *is an exact sequence such that both end terms are weakly Laskerian R-modules, then M is also weakly Laskerian.*

(ii) *Let N be a finitely generated R-module and M be a weakly Laskerian R-module. Then* $\mathrm{Ext}^i_R(N, M)$ *and* $\mathrm{Tor}^R_i(N, M)$ *are weakly Laskerian for all* $i \in \mathbb{N}_0$.

(iii) *Suppose that M is a weakly Laskerian R-module with* $\mathrm{Supp}_R(M) \subseteq V(\mathfrak{a})$. *Then M is $\mathfrak{a}$-weakly cofinite.*

(iv) *If* $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ *is an exact sequence and two of modules in the sequence are $\mathfrak{a}$-weakly cofinite, then so is the third one.*

(v) *The set of associated primes of an $\mathfrak{a}$-weakly cofinite module is finite.*

Remark 2.3. (i) *In the light of* [12, Example 2.2], *the class of weakly Laskerian R-modules includes all finitely generated, Artinian and linearly compact R-modules.*

(ii) *Let E be the minimal injective cogenerator of the category of R-modules. For an R-module M, we let* $D(M) = \mathrm{Hom}_R(M, E)$. *If the canonical map* $M \to D(D(M))$ *is an isomorphism, then M is called Matlis reflexive. Now, by* [3, Theorem 12] *and* (i), *in conjunction with Lemma* 2.2(i), *every Matlis reflexive module is weakly Laskerian.*

Recall that a sequence $x_1, \ldots, x_n$ of elements of R is an $\mathfrak{a}$-filter regular sequence on M if $x_1, \ldots, x_n \in \mathfrak{a}$ and $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}_R(M/(x_1, \ldots, x_{i-1})M) \setminus V(\mathfrak{a})$ and for all $i = 1, \ldots, n$. When $i = 1$, this is to be interpreted as

$$x_i \notin \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(M) \setminus V(\mathfrak{a})} \mathfrak{p}.$$

The concept of an $\mathfrak{a}$-filter regular sequence on M is a generalization of the one of a filter regular sequence which has been studied in [21], [23] and has led to some interesting results. Note that both concepts coincide if $\mathfrak{a}$ is the maximal ideal in local ring. Also, note that $x_1, \ldots, x_n$ is a weak M-sequence if and only if it is an R-filter regular sequence on M. The following proposition enables one to see quickly that, for a weakly Laskerian R-module M, there exist $\mathfrak{a}$-filter regular sequences on it of any length.

Proposition 2.4. *Let M be a weakly Laskerian R-module and n be a positive integer. Assume that $x_1, \ldots, x_n$ is an $\mathfrak{a}$-filter regular sequence on M. Then there exists an element $x_{n+1} \in \mathfrak{a}$ such that $x_1, \ldots, x_n, x_{n+1}$ is an $\mathfrak{a}$-filter regular sequence on M.*

Proof. In contrary, suppose that $x_1, \ldots, x_n$ is an $\mathfrak{a}$-filter regular sequence on M such that

$$\mathfrak{a} \subseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(M/(x_1,\ldots,x_n)M) \setminus V(\mathfrak{a})} \mathfrak{p}.$$

Then, since M is weakly Laskerian R-module, $\mathrm{Ass}_R(M/(x_1, \ldots, x_n)M)$ is a finite set. So, Prime Avoidance Theorem provides that $\mathfrak{a} \subseteq \mathfrak{p}$ for some $\mathfrak{p}$ in the set $\mathrm{Ass}_R(M/(x_1, \ldots, x_n)M) \setminus V(\mathfrak{a})$ which is a required contradiction. □

3. Extension functors and local cohomology of weakly Laskerian modules

The first present author, in [18], by using filter regular sequences, established some results about finiteness properties of $\mathrm{Ext}^j_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M))$ and $\mathrm{Tor}^R_j(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M))$ for fixed integers $j \in \mathbb{N}_0$ and $n \in \mathbb{N}$ when $\mathfrak{b}$ is an ideal of R containing $\mathfrak{a}$. Now, in view of Lemma 2.2(i)-(ii), in conjunction with Proposition 2.4, by employing the methods of proofs which are similar to those used in [18], one can establish the following theorem which is generalization of [18, Theorem 3.3], in some sense.

