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Generalized projectors and the saturated closure of a π-homomorph of finite π-solvable groups

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Abstract. The paper introduces and studies the notion of *generalized projector*, which generalizes the well-known notion of *projector* defined by W. Gaschütz in [8] as a generalization of the *covering subgroups* introduced by the same author in [7]. Let π be an arbitrary set of primes. A new definition for the *saturated closure* of a π-homomorph of finite π-solvable groups, equivalent to that in [3], is given. A property connected with the notion of generalized projector on a class X of finite π-solvable groups, called the *GP-property*, is also introduced. The main results of the paper are the following: 1) a characterization theorem for the saturated closure of the π-homomorphs of finite π-solvable groups with the GP-property by means of the generalized projectors; 2) a theorem showing that if X is a π-homomorph of finite π-solvable groups with the GP-property and $\overline{X}$ is its saturated closure, then X is a Schunck class if and only if $X = \overline{X}$. These results prove that theorems similar to those obtained by J. Weidner in [10] for finite solvable groups can be also obtained in the more general case of finite π-solvable groups.

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1. Preliminaries

In [3], we generalized in the more general case of finite π-solvable groups the results established by J. Weidner in [10] for finite solvable groups, obtaining a characterization of the saturated closure of a homomorph of finite π-solvable groups by means of the semicovering subgroups (introduced by J. Weidner in [10] as a generalization of the covering subgroups defined by W. Gaschütz in

[7]). Following the ideas from [10] and [3], the present paper introduces and studies the notion of *generalized projector*, which generalizes the well-known notion of *projector* defined by W. Gaschütz in [8] as a generalization of the covering subgroups. Using the projectors, a new definition for the *saturated closure* of a π-homomorph of finite π-solvable groups, equivalent to that in [3], is given. We define for a class X of finite π-solvable groups the *GP-property*, which is connected with the generalized projectors. A characterization theorem for the saturated closure of the π-homomorphs of finite π-solvable groups with the GP-property and an important consequence of this characterization are the main results of the paper.

All groups considered in the paper are finite. Denote by π an arbitrary set of primes and by π' the complement to π in the set of all primes.

We remind some definitions and theorems which will be useful for our considerations.

Definition 1.1. *a)* ([9]) *A class X of groups is a* **homomorph** *if X is closed under homomorphisms, i.e. if $G \in X$ and N is a normal subgroup of G, then $G/N \in X$.*

b) A group G is said to be **primitive** *if there exists a stabilizer W of G, i.e. W is a maximal subgroup of G and $core_G W = 1$, where*

$$core_G W = \cap\{W^g \mid g \in G\}.$$

c) ([9]) *A homomorph X is a* **Schunck class** *if X is primitively closed, i.e. if any group G, all of whose primitive factor groups are in X, is itself in X.*

Definition 1.2. *Let X be a class of groups, G a group and H a subgroup of G.*

a) ([8]) *H is an X-***maximal subgroup** *of G if:*

(i) $H \in X$;

(ii) $H \leq H^* \leq G,\ H^* \in X \Rightarrow H = H^*$.

b) ([8]) *H is an X-***projector** *of G if for any normal subgroup N of G, HN/N is X-maximal in G/N.*

c) ([7]) *H is an X-***covering subgroup** *of G if:*

(i) $H \in X$;

(ii) $H \leq K \leq G,\ K_0 \trianglelefteq K,\ K/K_0 \in X \Rightarrow K = HK_0$.

Remark 1.3. *a) Let X be a class of groups and G a group. Then: i) $G \in X$ if and only if G is X-maximal in G ; ii) if G is an X-projector of G, then $G \in X$.*

b) Let X be a homomorph and G a group. Then G is an X-projector of G if and only if $G \in X$.

