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FIXED POINT AND INTERPOLATION POINT SET OF A POSITIVE LINEAR OPERATOR ON $C(\overline{D})$

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Abstract. Let $D \subset \mathbb{R}^p$ be a compact convex subset with nonempty interior. If $A : C(D) \to C(D)$ is a positive linear operator with $\Pi_0(D) \subset F_A$ or $\Pi_1(D) \subset F_A$ then we establish some relations between the mixed-extremal point set of D and the interpolation point set of A. Our results include some well known results (see I. Raşa, *Positive linear operators preserving linear functions*, Ann. T. Popoviciu Seminar of Funct. Eq. Approx. Conv., **7**(2009), 105-109) and the proofs are directly and elementarely.

1. **Introduction**

In the iteration theory of a positive linear operator on a linear space of functions, the interpolation set of the operator has a fundamental part (U. Abel and M. Ivan [1], O. Agratini [2], [3], O. Agratini and I.A. Rus [5], [6], S. Andras and I.A. Rus [8], I. Gavrea and M. Ivan [12], H. Gonska and P. Piţul [14], I. Raşa [17], I.A. Rus [19], [20]).

A well known result is the following ([12],[14],[17], ...)

Theorem 1.1. *Let $L : C[0,1] \to C[0,1]$ be a positive linear operator such that*

$$L(e_i) = e_i,\ i = 0,1$$

where $e_i(x) = x^i$, $x \in [0,1]$.

Then:

$$L(f)(0) = f(0) \text{ and } L(f)(1) = f(1),\ \forall f \in C[0,1].$$

There exist different proofs of this result. One proof uses some estimations (Mamedov [16], Raşa [17], Gonska and Piţul [14], ...). Another proof uses a theorem by H. Bauer (H. Bauer [9], N. Boboc and Gh. Bucur [10], F. Altomare and M. Campiti [7], I. Raşa [17], ...). In [17], I. Raşa gives a directly and elementary proof.

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Let $D \subset \mathbb{R}^p$ be a bounded open convex subset and $A : C(\overline{D}) \to C(\overline{D})$ be a positive linear operator. The aim of this paper is to establish some relations between the mixed-extremal point set of D, the fixed point set and the interpolation point set of A. In this paper we shall use the notations in [7] and [20].

2. Mixed-extremal point set: Examples

Let $D \subset \mathbb{R}^p$ be a convex closed subset of $\mathbb{R}^p$ with nonempty interior.

Definition 2.1. A point $x^0 = (x_1^0, \ldots, x_p^0) \in \partial D$ is mixed-extremal point of D iff for each $i \in \{1, \ldots, p\}$, x_i^0 is an extremal (i.e., maximal or minimal) point of the ordered set

$$\big(\{x_i \mid (x_1, \ldots, x_p) \in D\}, \leq_{\mathbb{R}}\big).$$

We shall denote by $(ME)_D$ the mixed-extremal point set of D.

For a better understanding of this notion we shall give some examples.

Example 2.2. If $D_1 := [0, 1] \subset \mathbb{R}$, then $(ME)_{D_1} = \{0, 1\}$.

Example 2.3. If $D_2 := \mathbb{R}_+$, then $(ME)_{D_2} = \{0\}$.

Example 2.4. If D_3 is the simplex $\overline{P_1P_2P_3}$ in $\mathbb{R}^2$ with $P_1 = (0, 0)$, $P_2 = (1, 0)$ and $P_3 = (0, 1)$, then $(ME)_{D_3} = \{P_1, P_2, P_3\}$.

Example 2.5. If D_4 is the simplex $\overline{P_1P_2P_3}$ in $\mathbb{R}^2$ with $P_1 = (0, 0)$, $P_2 = (2, 0)$ and $P_3 = (1, 1)$, then $(ME)_{D_4} = \{P_1, P_2\}$.

Example 2.6. If D_5 is the polytope $\overline{P_1P_2P_3P_4}$ with $P_1 = (0, 0)$, $P_2 = (1, 0)$, $P_3 = (2, 1)$ and $P_4 = (1, 1)$, then $(ME)_{D_5} = \{P_1, P_3\}$.