Theorem 3.1. *Fix $j \in \mathbb{N}_0$, $n \in \mathbb{N}$, a finitely generated R-module N with* $\mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$ *and a weakly Laskerian R-module M of dimension d. Assume that*

(i) $\mathrm{Ext}^{j+t+1}_R(N, H^{n-t}_{\mathfrak{a}}(M))$ *is weakly Laskerian for all $t = 1, \ldots, n$, and*
(ii) $\mathrm{Ext}^{j-s-1}_R(N, H^{n+s}_{\mathfrak{a}}(M))$ *is weakly Laskerian for all $s = 1, \ldots, d-n$.*

Then $\mathrm{Ext}^j_R(N, H^n_{\mathfrak{a}}(M))$ *is weakly Laskerian.*

Proof. In view of Proposition 2.4, let $x_1, \ldots, x_{n+1}$ be an $\mathfrak{a}$-filter regular sequence on M. By means of [18], for each integer i with $1 \le i \le n$ there exists an exact sequence

$$0 \longrightarrow H^i_{\mathfrak{a}}(M) \longrightarrow H^i_{(x_1,\ldots,x_i)}(M) \longrightarrow (H^i_{(x_1,\ldots,x_i)}(M))_{x_{i+1}} \longrightarrow H^{i+1}_{(x_1,\ldots,x_{i+1})}(M) \longrightarrow 0.$$

One can break the above exact sequence into two exact sequences

$$(1) \quad 0 \longrightarrow H^i_{\mathfrak{a}}(M) \longrightarrow H^i_{(x_1,\ldots,x_i)}(M) \longrightarrow L_i \longrightarrow 0 \quad \text{and}$$
$$(2) \quad 0 \longrightarrow L_i \longrightarrow (H^i_{(x_1,\ldots,x_i)}(M))_{x_{i+1}} \longrightarrow H^{i+1}_{(x_1,\ldots,x_{i+1})}(M) \longrightarrow 0.$$

On the other hand, it is a fact that, for each R-module N and each element x of R, multiplication by x provides an automorphism on N_x. In this regard, since x_i's belong to $\mathfrak{a}$ and N is a finitely generated R-module with $\mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$, applying the long exact sequences of $\mathrm{Ext}^j_R(N,-)$ on the exact sequence (2) induces the isomorphism

$$\mathrm{Ext}^j_R(N, L_i) \cong \mathrm{Ext}^{j-1}_R(N, H^{i+1}_{(x_1,\ldots,x_{i+1})}(M)).$$

Now, several uses of the long exact sequences of $\mathrm{Ext}^j_R(N,-)$ on the exact sequence (1) and our assumptions of the theorem in conjunction with parts (i) and (ii) of Lemma 2.2 imply the result. □

Suppose that M is a finitely generated R-module and n is a positive integer. Marley and Vassilev, in [20, Proposition 2.5], showed that if $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all i with $i \neq n$, then $H^n_{\mathfrak{a}}(M)$ is also $\mathfrak{a}$-cofinite. By using the spectral sequence method, Divaani-Aazar and Mafi established the analogue result for weakly Laskerian modules (see [13, Theorem 3.1]). The following corollary which is a slight generalization of [13, Theorem 3.1], is an immediate consequence of Theorem 3.1.

Corollary 3.2. *Let M be a weakly Laskerian R-module and N be a finitely generated R-module with $\mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$. Assume that n is a fixed integer such that the R-module $\mathrm{Ext}^s_R(N, H^i_{\mathfrak{a}}(M))$ is weakly Laskerian for all $s \in \mathbb{N}_0$ and all i with $i \neq n$. Then $\mathrm{Ext}^s_R(N, H^n_{\mathfrak{a}}(M))$ is also weakly Laskerian for all $s \in \mathbb{N}_0$.*

The following results are consequences of Theorem 3.1 for special choices of j and n.