Theorem 1.4. ([8]) *Let X be a class of groups, G a group and H a subgroup of G.*

a) If H is an X-projector of G and N is a normal subgroup of G, then HN/N is an X-projector of G/N.

b) H is an X-projector of G if and only if:

(i) H is X-maximal in G;

(ii) HM/M is an X-projector of G/M for all minimal normal subgroups M of G.

Theorem 1.5. *Let X be a class of groups, G a group and H a subgroup of G.*

a) If H is an X-covering subgroup or an X-projector of G, then H is X-maximal in G.

b) ([4]) *If X is a homomorph, then H is an X-covering subgroup of G if and only if H is an X-projector in any subgroup K with $H \leq K \leq G$. In particular, any X-covering subgroup of G is an X-projector of G.*

Theorem 1.6. ([1]) *A solvable minimal normal subgroup of a finite group is abelian.*

Introduced by S.A. Čunihin in [6], the π-solvable groups are more general than the solvable groups.

Definition 1.7. *a)* ([6]) *A group G is π-**solvable** if every chief factor M/N of G (i.e. M/N is a minimal normal subgroup of G/N) is either a solvable π-group or a π'-group. In particular, if π is the set of all primes, we obtain the notion of* **solvable group**.

b) ([2]) *A class X of groups is said to be π-**closed** if*

$$G/O_{\pi'}(G) \in X \Rightarrow G \in X,$$

where $O_{\pi'}(G)$ denotes the largest normal π'-subgroup of G.

*c) We say that X is a π-**homomorph** (respectively a π-**Schunck class**) if X is a π-closed homomorph (respectively X is a π-closed Schunck class).*

Theorem 1.8. ([6]) *a) If G is a π-solvable group and N is a normal subgroup of G, then G/N is π-solvable.*

b) If G is a group and N is a normal subgroup of G, such that N and G/N are π-solvable, then G is π-solvable.

Theorem 1.9. ([5]) *Let X be a π-homomorph. The following conditions are equivalent:*

(i) X is a Schunck class;

(ii) if G is a π-solvable group, $G \notin X$ and M is a minimal normal subgroup of G such that $G/M \in X$, then M has a complement in G;

(iii) any π-solvable group G has X-covering subgroups;

(iv) any π-solvable group G has X-projectors.

2. Generalized projectors

In [10], J. Weidner generalizes the notion of covering subgroup given in Definition 1.2.c) by renouncing to the condition (i). In [3], this generalized covering subgroup is called *semicovering subgroup*. Similarly, we will introduce o notion which generalizes the notion of projector.

Definition 2.1. *Let X be a class of groups, G a group and H a subgroup of G. H is called a* **generalized** X-**projector** *of G if for any normal subgroup N of G, $N \neq 1$, HN/N is X-maximal in G/N.*

It is the aim of this section to prove some properties of the generalized projectors.

Everywhere in this section we denote by X a class of groups, by G an arbitrary finite group and by H a subgroup of G.

Remark 2.2. *If H is an X-projector of G, then H is a generalized X-projector of G.*

Theorem 2.3. *H is an X-projector of G if and only if the following two conditions hold:*

(i) H is X-maximal in G;

(ii) H is a generalized X-projector of G.

Proof. Let H be an X-projector of G. By Definition 1.2.b), for any normal subgroup N of G we have that HN/N is X-maximal in G/N. In particular, for $N = 1$ we obtain that H is X-maximal in G, and so condition (i) holds. If we take $N \neq 1$ a normal subgroup of G, then HN/N is X-maximal in G/N, and, by Definition 2.1, H is a generalized X-projector of G, which mean that condition (ii) also holds.

Conversely, suppose that conditions (i) and (ii) hold. From (i) follows that for $N = 1$ we have HN/N is X-maximal in G/N. Let now $N \neq 1$ be a normal subgroup of G. By (ii) and Definition 2.1, HN/N is X-maximal in G/N. So HN/N is X-maximal in G/N for any normal subgroup N of G. This means by Definition 1.2.b) that H is an X-projector of G. $\square$

Theorem 2.4. *If H is a generalized X-projector of G and N is a normal subgroup of G, then HN/N is a generalized X-projector of G/N.*

Proof. Let H be a generalized X-projector of G and N a normal subgroup of G. We distinguish two cases:

1° $N = 1$. Since H is a generalized X-projector of G, we have for $N = 1$ that HN/N is a generalized X-projector of G/N.

