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VORONOVSKAYA TYPE THEOREMS FOR SMOOTH PICARD AND GAUSS-WEIERSTRASS SINGULAR OPERATORS

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Abstract. In this article we continue with the study of approximation properties of smooth Picard singular integral operators and smooth Gauss-Weierstrass singular integral operators over the real line. We produce some Voronovskaya type theorems and give some quantitative results regarding the rate of convergence of the above mentioned singular integral operators.

1. **Introduction**

We are motivated by the approximation properties of *Picard* and *Gauss-Weierstrass singular integrals* of a function f defined by the following

$$P_\xi(f;x) := \frac{1}{2\xi}\int_{-\infty}^{\infty} f(x+y)e^{-|y|/\xi}dy, \tag{1.1}$$

$$W_\xi(f;x) := \frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty} f(x+y)e^{-y^2/\xi}dy, \tag{1.2}$$

for all $x \in \mathbb{R}$, $\xi > 0$, see [5], chp. 16, 17, and [4], chp. 21, and [6].

Next we mention the *smooth Picard singular integral operators* $P_{r,\xi}(f;x)$ and the *smooth Gauss-Weierstrass singular integral operators* $W_{r,\xi}(f;x)$ defined next, basic approximation properties of them were studied in [1], [2], [3], [7], [8] and [9].

For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$ we set

$$\alpha_j = \begin{cases} (-1)^{r-j}\binom{r}{j}j^{-n}, & j = 1,\dots,r, \\ 1 - \displaystyle\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}j^{-n}, & j = 0, \end{cases} \tag{1.3}$$

that is $\sum_{j=0}^{r}\alpha_j = 1$.

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Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f^{(n)}$ exists and it is bounded and Lebesgue measurable. We define for $x \in \mathbb{R}$, $\xi > 0$ the Lebesgue singular integrals

$$P_{r,\xi}(f;x) = \frac{1}{2\xi} \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x + jt) \right) e^{-|t|/\xi} dt, \tag{1.4}$$

and

$$W_{r,\xi}(f;x) = \frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x + jt) \right) e^{-t^2/\xi} dt. \tag{1.5}$$

Note 1.1. The operators $P_{r,\xi}$ and $W_{r,\xi}$ are not, in general, positive, see [1] and [7] respectively.
Note 1.2. In particular we have $P_{1,\xi} = P_\xi$ and $W_{1,\xi} = W_\xi$.

In Section 2 we will give some elementary properties of the integrals defined in (1.4) and (1.5). Then, in Section 3, we will prove some Voronovskaya type asymptotic theorems, see also [11].

2. **Auxiliary result**

From (1.4) and (1.5) we also obtain

$$P_{r,\xi}(f;x) = \sum_{j=0}^{r} \frac{1}{2\xi} \alpha_j \int_{-\infty}^{\infty} f(x + jt) e^{-|t|/\xi} dt, \tag{2.1}$$

and

$$W_{r,\xi}(f;x) = \sum_{j=0}^{r} \frac{1}{\sqrt{\pi\xi}} \alpha_j \int_{-\infty}^{\infty} f(x + jt) e^{-t^2/\xi} dt. \tag{2.2}$$

By means of elementary calculations, we obtain
Lemma 2.1. For every $n \in \mathbb{N}_0$, and $\xi > 0$, we have

$$I_n := \int_0^{\infty} t^n e^{-t/\xi} dt = n! \xi^{n+1}, \tag{2.3}$$

and

$$I_n^* := \int_0^{\infty} t^n e^{-t^2/\xi} dt = \begin{cases} \xi^{\frac{n+1}{2}} \cdot \frac{1}{2} \cdot \left(\frac{n-1}{2}\right)!, & n - odd \\ \xi^{\frac{n+1}{2}} \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{1}{2^n} \cdot \frac{n!}{\left(\frac{n}{2}\right)!}, & n - even. \end{cases} \tag{2.4}$$

Proof. Easy. □

3. Results

We present first our main result.