Corollary 3.3. *Let M be a weakly Laskerian R-module and N be a finitely generated R-module with $\mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$. Assume that for a fixed integer n, the R-module $\mathrm{Ext}^s_R(N, H^i_{\mathfrak{a}}(M))$ is weakly Laskerian for all $s \in \mathbb{N}$ and all i with $i < n$. Then*

(i) $\mathrm{Hom}_R(N, H^n_{\mathfrak{a}}(M))$ *is weakly Laskerian and so*

$$\mathrm{Ass}_R(H^n_{\mathfrak{a}}(M)) \cap \mathrm{Supp}_R(N)$$

is finite, and

(ii) $\mathrm{Ext}^1_R(N, H^n_{\mathfrak{a}}(M))$ *is weakly Laskerian.*

Proof. (i) Applying Theorem 3.1 when $j = 0$ ensures that the R-module $\mathrm{Hom}_R(N, H^n_{\mathfrak{a}}(M))$ is weakly Laskerian. The second assertion now follows from the fact that

$$\mathrm{Ass}_R(\mathrm{Hom}_R(N, H^n_{\mathfrak{a}}(M))) = \mathrm{Ass}_R(H^n_{\mathfrak{a}}(M)) \cap \mathrm{Supp}_R(N).$$

(ii) Apply Theorem 3.1 when $j = 1$. □

Note that, by Remarks 2.3(i), the first part of Corollary 3.3 is a generalization of the main results of [5] and [19].

Corollary 3.4. (Compare [1, Theorem 1.2] and [11, Theorem B].) *Let M be a weakly Laskerian R-module and N be a finitely generated R-module with* $\mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$. *Let t be a non-negative integer such that*

$$\mathrm{Ext}^s_R(N, H^i_{\mathfrak{a}}(M))$$

is weakly Laskerian for all $s \in \mathbb{N}$ and all i with $i < t$. Then the following statements are equivalent:

(i) $\mathrm{Hom}_R(N, H^{t+1}_{\mathfrak{a}}(M))$ *is weakly Laskerian.*
(ii) $\mathrm{Ext}^2_R(N, H^t_{\mathfrak{a}}(M))$ *is weakly Laskerian.*

Proof. (i)⇒(ii) Apply Theorem 3.1 with $j = 2$ and $n = t$.
(ii)⇒(i) Apply Theorem 3.1 with $j = 0$ and $n = t + 1$. □

By using Theorem 3.1, in conjunction with [6, Corollary 3.3.3], we have the following corollary.

Corollary 3.5. *Let M be a weakly Laskerian R-module, N be a finitely generated R-module and $x, y \in R$ such that $(x, y) \subseteq \sqrt{(0 :_R N)}$. Then, for a fixed integer j, the following statements are equivalent:*

(i) $\mathrm{Ext}^j_R(N, H^2_{(x,y)}(M))$ *is weakly Laskerian.*
(ii) $\mathrm{Ext}^{j+2}_R(N, H^1_{(x,y)}(M))$ *is weakly Laskerian.*

4. Torsion functors and local cohomology of weakly Laskerian modules

In the light of Lemma 2.2(i)-(ii), in conjunction with Proposition 2.4, the methods of proofs used in [18] may be adapted. So, one can establish the following theorem which is a generalization of Theorem 4.1 in [18], in some sense.