2° $N \neq 1$. In order to prove that HN/N is a generalized X-projector of G/N, by Definition 2.1 we have to prove that for any normal subgroup L/N of G/N, $L/N \neq 1$, $(HN/N \cdot L/N)/(L/N)$ is X-maximal in $(G/N)/(L/N)$. But

$$(HN/N \cdot L/N)/(L/N) = (HNL/N)/(L/N) = (HL/N)/(L/N) \simeq HL/L$$

and

$$(G/N)/(L/N) \simeq G/L,$$

and so we have to prove that

$$HL/L \text{ is } X\text{-maximal in } G/L.$$

Indeed, from the hypothesis that H is a generalized X-projector of G, by using Definition 2.1 for the normal subgroup L of G, where $L \neq 1$ (since $1 \neq N < L$), we obtain that HL/L is X-maximal in G/L. $\square$

Our last theorem concerning some properties of the generalized projectors is a characterization theorem for the generalized projectors.

Theorem 2.5. *H is a generalized X-projector of G if and only if HM/M is an X-projector of G/M for any minimal normal subgroup M of G.*

Proof. Let H be a generalized X-projector of G and let M be a minimal normal subgroup of G. In order to prove that HM/M is an X-projector of G/M, we use Theorem 2.3 and verify conditions (i) and (ii) from this theorem.

(i) HM/M is X-maximal in G/M. Indeed, H being a generalized X-projector of G and M being normal in G with $M \neq 1$, Definition 2.1 leads to the conclusion that HM/M is X-maximal in G/M.

(ii) HM/M is a generalized X-projector of G/M. Indeed, from the facts that H is a generalized X-projector of G and M is a normal subgroup of G, Theorem 2.4 leads to the conclusion that HM/M is a generalized X-projector of G/M.

Conversely, suppose that HM/M is an X-projector of G/M for any minimal normal subgroup M of G. In order to prove that H is a generalized X-projector of G, we use Definition 2.1. Let N be a normal subgroup of G such that $N \neq 1$. Then there exists a minimal normal subgroup M of G such that $M \subseteq N$. By our hypothesis, HM/M is an X-projector of G/M. From this and from $N/M \trianglelefteq G/M$, we obtain by applying Theorem 1.4.a) that $(HM/M \cdot N/M)/(N/M)$ is an X-projector of $(G/M)/(N/M)$. But

$$(HM/M \cdot N/M)/(N/M) = (HMN/M)/(N/M) = (HN/M)/(N/M) \simeq HN/N$$

and

$$(G/M)/(N/M) \simeq G/N,$$

and so HN/N is an X-projector of G/N, which leads by Theorem 1.5.a) to the conclusion that HN/N is X-maximal in G/N. This means, by Definition 2.1, that H is a generalized X-projector of G. □

Finally in this section, two remarks.

From Theorem 1.5.b) and Remark 2.2, we obtain:

Remark 2.6. *If X is a homomorph, G is a group and H is a subgroup of G, then the following implications hold:*

$$H \text{ is an } X\text{-covering subgroup of } G \Rightarrow H \text{ is an } X\text{-projector of } G \Longrightarrow$$

$$H \text{ is a generalized } X\text{-projector of } G.$$

This shows that if X is a homomorph, then the notion of generalized projector generalizes both the projectors and the covering subgroups.

From the Remarks 1.3.b) and 2.2, follows immediately:

Remark 2.7. *If X is a homomorph and G is a group, then:*

(i) $G \in X \Longleftrightarrow G$ is an X-projector of G;

(ii) $G \in X \Rightarrow G$ is a generalized X-projector of G.