Example 2.7. If $D_6 := \big\{x \in \mathbb{R}^p \mid x_1^2 + \ldots + x_p^2 \leq 1\big\}$, then $(ME)_{D_6} = \emptyset$.

3. Interpolation points and fixed points of positive linear operators

Let $D \subset \mathbb{R}^p$ be a bounded open convex subset of $\mathbb{R}^p$. Let $A : C(\overline{D}) \to C(\overline{D})$ be a positive linear (i.e., increasing linear) operator.

Definition 3.1. A point $x \in \overline{D}$ is an interpolation point of A iff $A(f)(x) = f(x)$, for all $f \in C(\overline{D})$. A subset $E \subset \overline{D}$ is an interpolation set of A iff $A(f)\big|_E = f\big|_E$. The subset

$$(IP)_D := \big\{x \in \overline{D} \mid A(f)(x) = f(x),\ \forall f \in C(\overline{D})\big\}$$

is by definition the interpolation point set of A.

Remark 3.2. Let us denote by $\xrightarrow{p}$, the pointwise convergence. Let $Y \subset C(\overline{D})$ be a dense subset of $(C(\overline{D}), \xrightarrow{p})$. If for a point $x \in \overline{D}$ we have

$$A(f)(x) = f(x),\ \forall f \in Y$$

then x is an interpolation point of A.

Remark 3.3. If $A : (C(\overline{D}), \xrightarrow{p}) \to (C(\overline{D}), \xrightarrow{p})$ is weakly Picard operator and $x \in \overline{D}$ is an interpolation point of A, then x is an interpolation point of A^∞.

The main results of this paper are the following

Theorem 3.4. *We suppose that:*

(i) *A is an increasing linear operator;*
(ii) $\Pi_1(\overline{D}) \subset F_A$.

Then $(ME)_D$ is an interpolation set of A.

Proof. Let us denote by $\Pi(\overline{D}) \subset C(\overline{D})$ the set of polynomial functions on $\overline{D}$.

Since $\Pi(\overline{D})$ is a dense subset of $(C(\overline{D}), \overset{unif}{\to})$, it is sufficient to prove that

$$A(f)\big|_{(ME)_D} = f\big|_{(ME)_D},\ \forall f \in \Pi(\overline{D}).$$

Let $x^0 \in (ME)_D$. From the mean-value theorem we have

$$f(x) - f(x^0) = \sum_{i=1}^{p}(x_i - x_i^0)\frac{\partial f(x_0 + \theta(x - x_0))}{\partial x_i},\ \forall x \in \overline{D}.$$

Since $\overline{D}$ is compact and x^0 is a mixed-extremal element of $\overline{D}$, there exist $\alpha_i, \beta_i \in \mathbb{R}$, $i \in \{1, \ldots, p\}$ such that

$$\sum_{i=1}^{p}\alpha_i(x_i - x_i^0) \le f(x) - f(x^0) \le \sum_{i=1}^{p}\beta_i(x_i - x_i^0),\ \forall x \in \overline{D}.$$

From this we have

$$\sum_{i=1}^{p}\alpha_i(q_i - x_i^0\tilde{1}) \le f - f(x^0)\tilde{1} \le \sum_{i=1}^{p}\beta_i(q_i - x_i^0\tilde{1}). \tag{3.1}$$

Here

$$q_i : \overline{D} \to \mathbb{R},\ x \mapsto x_i,\ i \in \{1, \ldots, p\},$$

and

$$\tilde{1} : \overline{D} \to \mathbb{R},\ x \mapsto 1.$$

Since A is an increasing linear operator and $\tilde{1}$, q_1, ... , $q_p \in F_A$, from (3.1) we have

$$\sum_{i=1}^{p}\alpha_i(q_i - x_i^0\tilde{1}) \le A(f) - f(x^0)\tilde{1} \le \sum_{i=1}^{p}\beta_i(q_i - x_i^0\tilde{1}).$$

For $x := x^0$, we have

$$A(f)(x^0) = f(x^0),\ \forall f \in \Pi(\overline{D})$$

and, from Remark 3.2, for all $f \in C(\overline{D})$. □

More general we have

Theorem 3.5. *We suppose that*

(i) *A is an increasing linear operator;*

(ii) $\Pi_0(\overline{D}) \subset F_A$.