Theorem 3.1. Let $f:\mathbb{R}\to\mathbb{R}$ be such that $f^{(n)}$exists, $n\in\mathbb{N}$, and is bounded and Lebesgue measurable on $\mathbb{R}$, and let $\xi\to 0+$, $0<\alpha\le 1$. Then

$$P_{r,\xi}(f;x)-f(x)=\sum_{m=1}^{\lfloor\frac{n-1}{2}\rfloor} f^{(2m)}(x)\left(\sum_{j=1}^{r}\alpha_j j^{2m}\right)\xi^{2m}+o(\xi^{n-\alpha}), \tag{3.1}$$

and

$$W_{r,\xi}(f;x)-f(x)=\sum_{m=1}^{\lfloor\frac{n-1}{2}\rfloor} \frac{f^{(2m)}(x)}{m!2^{2m}}\left(\sum_{j=1}^{r}\alpha_j j^{2m}\right)\xi^{m}+o(\xi^{\frac{n-\alpha}{2}}). \tag{3.2}$$

Proof. We notice by $\frac{1}{2\xi}\int_{-\infty}^{\infty}e^{-|t|/\xi}dt=1$ and $\frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty}e^{-t^2/\xi}dt=1$, that $P_{r,\xi}(c,x)=c, W_{r,\xi}(c,x)=c$, for any c constant, and therefore (see also [7] and [2], formula (3.3) there) we have

$$P_{r,\xi}(f;x)-f(x)=\frac{1}{2\xi}\left(\sum_{j=0}^{r}\alpha_j\int_{-\infty}^{\infty}(f(x+jt)-f(x))e^{-|t|/\xi}dt\right), \tag{3.3}$$

and

$$W_{r,\xi}(f;x)-f(x)=\frac{1}{\sqrt{\pi\xi}}\left(\sum_{j=0}^{r}\alpha_j\int_{-\infty}^{\infty}(f(x+jt)-f(x))e^{-t^2/\xi}dt\right). \tag{3.4}$$

Using Taylor's formula for f, we have

$$f(x+jt)=\sum_{k=0}^{n-1}\frac{f^{(k)}(x)}{k!}(jt)^k+\frac{f^{(n)}(\gamma)}{n!}(jt)^n, \tag{3.5}$$

with γ between x and $x+jt$.

We obtain

$$P_{r,\xi}(f;x)-f(x)\overset{(3.3)}{=}\frac{1}{2\xi}\left(\sum_{j=1}^{r}\alpha_j\int_{-\infty}^{\infty}\left(\left[\sum_{k=0}^{n-1}\frac{f^{(k)}(x)}{k!}(jt)^k+\frac{f^{(n)}(\gamma)}{n!}(jt)^n\right]\right.\right.$$
$$\left.\left.-f(x)\right)e^{-|t|/\xi}dt\right)$$
$$=\frac{1}{2\xi}\left(\sum_{j=1}^{r}\alpha_j\int_{-\infty}^{\infty}\left[\sum_{k=1}^{n-1}\frac{f^{(k)}(x)}{k!}j^kt^k+\frac{f^{(n)}(\gamma)}{n!}j^nt^n\right]e^{-|t|/\xi}dt\right)$$
$$=\frac{1}{2\xi}\left(\sum_{j=1}^{r}\alpha_j\left[\sum_{k=1}^{n-1}\frac{f^{(k)}(x)}{k!}j^k\left(\int_{-\infty}^{\infty}t^ke^{-|t|/\xi}dt\right)\right.\right.$$
$$\left.\left.+\frac{j^n}{n!}\int_{-\infty}^{\infty}f^{(n)}(\gamma)t^ne^{-|t|/\xi}dt\right]\right)$$
$$\overset{(2.3)}{=}\frac{1}{2\xi}\left(\sum_{j=1}^{r}\alpha_j\left[\sum_{m=1}^{\lfloor\frac{n-1}{2}\rfloor}f^{(2m)}(x)j^{2m}2\xi^{2m+1}\right.\right.$$
$$\left.\left.+\frac{j^n}{n!}\int_{-\infty}^{\infty}f^{(n)}(\gamma)t^ne^{-|t|/\xi}dt\right]\right)$$
$$=\sum_{j=1}^{r}\alpha_j\left[\left(\sum_{m=1}^{\lfloor\frac{n-1}{2}\rfloor}f^{(2m)}(x)j^{2m}\xi^{2m}\right)+\frac{j^n}{2\xi n!}\int_{-\infty}^{\infty}f^{(n)}(\gamma)t^ne^{-|t|/\xi}dt\right],$$