3. The saturated closure of a π-homomorph

Let π be an arbitrary set of primes. From now on, all groups used in our considerations will be finite π-solvable groups.

Definition 3.1. *Let X be a π-homomorph. We call* **the saturated closure** *of X the smallest π-homomorph $\overline{X}$ of finite π-solvable groups such that the following two conditions hold:*

(i) $X \subseteq \overline{X}$;

(ii) any finite π-solvable group has $\overline{X}$-projectors.

Remark 3.2. *a) Theorem 1.9 shows that Definition 3.1 is equivalent with that given in* [3].

b) If X is a π-homomorph and $\overline{X}$ is its saturated closure, then $\overline{X}$ is a π-homomorph and any finite π-solvable group has $\overline{X}$-projectors. It follows by Theorem 1.9 that the saturated closure $\overline{X}$ is a Schunck class. Since $\overline{X}$ is π-closed, we conclude that $\overline{X}$ is a π-Schunck class.

Notation 3.3. *Let X be a class of finite π-solvable groups. We denote by X^* the class of all finite π-solvable groups G such that G is a generalized X-projector of G.*

Let us give some properties of the class X^*, which will be used to prove the main results of the paper. Everywhere X will denote a class of finite π-solvable groups.

Theorem 3.4. *If X is a homomorph, then $X \subseteq X^*$.*

Proof. Let $G \in X$. By Remark 2.7.(ii), G is a generalized X-projector of G. It follows that $G \in X^*$. □

Theorem 3.5. *If X is a class of finite π-solvable groups, then X^* is a homomorph.*

Proof. Let $G \in X^*$ and let N be a normal subgroup of G. We show that $G/N \in X^*$. Indeed, from $G \in X^*$ we have that G is a finite π-solvable group and G is a generalized X-projector of G. G being a finite π-solvable group and N being normal in G, it follows by Theorem 1.8.a) that G/N is also a finite π-solvable group. Furthermore, from the facts that G is a generalized X-projector of G and N is a normal subgroup of G, Theorem 2.4 leads to the conclusion that G/N is a generalized X-projector of G/N. It follows that $G/N \in X^*$. □

The property of a class X of finite π-solvable groups we define below is connected with the generalized projectors introduced in Definition 2.1 and will be called therefore the *GP-property.*

Definition 3.6. *A class X of finite π-solvable groups is said to have the* **GP-property** *if X satisfies the following two conditions:*

(i) every finite π-solvable group has generalized X-projectors;

(ii) if G is a finite π-solvable group, then for any generalized X-projector H of G there exists a minimal normal subgroup M of G such that $M \subseteq H$.

Theorem 3.7. *Let X be a class of finite π-solvable groups with the GP-property and G a finite π-solvable group. The following two conditions are equivalent:*
(i) $G \in X^*$;
(ii) if H is a generalized X-projector of G, then $H = G$.

Proof. Let X be a class with the GP-property and G a finite π-solvable group.

$(i) \Rightarrow (ii)$: Let $G \in X^*$ and H be a generalized X-projector of G. From $G \in X^*$ follows that G is a generalized X-projector of G, which implies by Theorem 2.5 that G/M is an X-projector of G/M for any minimal normal subgroup M of G. By Theorem 1.5.a), we deduce that G/M is X-maximal in G/M, hence $G/M \in X$. On the other side, by applying Theorem 2.5 for the generalized X-projector H of G, we obtain that HM/M is an X-projector of G/M for any minimal normal subgroup M of G, hence HM/M is X-maximal in G/M. From this, since $G/M \in X$, we deduce that $HM/M = G/M$. It follows that $HM = G$ for any minimal normal subgroup M of G. But X is a class with the GP-property and so for the generalized X-projector H of G, there exists a minimal normal subgroup M_0 of G such that $M_0 \subseteq H$. Then $H = HM_0$. But, as we saw above, $HM_0 = G$. It follows that $H = G$.