Then

$$E := \left\{x \in (ME)_D \mid A(q_i)(x) = x_i\right\}$$

is an interpolation set of A.

Proof. Let $x^0 \in E$. From (3.1) we have

$$\sum_{i=1}^{p} \alpha_i (A(q_i) - x_i^0 \tilde{1}) \leq A(f) - f(x^0)\tilde{1} \leq \sum_{i=1}^{p} \beta_i (A(q_i) - x_i^0 \tilde{1})$$

For $x := x^0$, it follows

$$A(f)(x^0) = f(x^0),\ \forall f \in C(\overline{D}),$$

□

In a similar way we have

Theorem 3.6. *We suppose that:*

(i) *A is an increasing linear operator;*

(ii) $q_1, \ldots, q_p \in F_A$.

Then

$$E := \left\{x \in (ME)_D \mid A(\tilde{1})(x) = 1\right\}$$

is an interpolation set of A.

Example 3.7. Let $\overline{\Omega} = [0,1] \times [0,1]$ and

$$A(f)(x_1, x_2) := f(0,0) + f(1,0)x_1 + f(0,1)x_2.$$

In this case A is an increasing linear operator with

$$\tilde{1} \notin F_A \text{ and } q_1, q_2 \in F_A$$

and

$$(IP)_A = \{(0,0)\}.$$

We remark that

$$A(\tilde{1})(0,0) = 1,\ A(\tilde{1})(0,1) = 2,\ A(\tilde{1})(1,0) = 2 \text{ and } A(\tilde{1})(1,1) = 3.$$

In the case $p = 1$ and $\overline{D} = [a, b]$, let us denote $e_i(x) := x^i$, $x \in [a, b]$, $i \in \mathbb{N}$. We have

Theorem 3.8. *We suppose that:*

(i) $A : C[a, b] \to C[a, b]$ *is an increasing linear operator;*
(ii) e_0 *and* $e_2 \in F_A$.

Then:

(1) *If* $A(e_1)(a) = a$, *then* a *is an interpolation point of* A.
(2) *If* $A(e_1)(b) = b$, *then* b *is an interpolation point of* A.

Example 3.9. Let us consider the following operator of J.P. King (see [14])

$$A : C[0, 1] \to C[0, 1],$$
$$A(f)(x) := (1 - x^2)f(0) + x^2 f(1),\ x \in [0, 1].$$

In this case:

(1) $e_0, e_2 \in F_A$;
(2) $(IP)_A = \{0, 1\}$;
(3) $A(e_1)(0) = 0$, $A(e_1)(1) = 1$.

4. **Open problems**

From the above considerations the following problems arise:

Problem 4.1. To extend the above results to the case when D is an open convex subset of $\mathbb{R}^p$, not necessarily bounded.

Problem 4.2. Let $D \subset \mathbb{R}^p$ be an open convex subset of $\mathbb{R}^p$. Let $A : C(\overline{D}) \to C(\overline{D})$ be an increasing linear operator. We suppose that $E \subset \overline{D}$ is a strong Volterra set of A ([20], [6]), i.e.,

$$f, g \in C(\overline{D}),\ f\big|_E = g\big|_E \Rightarrow A(f) = A(g).$$

We consider the operator

$$A_{\overline{co}E} : C(\overline{co}E) \to C(\overline{co}E),\ A_{\overline{co}E}\big(f\big|_{\overline{co}E}\big) := A(f)\big|_{\overline{co}E}.$$

It is clear that $A_{\overline{co}E}$ is an increasing linear operator.
If $\Pi_0(\overline{D}) \subset F_A$ or $\Pi_1(\overline{D}) \subset F_A$, in which conditions we have that $(IP)_{A_{\overline{co}E}} \neq \emptyset$?

Problem 4.3. Could our results be derived from the H. Bauer principle of the barycenter of a probability Radon measure (Theorem 2.1 in Raşa [17])?

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