and

$$W_{r,\xi}(f;x)-f(x)\overset{(3.4)}{=}\frac{1}{\sqrt{\pi\xi}}\left(\sum_{j=1}^{r}\alpha_j\int_{-\infty}^{\infty}\left(\left[\sum_{k=0}^{n-1}\frac{f^{(k)}(x)}{k!}(jt)^k+\frac{f^{(n)}(\gamma)}{n!}(jt)^n\right]\right.\right.$$
$$\left.\left.-f(x)\right)e^{-t^2/\xi}dt\right)$$
$$=\frac{1}{\sqrt{\pi\xi}}\left(\sum_{j=1}^{r}\alpha_j\int_{-\infty}^{\infty}\left[\sum_{k=1}^{n-1}\frac{f^{(k)}(x)}{k!}(jt)^k+\frac{f^{(n)}(\gamma)}{n!}(jt)^n\right]e^{-t^2/\xi}dt\right)$$
$$=\frac{1}{\sqrt{\pi\xi}}\left(\sum_{j=1}^{r}\alpha_j\left[\sum_{k=1}^{n-1}\frac{f^{(k)}(x)}{k!}j^k\left(\int_{-\infty}^{\infty}t^ke^{-t^2/\xi}dt\right)\right.\right.$$
$$\left.\left.+\frac{j^n}{n!}\int_{-\infty}^{\infty}f^{(n)}(\gamma)t^ne^{-t^2/\xi}dt\right]\right)$$

$$\overset{(2.4)}{=} \frac{1}{\sqrt{\pi\xi}} \left(\sum_{j=1}^{r} \alpha_j \left[\sum_{m=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{f^{(2m)}(x)}{m!} j^{2m} \xi^{\frac{2m+1}{2}} \frac{\sqrt{\pi}}{2^{2m}} \right.\right.$$

$$\left.\left. + \frac{j^n}{n!} \int_{-\infty}^{\infty} f^{(n)}(\gamma) t^n e^{-t^2/\xi} dt \right] \right)$$

$$= \sum_{j=1}^{r} \alpha_j \left[\sum_{m=1}^{\lfloor \frac{n-1}{2} \rfloor} f^{(2m)}(x) \frac{j^{2m}\xi^m}{m!2^{2m}} + \frac{j^n}{\sqrt{\pi\xi} n!} \int_{-\infty}^{\infty} f^{(n)}(\gamma) t^n e^{-t^2/\xi} dt \right].$$

Therefore we have obtained

$$P_{r,\xi}(f;x) - f(x) - \sum_{m=1}^{\lfloor \frac{n-1}{2} \rfloor} f^{(2m)}(x) \left(\sum_{j=1}^{r} \alpha_j j^{2m} \right) \xi^{2m}$$
$$= \sum_{j=1}^{r} \alpha_j \frac{j^n}{2\xi n!} \int_{-\infty}^{\infty} f^{(n)}(\gamma) t^n e^{-|t|/\xi} dt, \tag{3.6}$$

and

$$W_{r,\xi}(f;x) - f(x) - \sum_{m=1}^{\lfloor \frac{n-1}{2} \rfloor} f^{(2m)}(x) \frac{1}{m!2^{2m}} \left(\sum_{j=1}^{r} \alpha_j j^{2m} \right) \xi^{m}$$
$$= \sum_{j=1}^{r} \alpha_j \frac{j^n}{\sqrt{\pi\xi} n!} \int_{-\infty}^{\infty} f^{(n)}(\gamma) t^n e^{-t^2/\xi} dt. \tag{3.7}$$