$(ii) \Rightarrow (i)$: Let H be an arbitrary generalized X-projector of G. Then, by (ii), $H = G$. Hence G is its own generalized X-projector and so $G \in X^*$. □

Theorem 3.8. *If X is a π-homomorph with the GP-property, then X^* is a π-homomorph.*

Proof. Let X be a π-homomorph with the GP-property. By Theorem 3.5, X^* is a homomorph. It remains to prove that X^* is π-closed, i.e. that $G/O_{\pi'}(G) \in X^*$ implies $G \in X^*$. Let $G/O_{\pi'}(G) \in X^*$. We first notice that from $G/O_{\pi'}(G) \in X^*$ follows that $G/O_{\pi'}(G)$ is a finite π-solvable group. Now, $G/O_{\pi'}(G)$ and $O_{\pi'}(G)$ being π-solvable groups, we deduce by Theorem 1.8.b) that G is also a π-solvable group. In order to prove that $G \in X^*$, we use Theorem 3.7. Let H be a generalized X-projector of G. Since $O_{\pi'}(G) \trianglelefteq G$, Theorem 2.4 leads to the conclusion that $HO_{\pi'}(G)/O_{\pi'}(G)$ is a generalized X-projector of $G/O_{\pi'}(G)$. But the class X has the GP-property and $G/O_{\pi'}(G) \in X^*$. By Theorem 3.7, it follows that

$$HO_{\pi'}(G)/O_{\pi'}(G) = G/O_{\pi'}(G).$$

Hence

$$HO_{\pi'}(G) = G. \tag{3.1}$$

We consider two cases:

1° $O_{\pi'}(G) = 1$. In this case, (3.1) gives that $H = G$. But H being a generalized X-projector of G, it follows that G is a generalized X-projector of G. Hence $G \in X^*$.

2° $O_{\pi'}(G) \neq 1$. Then H being a generalized X-projector of G and $O_{\pi'}(G) \trianglelefteq G$, $O_{\pi'}(G) \neq 1$, Definition 2.1 leads to the conclusion that $HO_{\pi'}(G)/O_{\pi'}(G)$ is X-maximal in $G/O_{\pi'}(G)$, which means by applying (3.1) that $G/O_{\pi'}(G)$ is X-maximal in $G/O_{\pi'}(G)$. Hence $G/O_{\pi'}(G) \in X$. But the class X being π-closed, it follows that $G \in X$. By Theorem 3.4, the homomorph X has the property that $X \subseteq X^*$. So $G \in X^*$. □

Theorem 3.9. *If X is a π-homomorph with the GP-property, then any finite π-solvable group has X^*-projectors.*

Proof. Let X be a π-homomorph with the GP-property. Then, by Theorem 3.8, X^* is a π-homomorph. We apply Theorem 1.9 for the π-homomorph X^* and conclude that instead of proving that any finite π-solvable group has X^*-projectors we can prove the equivalent condition (ii) from Theorem 1.9, which becomes in our case: if G is a π-solvable group, $G \notin X^*$ and M is a minimal normal subgroup of G such that $G/M \in X^*$, then M has a complement in G. Let G be a π-solvable group, $G \notin X^*$ and M a minimal normal subgroup of G such that $G/M \in X^*$. We first observe that there exists a subgroup H of G such that H is a generalized X-projector of G and $H \neq G$. Indeed, if we suppose the contrary, then every generalized X-projector H of G is equal to G, which means by Theorem 3.7 that $G \in X^*$, a contradiction with the hypothesis $G \notin X^*$. We complete the proof of the present theorem by showing that H is a complement of M in G, i.e. $HM = G$ and $H \cap M = 1$. Indeed, since H is a generalized X-projector of G and M is normal in G, we conclude by Theorem 2.4 that HM/M is a generalized X-projector of G/M. This and $G/M \in X^*$ imply by Theorem 3.7 that $HM/M = G/M$. Hence $HM = G$. It remains to prove that $H \cap M = 1$. Since M is a minimal normal subgroup of the π-solvable group G, M is either a solvable π-group or a π'-group. Suppose that M is a π'-group. Then $M \leq O_{\pi'}(G)$ and so