Theorem 4.1. *Fix $j \in \mathbb{N}_0$, $n \in \mathbb{N}$, a finitely generated R-module N with* $\mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$, *and a weakly Laskerian R-module M of dimension d. Assume that*

(i) $\mathrm{Tor}^R_{j-t-1}(N, H^{n-t}_{\mathfrak{a}}(M))$ *is weakly Laskerian for all $t = 1, \ldots, n$, and*
(ii) $\mathrm{Tor}^R_{j+s+1}(N, H^{n+s}_{\mathfrak{a}}(M))$ *is weakly Laskerian for all $s = 1, \ldots, d - n$.*

Then $\mathrm{Tor}^R_j(N, H^n_{\mathfrak{a}}(M))$ *is weakly Laskerian.*

Proof. The proof is similar to that used in the proof of Theorem 3.1 by replacing the functor $\mathrm{Tor}^R_j(N, -)$ in stead of the functor $\mathrm{Ext}^j_R(N, -)$. □

Now, we recall the definition of coassociated prime ideals which is needed in the sequel.

Definition 4.2. (*See* [25].) *Let* $(R, \mathfrak{m})$ *be a local ring and* K *be an* R*-module. A prime ideal* $\mathfrak{p}$ *of* R *is said to be a coassociated prime of* K *if* $\mathfrak{p}$ *is an associated prime of* $D(K)$*. We denote the set of coassociated primes of* K *by* $\mathrm{Coass}_R(K)$ *(or simply* $\mathrm{Coass}(K)$*, if there is no ambiguity about the underlying ring).*

Note that $\mathrm{Coass}(K) = \emptyset$ if and only if $K = 0$. Also, for a finitely generated R-module K and arbitrary R-module L, in the light of [24, Theorem 1.22] or [9, Remark p. 50], we have that $\mathrm{Coass}(K \otimes_R L) = \mathrm{Supp}_R(K) \cap \mathrm{Coass}_R(L)$. Now, we present a dual of Corollary 3.3, in some sense.

Corollary 4.3. *Let* n *be a non-negative integer. Let* $(R, \mathfrak{m})$ *be a local ring,* M *a weakly Laskerian* R*-module and* N *be a finitely generated* R*-module with* $\mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$*. Suppose that* $\mathrm{Tor}_j^R(N, H_{\mathfrak{a}}^i(M))$ *is weakly Laskerian for all* $j \in \mathbb{N}_0$ *and all* i *with* $i > n$*. Then*

(i) $N \otimes_R H_{\mathfrak{a}}^n(M)$ *is weakly Laskerian and so the set*

$$\mathrm{Supp}_R(N) \cap \mathrm{Coass}_R(H_{\mathfrak{a}}^n(M))$$

is finite, and

(ii) $\mathrm{Tor}_1^R(N, H_{\mathfrak{a}}^n(M))$ *is weakly Laskerian.*

Recall that for an R-module K, the cohomological dimension of K with respect to $\mathfrak{a}$ is defined as

$$\mathrm{cd}(\mathfrak{a}, K) = max\{i \in \mathbb{N}_0 \mid H_{\mathfrak{a}}^i(K) \neq 0\}.$$

Now, the following corollary is an immediate consequence of Theorem 4.1.

Corollary 4.4. *Let* M *be a weakly Laskerian* R*-module and* N *be a finitely generated* R*-module with* $\mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$*. Then for a fixed integer* n*,*

(i) *if* $\mathrm{Tor}_s^R(N, H_{\mathfrak{a}}^i(M))$ *is weakly Laskerian for all* i *with* $i \neq n$*, then* $\mathrm{Tor}_s^R(N, H_{\mathfrak{a}}^i(M))$ *is weakly Laskerian for all integers* i *and* s*.*

(ii) *if* $\mathrm{Tor}_s^R(N, H_{\mathfrak{a}}^i(M))$ *is weakly Laskerian for all* i *with* $i < \mathrm{cd}(\mathfrak{a}, M)$*, then* $\mathrm{Tor}_s^R(N, H_{\mathfrak{a}}^i(M))$ *is weakly Laskerian for all integers* i *and* s*.*

Applying Theorem 4.1 for special integers j and n yields the following corollary.