Hence

$$\Delta_\xi \quad : \quad = \frac{1}{\xi^n} \left[(P_{r,\xi}(f;x) - f(x)) - \sum_{m=1}^{\lfloor \frac{n-1}{2} \rfloor} \xi^{2m} f^{(2m)}(x) \left(\sum_{j=1}^{r} \alpha_j j^{2m} \right) \right]$$

$$= \sum_{j=1}^{r} \alpha_j \frac{j^n}{2\xi^{n+1} n!} \int_{-\infty}^{\infty} f^{(n)}(\gamma) t^n e^{-|t|/\xi} dt$$

$$= \frac{1}{2n!\xi^{n+1}} \int_{-\infty}^{\infty} \left(\sum_{j=1}^{r} \alpha_j j^n f^{(n)}(\gamma) \right) t^n e^{-|t|/\xi} dt$$

$$= \frac{1}{2n!\xi^{n+1}} \left[\int_{-\infty}^{\infty} \left(\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} f^{(n)}(\gamma) \right) t^n e^{-|t|/\xi} dt \right], \tag{3.8}$$

and

$$\begin{aligned}
\Delta_\xi^* \quad : \quad &= \frac{1}{\xi^{\frac{n-1}{2}}}\left[(W_{r,\xi}(f;x)-f(x)) - \sum_{m=1}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \xi^m f^{(2m)}(x)\frac{1}{m!2^{2m}}\left(\sum_{j=1}^{r}\alpha_j j^{2m}\right)\right] \\
&= \sum_{j=1}^{r}\alpha_j \frac{j^n}{\sqrt{\pi\xi}\xi^{\frac{n-1}{2}}n!}\int_{-\infty}^{\infty} f^{(n)}(\gamma)t^n e^{-t^2/\xi}dt \\
&= \frac{1}{\sqrt{\pi\xi}\xi^{\frac{n-1}{2}}n!}\int_{-\infty}^{\infty}\left(\sum_{j=1}^{r}\alpha_j j^n f^{(n)}(\gamma)\right) t^n e^{-t^2/\xi}dt \\
&= \frac{1}{\sqrt{\pi\xi}\xi^{\frac{n-1}{2}}n!}\left[\int_{-\infty}^{\infty}\left(\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}f^{(n)}(\gamma)\right)t^n e^{-t^2/\xi}dt\right]. \qquad (3.9)
\end{aligned}$$

Call

$$\Phi_n(x,t) := \sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}f^{(n)}(\gamma). \qquad (3.10)$$

Thus

$$\Delta_\xi = \frac{1}{2n!\xi^{n+1}}\left[\int_{-\infty}^{\infty}\Phi_n(x,t)t^n e^{-|t|/\xi}dt\right], \qquad (3.11)$$

and

$$\Delta_\xi^* = \frac{1}{\sqrt{\pi\xi}\xi^{\frac{n-1}{2}}n!}\left[\int_{-\infty}^{\infty}\Phi_n(x,t)t^n e^{-t^2/\xi}dt\right]. \qquad (3.12)$$

Using Hölder's inequality we obtain

$$\begin{aligned}
|\Delta_\xi| &\le \frac{1}{2n!\xi^{n+1}}\left[\int_{-\infty}^{\infty}\left|\Phi_n(x,t)t^n e^{-|t|/(2\xi)}e^{-|t|/(2\xi)}\right|dt\right] \\
&\le \frac{1}{2n!\xi^{n+1}}\left(\int_{-\infty}^{\infty}\left(\Phi_n(x,t)e^{-|t|/(2\xi)}\right)^2 dt\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}\left(t^n e^{-|t|/(2\xi)}\right)^2 dt\right)^{\frac{1}{2}} \\
&= \frac{1}{2n!\xi^{n+1}}\left(\int_{-\infty}^{\infty}\Phi_n^2(x,t)e^{-|t|/\xi}dt\right)^{\frac{1}{2}}\left(2(2n)!\xi^{2n+1}\right)^{\frac{1}{2}} \\
&= \frac{\sqrt{(2n)!}}{n!}\left(\frac{1}{2\xi}\int_{-\infty}^{\infty}\Phi_n^2(x,t)e^{-|t|/\xi}dt\right)^{\frac{1}{2}} \\
&= \sqrt{\binom{2n}{n}}\left(\frac{1}{2\xi}\int_{-\infty}^{\infty}\Phi_n^2(x,t)e^{-|t|/\xi}dt\right)^{\frac{1}{2}}, \qquad (3.13)
\end{aligned}$$