$$G/O_{\pi'}(G) \simeq (G/M)/(O_{\pi'}(G)/M). \tag{3.2}$$

Since $G/M \in X^*$ and X^* is a homomorph, (3.2) leads to $G/O_{\pi'}(G) \in X^*$, which implies by the π-closure of X^* that $G \in X^*$, a contradiction with the hypothesis $G \notin X^*$. It follows that M is a solvable π-group. Then, by Theorem 1.6, M is abelian. Let us prove that $H \cap M$ is normal in G. We know that $H \leq G$ and $M \trianglelefteq G$ imply $H \cap M \trianglelefteq H$. Let now $g \in G = HM$ and $x \in H \cap M$. Then $g = hm$, with $h \in H$ and $m \in M$, and we have

$$g^{-1}xg = (hm)^{-1}x\ (hm) = (m^{-1}h^{-1})\ x\ (hm) = m^{-1}(h^{-1}x\ h)\ m. \tag{3.3}$$

From $H \cap M \trianglelefteq H$, we conclude that $h^{-1}x\ h \in H \cap M$. Furthermore, M being abelian, we can commute in (3.3) the elements $h^{-1}x\ h$ and m, both in M, and obtain

$$g^{-1}xg = m^{-1}(h^{-1}x\ h)\ m = m^{-1}m\ (h^{-1}x\ h) = h^{-1}xh \in H \cap M.$$

We proved that $H \cap M$ is normal in G. From this and from $H \cap M \subseteq M$, by using that M is a minimal normal subgroup of G, it follows that $H \cap M = 1$ or $H \cap M = M$. But $H \cap M = M$ leads to $M \subseteq H$, hence $G = HM = H$, a contradiction with $H \neq G$. It follows that $H \cap M = 1$, and the theorem is proved. □

Theorem 3.10. *If X is a π-homomorph with the GP-property, then X^* is a π-Schunck class.*

Proof. Since X is a π-homomorph with the GP-property, Theorem 3.8 shows that X^* is a π-homomorph and Theorem 3.9 shows that any finite π-solvable group has X^*-projectors. By applying Theorem 1.9, we conclude that X^* is a π-Schunck class. □

Theorem 3.11. *Let X be a π-homomorph with the GP-property. If Y is a π-homomorph satisfying the conditions*

(i) $X \subseteq Y$;

(ii) any finite π-solvable group has Y-projectors, then $X^* \subseteq Y$.

Proof. Let $G \in X^*$. Then G is a finite π-solvable group and so, by (ii), there exists an Y-projector H of G. We will prove that H is a generalized X-projector of G. For this, we use Theorem 2.5. and prove that HM/M is an X-projector of G/M for any minimal normal subgroup M of G. Let M be a minimal normal subgroup of G. From $G \in X^*$ follows that G is its own generalized X-projector, and by Theorem 2.5 we have that G/M is an X-projector of G/M, hence by Theorem 1.5.a) G/M is X-maximal in G/M, and so $G/M \in X$. But (i) claims that $X \subseteq Y$. It follows that $G/M \in Y$. Now, H being an Y-projector of G and M being normal in G, Definition 1.2.b) leads to the conclusion that HM/M is Y-maximal in G/M. This and $G/M \in Y$ imply $HM/M = G/M$, hence $HM = G$. But we saw that G/M is an X-projector of G/M, which together with $HM = G$ gives that HM/M is an X-projector of G/M, what we had to prove. It follows that H is a generalized X-projector of G. But $G \in X^*$ and the class X has the GP-property. So we can apply Theorem 3.7 and obtain that $H = G$. From the choice of H as an Y-projector of G, we deduce by Theorem 1.5.a) that H is Y-maximal in G, which implies that $H \in Y$. This and $H = G$ lead to $G \in Y$. The inclusion $X^* \subseteq Y$ is proved. □