Corollary 4.5. *Let* n *be a positive integer,* M *a weakly Laskerian* R*-module and* N *a finitely generated* R*-module with* $\mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$*. Let the* R*-module* $\mathrm{Tor}_s^R(N, H_{\mathfrak{a}}^i(M))$ *be weakly Laskerian for all* i *with* $i > n$ *and all* $s \in \mathbb{N}_0$*. Then the following statements are equivalent:*

(i) $N \otimes_R H_{\mathfrak{a}}^{n-1}(M)$ *is weakly Laskerian.*

(ii) $\mathrm{Tor}_2^R(N, H_{\mathfrak{a}}^n(M))$ *is weakly Laskerian.*

The following corollary is an immediate consequence of Theorem 4.1 which is a dual of Corollary 3.5, in some sense.

Corollary 4.6. *Let* M *be a weakly Laskerian* R*-module,* N *a finitely generated* R*-module and* $x, y \in R$ *such that* $(x, y) \subseteq \sqrt{(0 :_R N)}$*. Then, for a fixed integer* j*, the following statements are equivalent:*

(i) $\mathrm{Tor}_j^R(N, H^2_{(x,y)}(M))$ *is weakly Laskerian.*
(ii) $\mathrm{Tor}_{j-2}^R(N, H^1_{(x,y)}(M))$ *is weakly Laskerian.*

5. Local cohomology of Matlis reflexive modules

Throughout this section, $(R, \mathfrak{m}, k)$ will denote a local complete ring with respect to $\mathfrak{m}$-adic topology. For the reminder of this paper, we focus our attention to Matlis reflexive modules. For basic theory concerning Matlis reflexive modules, the reader is referred to [22, §3.2] and [6, §10].

Remark 5.1. (i) *In view of Matlis duality theorem, the class of Matlis reflexive modules over a complete local ring includes all finitely generated and Artinian modules.*

(ii) *By* [14, Proposition 1.3] *or* [22, Theorem 3.4.13], *M is Matlis reflexive if and only if there is an exact sequence*

$$0 \longrightarrow S \longrightarrow M \longrightarrow A \longrightarrow 0$$

with S finitely generated and A Artinian.

(iii) *Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be an exact sequence of R-modules and R-homomorphisms. Then B is Matlis reflexive if and only if A and C are Matlis reflexive. This follows by mapping the exact sequence into its double dual and applying the snake lemma.*

In view of Remarks 5.1(iii) and Theorem 3 in [2], one can also gain the following results.

Theorem 5.2. (Compare [18, Theorem 3.3].) *Fix $j \in \mathbb{N}_0$, $n \in \mathbb{N}$, a Matlis reflexive R-module N with $\mathfrak{a} \subseteq \sqrt{(0 :_R N)}$ and a finitely generated R-module M of dimension d. Assume that*

(i) $\mathrm{Ext}_R^{j+t+1}(N, H^{n-t}_{\mathfrak{a}}(M))$ *is Matlis reflexive for all $t = 1, \ldots, n$, and*
(ii) $\mathrm{Ext}_R^{j-s-1}(N, H^{n+s}_{\mathfrak{a}}(M))$ *is Matlis reflexive for all $s = 1, \ldots, d - n$.*

Then $\mathrm{Ext}_R^j(N, H^n_{\mathfrak{a}}(M))$ is Matlis reflexive.

Theorem 5.3. (Compare [18, Theorem 4.1].) *Fix $j \in \mathbb{N}_0$, $n \in \mathbb{N}$, a Matlis reflexive R-module N with $\mathfrak{a} \subseteq \sqrt{(0 :_R N)}$ and a finitely generated R-module M of dimension d. Assume that*

(i) $\mathrm{Tor}_{j-t-1}^R(N, H^{n-t}_{\mathfrak{a}}(M))$ *is Matlis reflexive for all $t = 1, \ldots, n$, and*
(ii) $\mathrm{Tor}_{j+s+1}^R(N, H^{n+s}_{\mathfrak{a}}(M))$ *is Matlis reflexive for all $s = 1, \ldots, d - n$.*

Then $\mathrm{Tor}_j^R(N, H^n_{\mathfrak{a}}(M))$ is Matlis reflexive.

Now, we are ready to present the main results of this section.