and

$$\begin{aligned}
\left|\Delta_{\xi}^{*}\right| &\leq \frac{1}{\sqrt{\pi\xi}\xi^{\frac{n-1}{2}}n!}\left[\int_{-\infty}^{\infty}\left|\Phi_{n}(x,t)t^{n}e^{-t^{2}/(2\xi)}e^{-t^{2}/(2\xi)}\right|dt\right] \\
&\leq \frac{1}{\sqrt{\pi\xi}\xi^{\frac{n-1}{2}}n!}\left(\int_{-\infty}^{\infty}\left(\Phi_{n}(x,t)e^{-t^{2}/(2\xi)}\right)^{2}dt\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}\left(t^{n}e^{-t^{2}/(2\xi)}\right)^{2}dt\right)^{\frac{1}{2}} \\
&= \frac{1}{\xi^{\frac{n-1}{2}}n!}\left(\frac{\xi^{n}}{2^{2n}}\frac{(2n)!}{(n)!}\right)^{\frac{1}{2}}\left(\frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty}\Phi_{n}^{2}(x,t)e^{-t^{2}/\xi}dt\right)^{\frac{1}{2}} \\
&= \frac{\sqrt{\xi}}{n!2^{n}}\left(\frac{(2n)!}{n!}\right)^{\frac{1}{2}}\left(\frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty}\Phi_{n}^{2}(x,t)e^{-t^{2}/\xi}dt\right)^{\frac{1}{2}}. \qquad (3.14)
\end{aligned}$$

So far we have obtained

$$|\Delta_{\xi}| \leq \sqrt{\binom{2n}{n}}\left(\frac{1}{2\xi}\int_{-\infty}^{\infty}\Phi_{n}^{2}(x,t)e^{-|t|/\xi}dt\right)^{\frac{1}{2}}, \qquad (3.15)$$

and

$$\left|\Delta_{\xi}^{*}\right| \leq \frac{\sqrt{\xi}}{n!2^{n}}\left(\frac{(2n)!}{n!}\right)^{\frac{1}{2}}\left(\frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty}\Phi_{n}^{2}(x,t)e^{-t^{2}/\xi}dt\right)^{\frac{1}{2}}. \qquad (3.16)$$

Since we assumed that $f^{(n)}$ exists and it is bounded and it is Lebesgue measurable, we obtain

$$\left\|f^{(n)}\right\|_{\infty} < M, \text{ for some } M \geq 0.$$

Therefore

$$\begin{aligned}
|\Phi_{n}(x,t)| &\leq \left(\sum_{j=1}^{r}\binom{r}{j}\right)M \\
&= \left(2^{r}-1\right)M. \qquad (3.17)
\end{aligned}$$

Hence

$$\begin{aligned}
\left(\frac{1}{2\xi}\int_{-\infty}^{\infty}\Phi_{n}^{2}(x,t)e^{-|t|/\xi}dt\right)^{\frac{1}{2}} &\leq \left(2^{r}-1\right)M\left(\frac{1}{2\xi}\int_{-\infty}^{\infty}e^{-|t|/\xi}dt\right)^{\frac{1}{2}} \\
&= \left(2^{r}-1\right)M, \qquad (3.18)
\end{aligned}$$

and

$$\begin{aligned}
\left(\frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty}\Phi_{n}^{2}(x,t)e^{-t^{2}/\xi}dt\right)^{\frac{1}{2}} &\leq \left(2^{r}-1\right)M\left(\frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty}e^{-t^{2}/\xi}dt\right)^{\frac{1}{2}} \\
&= \left(2^{r}-1\right)M. \qquad (3.19)
\end{aligned}$$