Theorem 3.12. *If X is a π-homomorph with the GP-property and $\overline{X}$ is its saturated closure, then*

$$X^* \subseteq \overline{X} \ .$$

Proof. Let X be a π-homomorph with the GP-property and $\overline{X}$ its saturated closure. We can take in Theorem 3.11: $Y = \overline{X}$. Indeed, by Definition 3.1, the saturated closure $\overline{X}$ satisfies conditions (i) and (ii) claimed in Theorem 3.11. By applying Theorem 3.11, we conclude that $X^* \subseteq \overline{X}$. □

From Theorems 3.4 and 3.12 immediately follows:

Corollary 3.13. *If X is a π-homomorph with the GP-property and $\overline{X}$ is its saturated closure, then*

$$X \subseteq X^* \subseteq \overline{X} \ .$$

4. The main results

The main results of this paper, which we prove below, are the following: 1) a characterization theorem for the saturated closure of the π-homomorphs of finite π-solvable groups with the GP-property by means of the generalized projectors; 2) a characterization theorem for Schunck classes of finite π-solvable groups by means of the saturated closure of π-homomorphs of finite π-solvable groups with the GP-property.

Theorem 4.1. *If X is a π-homomorph with the GP-property and $\overline{X}$ is its saturated closure, then*

$$\overline{X} = X^*.$$

Proof. Let X be a π-homomorph with the GP-property and $\overline{X}$ its saturated closure. By applying Theorem 3.12, we obtain that $X^* \subseteq \overline{X}$. In order to prove that $\overline{X} \subseteq X^*$, we use the Definition 3.1 of the saturated closure of X. If we show that X^* verifies conditions (i) and (ii) given in Definition 3.1, then, $\overline{X}$ being the smallest π-homomorph which verifies (i) and (ii), we conclude that $\overline{X} \subseteq X^*$. It is easy to see that X^* verifies condition (i), namely $X \subseteq X^*$, because X is a homomorph and we apply Theorem 3.4. Furthermore, X^* verifies condition (ii), namely any finite π-solvable group has X^*-projectors, as Theorem 3.9 shows. □

Theorem 4.2. *Let X be a π-homomorph with the GP-property and $\overline{X}$ its saturated closure. The following two conditions are equivalent:*

(i) X is a Schunck class;

(ii) $X = \overline{X}$.

Proof. Let X be a π-homomorph with the GP-property and $\overline{X}$ its saturated closure.

$(i) \Rightarrow (ii)$: Let X be a Schunck class. We first prove that $X = X^*$. Indeed, X being a homomorph, Theorem 3.4 leads to $X \subseteq X^*$. Furthermore, by applying Theorem 1.9 for the π-homomorph X which is a Schunck class, we conclude that any finite π-solvable group has X-projectors. Let us take in Theorem 3.11 $Y = X$, which is a π-homomorph satisfying the two conditions claimed in this theorem, namely: $X \subseteq X$ and any finite π-solvable group has X-projectors. By applying Theorem 3.11, we obtain that $X^* \subseteq X$. From $X \subseteq X^*$ and $X^* \subseteq X$ follows that

$$X = X^*. \tag{4.1}$$

On the other side, we are in the hypotheses of Theorem 4.1 and so we conclude that

$$\overline{X} = X^*. \tag{4.2}$$

From (4.1) and (4.2) follows that $X = \overline{X}$.

$(ii) \Rightarrow (i)$: Let $X = \overline{X}$. By the Definition 3.1 of the saturated closure $\overline{X}$, any π-solvable group G has $\overline{X}$-projectors. But $X = \overline{X}$. Then any π-solvable group G has X-projectors. We can now apply Theorem 1.9 for the π-homomorph X, and it follows that X is a Schunck class. □

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