Theorem 5.4. *Fix $j \in \mathbb{N}_0$ and $n \in \mathbb{N}$. Let M and N be two Matlis reflexive R-modules with $\mathfrak{a} \subseteq \sqrt{(0 :_R N)}$ such that*

(i) $\mathrm{Ext}_R^{j+t+1}(N, H^{n-t}_{\mathfrak{a}}(M))$ *is Matlis reflexive for all $t = 1, \ldots, n$, and*
(ii) $\mathrm{Ext}_R^{j-s-1}(N, H^{n+s}_{\mathfrak{a}}(M))$ *is Matlis reflexive for all $s = 1, \ldots, \dim M - n$.*

Then $\mathrm{Ext}^j_R(N, H^n_{\mathfrak{a}}(M))$ *is Matlis reflexive.*

Proof. Since M is Matlis reflexive, in the light of Remarks 5.1(i), there exists an exact sequence

$$0 \longrightarrow S \longrightarrow M \longrightarrow A \longrightarrow 0,$$

with S finitely generated and A Artinian. So, by applying the local cohomology functor $H^0_{\mathfrak{a}}(-)$, one can deduce the exact sequence

$$0 \longrightarrow H^0_{\mathfrak{a}}(S) \longrightarrow H^0_{\mathfrak{a}}(M) \longrightarrow A \xrightarrow{f} H^1_{\mathfrak{a}}(S) \longrightarrow H^1_{\mathfrak{a}}(M) \longrightarrow 0 \qquad (1)$$

and the isomorphism

$$H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{a}}(S) \qquad (2)$$

for all $i \geqslant 2$. Hence, we have the exact sequence

$$\mathrm{Ext}^j_R(N, H^1_{\mathfrak{a}}(S)) \longrightarrow \mathrm{Ext}^j_R(N, H^1_{\mathfrak{a}}(M)) \longrightarrow \mathrm{Ext}^{j+1}_R(N, \mathrm{Im}f). \qquad (3)$$

Since A is Artinian, $\mathrm{Im}f$ is also Artinian. Therefore, by [2, Theorem 3], $\mathrm{Ext}^{j+1}_R(N, \mathrm{Im}f)$ is Matlis reflexive. On the other hand, in view of the isomorphism (2) and [4, Lemma 1], Theorem 5.2 ensures that the R-module $\mathrm{Ext}^j_R(N, H^1_{\mathfrak{a}}(S))$ is Matlis reflexive. So, by Remarks 5.1(iii), the exact sequence (3) proves the theorem when $n = 1$. It remains to prove the claim when $n \geqslant 2$. By means of the isomorphism (2), we need to establish the claim for finitely generated R-module S. Now, in the light of (2), [4, Lemma 1] and Theorem 5.2, we only need to prove that $\mathrm{Ext}^{j+n}_R(N, H^1_{\mathfrak{a}}(S))$ is Matlis reflexive. To do this, we use the exact sequence (1) to get the following exact sequence

$$\mathrm{Ext}^{j+n}_R(N, \mathrm{Im}f) \longrightarrow \mathrm{Ext}^{j+n}_R(N, H^1_{\mathfrak{a}}(S)) \longrightarrow \mathrm{Ext}^{j+n}_R(N, H^1_{\mathfrak{a}}(M)).$$

Note that $\mathrm{Im}f$ is Artinian, so the R-module $\mathrm{Ext}^{j+n}_R(N, \mathrm{Im}f)$ is Matlis reflexive. Now, since both end terms are Matlis reflexive, the R-module $\mathrm{Ext}^{j+n}_R(N, H^1_{\mathfrak{a}}(S))$ is also Matlis reflexive, as desired. So, the proof is complete. □

By using Theorem 5.3 together with straightforward modifications to the arguments in the proof of Theorem 5.4, we can earn the same result for the R-module $\mathrm{Tor}^R_j(N, H^n_{\mathfrak{a}}(M))$ as follows.