Therefore

$$|\Delta_{\xi}| \leq \left(2^{r}-1\right)M\sqrt{\binom{2n}{n}} =: \lambda, \qquad (3.20)$$

and

$$\left|\Delta_{\xi}^{*}\right| \leq (2^{r}-1)\, M \frac{\sqrt{\xi}}{n! 2^{n}} \left(\frac{(2n)!}{n!}\right)^{\frac{1}{2}} =: \sqrt{\xi}\lambda^{*}. \tag{3.21}$$

Consequently we get $(0 < \alpha \leq 1)$

$$\frac{1}{\xi^{n-\alpha}} \left| (P_{r,\xi}(f;x) - f(x)) - \sum_{m=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \xi^{2m} f^{(2m)}(x) \left(\sum_{j=1}^{r} \alpha_j j^{2m} \right) \right| \leq \lambda \xi^{\alpha} \to 0, \tag{3.22}$$

as $\xi \to 0+$, and

$$\frac{1}{\xi^{\frac{n-\alpha}{2}}} \left| (W_{r,\xi}(f;x) - f(x)) - \sum_{m=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \xi^{m} \frac{f^{(2m)}(x)}{m! 2^{2m}} \left(\sum_{j=1}^{r} \alpha_j j^{2m} \right) \right| \leq \lambda^{*} \xi^{\frac{\alpha}{2}} \to 0, \tag{3.23}$$

as $\xi \to 0+$.

Notice that $n-1-2m \geq 0$ and also $\frac{n-1}{2} - m \geq 0$.

From the last we conclude the claims of the theorem. □

Corollary 3.2. **($n = 1$ case)** Let f such that f'exists and it is bounded and Lebesgue measurable on $\mathbb{R}$. Let $\xi \to 0+$, $0 < \alpha \leq 1$. Then

$$P_{r,\xi}(f;x) - f(x) = o\left(\xi^{1-\alpha}\right), \tag{3.24}$$

and

$$W_{r,\xi}(f;x) - f(x) = o\left(\xi^{\frac{1-\alpha}{2}}\right). \tag{3.25}$$

Proof. In Theorem 3.1, we place $n = 1$. □

Corollary 3.3. **($n = 2$ case)** Let f such that f''exists and it is bounded and Lebesgue measurable on $\mathbb{R}$. Let $\xi \to 0+$, $0 < \alpha \leq 1$. Then

$$P_{r,\xi}(f;x) - f(x) = o\left(\xi^{2-\alpha}\right), \tag{3.26}$$

and

$$W_{r,\xi}(f;x) - f(x) = o\left(\xi^{1-\frac{\alpha}{2}}\right). \tag{3.27}$$

Proof. In Theorem 3.1, we place $n = 2$. □

Corollary 3.4. **($n = 3$ case)** Let f such that $f^{(3)}$ exists and it is bounded and Lebesgue measurable on $\mathbb{R}$. Let $\xi \to 0+$, $0 < \alpha \leq 1$. Then

$$P_{r,\xi}(f;x) - f(x) = \xi^{2} f''(x) \left(\sum_{j=1}^{r} \alpha_j j^{2} \right) + o(\xi^{3-\alpha}), \tag{3.28}$$

and

$$W_{r,\xi}(f;x) - f(x) = \frac{\xi f''(x)}{4} \left(\sum_{j=1}^{r} \alpha_j j^2 \right) + o(\xi^{\frac{3-\alpha}{2}}). \tag{3.29}$$

Proof. In Theorem 3.1, we place $n = 3$. □

Corollary 3.5. **($n = 4$ case)** Let f such that $f^{(14)}$ exists and it is bounded and Lebesgue measurable on $\mathbb{R}$. Let $\xi \to 0+$, $0 < \alpha \leq 1$. Then

$$P_{r,\xi}(f;x) - f(x) = \xi^2 f''(x) \left(\sum_{j=1}^{r} \alpha_j j^2 \right) + o(\xi^{4-\alpha}), \tag{3.30}$$

and

$$W_{r,\xi}(f;x) - f(x) = \frac{\xi f''(x)}{4} \left(\sum_{j=1}^{r} \alpha_j j^2 \right) + o(\xi^{2-\frac{\alpha}{2}}). \tag{3.31}$$

Proof. In Theorem 3.1, we place $n = 4$. □

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