Theorem 5.5. *Fix* $j \in \mathbb{N}_0$ *and* $n \in \mathbb{N}$. *Let* M *and* N *be two Matlis reflexive* R*-modules with* $\mathfrak{a} \subseteq \sqrt{(0 :_R N)}$ *such that the following conditions hold:*

(i) $\mathrm{Tor}^R_{j-t-1}(N, H^{n-t}_{\mathfrak{a}}(M))$ *is Matlis reflexive for all* $t = 1, \ldots, n$, *and*
(ii) $\mathrm{Tor}^R_{j+s+1}(N, H^{n+s}_{\mathfrak{a}}(M))$ *is Matlis reflexive for all* $s = 1, \ldots, \mathrm{dim}M - n$.

Then $\mathrm{Tor}^R_j(N, H^n_{\mathfrak{a}}(M))$ *is Matlis reflexive.*

Corollary 5.6. *Fix* $j \in \mathbb{N}_0$ *and* $n \in \mathbb{N}$. *Let* M *be a Matlis reflexive* R*-module of dimension* d *and* N *be a finitely generated* R*-module with* $\mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$ *such that*

(i) $\mathrm{Ext}^{j+t+1}_R(N, H^{n-t}_{\mathfrak{a}}(M))$ *is Matlis reflexive for all* $t = 1, \ldots, n$, *and*
(ii) $\mathrm{Ext}^{j-s-1}_R(N, H^{n+s}_{\mathfrak{a}}(M))$ *is Matlis reflexive for all* $s = 1, \ldots, d - n$.

Then $\mathrm{Tor}_j^R(N, D(H_{\mathfrak{a}}^n(M)))$ *is Matlis reflexive.*

Proof. By [6, Theorem 10.2.5], E is Artinian and so is Matlis reflexive. Also, we have the following isomorphism

$$\mathrm{Tor}_j^R(N, D(H_{\mathfrak{a}}^n(M))) \cong \mathrm{Hom}_R(\mathrm{Ext}_R^j(N, H_{\mathfrak{a}}^n(M)), E).$$

This fact together with Theorem 5.4 and [2, Theorem 3] proves the claim. □

We end the paper by the following result about the Bass numbers of local cohomology module $H_{\mathfrak{a}}^n(M)$.

Corollary 5.7. *Let M be a Matlis reflexive R-module of dimension d and $\mathfrak{p}$ be a prime ideal of R. Assume that*

(i) $\mathrm{Ext}_R^{j+t+1}(R/\mathfrak{p}, H_{\mathfrak{a}}^{n-t}(M))$ *is Matlis reflexive for all* $t = 1, \ldots, n$, *and*
(ii) $\mathrm{Ext}_R^{j-s-1}(R/\mathfrak{p}, H_{\mathfrak{a}}^{n+s}(M))$ *is Matlis reflexive for all* $s = 1, \ldots, d-n$.

Then the j-th Bass number of $H_{\mathfrak{a}}^n(M)$ with respect to $\mathfrak{p}$ is finite.

Proof. If $\mathfrak{p} \not\supseteq \mathfrak{a}$, then $\mathfrak{p} \notin \mathrm{Supp}_R(H_{\mathfrak{a}}^n(M))$. So, there is nothing to prove in this case. In other wise, Theorem 5.4 tells us that $\mathrm{Ext}_R^j(R/\mathfrak{p}, H_{\mathfrak{a}}^n(M))$ is Matlis reflexive. Now, in the case $\mathfrak{p} = \mathfrak{m}$, since $\mathrm{Ext}_R^j(R/\mathfrak{p}, H_{\mathfrak{a}}^n(M))$ is also a k-vector space, it must be finitely generated. Also, If $\mathfrak{p}$ is any non-maximal prime, it follows from Remarks 5.1(ii) that $(\mathrm{Ext}_R^j(R/\mathfrak{p}, H_{\mathfrak{a}}^n(M)))_{\mathfrak{p}}$ is finitely generated over $R_{\mathfrak{p}}$. Thus, in either case, the claim is true. □

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