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SUBORDINATION RESULTS AND INTEGRAL MEANS INEQUALITIES FOR $K-$UNIFORMLY STARLIKE FUNCTIONS DEFINED BY CONVOLUTION INVOLVING THE HURWITZ-LERCH ZETA FUNCTION

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Abstract. In this paper, we introduce a generalized class of $k-$uniformly starlike functions and obtain the subordination results and integral means inequalities. Some interesting consequences of our results are also pointed out.

1. **Introduction**

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the open disc $U = \{z : z \in \mathcal{C};\ |z| < 1\}$. For functions $f \in A$ given by (1.1) and $g \in A$ given by $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in U. \tag{1.2}$$

In terms of the Hadamard product (or convolution), we choose g as a fixed function in A such that $(f * g)(z)$ exists for any $f \in\ A$, and for various choices of g we get different linear operators which have been studied in recent past. To illustrate some of these cases which arise from the convolution structure (1.2), we consider the following examples.

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The following we recall a general Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ defined by (cf., e.g., [28], p. 121 et sep.])

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{1.3}$$

$$(a \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; s \in \mathbb{C}, \Re(s) > 1 \text{ and } |z| = 1)$$

where, as usual, $\mathbb{Z}_0^- := \mathbb{Z} \setminus \{\mathbb{N}\}$, $(\mathbb{Z} := \{\pm 1, \pm 2, \pm 3, ...\}); \mathbb{N} := \{1, 2, 3, ...\}$.

Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ can be found in the recent investigations by Choi and Srivastava [4], Ferreira and Lopez [5], Garg et al. [8], Lin and Srivastava [15], Lin et al. [16], and others. In 2007, Srivastava and Attiya [27] (see also Raducanu and Srivastava [20], and Prajapat and Goyal [19]) introduced and investigated the linear operator:

$$\mathcal{J}_{\mu,b} : \mathcal{A} \to \mathcal{A}$$

defined, in terms of the Hadamard product (or convolution), by

$$\mathcal{J}_{\mu,b} f(z) = \mathcal{G}_{\mu,b} * f(z) \tag{1.4}$$

$(z \in U; b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; \mu \in \mathbb{C}; f \in \mathcal{A})$, where, for convenience,

$$G_{\mu,b}(z) := (1+b)^{\mu}[\Phi(z,\mu,b) - b^{-\mu}] \quad (z \in U). \tag{1.5}$$

It is easy to observe from (1.4) and (1.5) that, for $f(z)$ of the form(1.1),we have

$$\mathcal{J}_{\mu,b} f(z) = z + \sum_{n=2}^{\infty} C_n(b,\mu) a_n z^n \tag{1.6}$$

$$C_n(b,\mu) = \left(\frac{1+b}{n+b}\right)^{\mu} \tag{1.7}$$

where (and throughout this paper unless otherwise mentioned) the parameters μ, b and $C_n(b,\mu)$ are constrained as follows:

$$b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; \mu \in \mathbb{C} \quad \text{and} \quad C_n(b,\mu) = \left(\frac{1+b}{n+b}\right)^{\mu}.$$

For $f(z) \in \mathcal{A}$ and $z \in \mathcal{U}$

$$\mathcal{J}_{\mu,b} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+b}{n+b}\right)^{\mu} a_n z^n \tag{1.8}$$

For various choices of μ we get different operators and are listed below.

$$\mathcal{J}_{0,b}(f)(z) := f(z), \tag{1.9}$$

$$\mathcal{J}_{1,b}(f)(z) := \int_0^z \frac{f(t)}{t}dt := A(f)(z), \tag{1.10}$$

$$\mathcal{J}_{1,\nu}(f)(z) := \frac{1+\nu}{z^\nu}\int_0^z t^{1-\nu}f(t)dt := \mathcal{F}_\nu(f)(z), (\nu > -1), \tag{1.11}$$

$$\mathcal{J}_{\sigma,1}(f)(z) := z + \sum_{n=2}^{\infty}\left(\frac{2}{n+1}\right)^\sigma a_n z^n = \mathcal{I}^\sigma(f)(z)(\sigma > 0), \tag{1.12}$$

where $\mathcal{A}(f)$ and $\mathcal{F}_\gamma$ are the integral operators introduced by Alexandor [1] and Bernardi [3], respectively, and $\mathcal{I}^\sigma(f)$ is the Jung-Kim-Srivastava integral operator [11] closely related to some multiplier transformation studied by Fleet [6].

In this paper, by making use of the operator$\mathcal{J}_{\mu,b}$ we introduced a new subclass of analytic functions with negative coefficients and discuss some interesting properties of this generalized function class.

For $0 \le \gamma < 1$ and $k \ge 0$, we let $\mathcal{J}_b^\mu(\gamma, k)$ be the subclass of A consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\text{Re}\left\{\frac{z(\mathcal{J}_b^\mu f(z))'}{\mathcal{J}_b^\mu f(z)} - \gamma\right\} > k\left|\frac{z(\mathcal{J}_b^\mu f(z))'}{\mathcal{J}_b^\mu f(z)} - 1\right|,\ z \in U, \tag{1.13}$$

where $\mathcal{J}_b^\mu f(z)$ is given by (1.4). We further let $T\mathcal{J}_b^\mu(\gamma, k) = \mathcal{J}_b^\mu(\gamma, k) \cap T$, where

$$T \ := \ \left\{f \in A : f(z) = z - \sum_{n=2}^{\infty}|a_n|z^n, \quad z \in U\right\} \tag{1.14}$$

is a subclass of A introduced and studied by Silverman [23].

By suitably specializing the values of μ, γ and k in the class $\mathcal{J}_b^\mu(\gamma, k)$, we obtain the various subclasses, we present some examples.

Example 1.1. If $\mu = 0$ then

$$\mathcal{J}_b^0(\gamma, k) \equiv \mathbb{S}(\gamma, k) := \left\{f \in A : \text{Re}\ \left\{\tfrac{zf'(z)}{f(z)} - \gamma\right\} > k\left|\tfrac{zf'(z)}{f(z)} - 1\right|,\ z \in U\right\}.$$

Further $T\mathbb{S}(\gamma, k) = \mathbb{S}(\gamma, k) \cap T$, where T is given by (1.14). The class $T\mathbb{S}(\gamma, k) \equiv UST(\gamma, k)$. A function in $UST(\gamma, k)$ is called $k-$uniformly starlike of order γ, $0 \le \gamma < 1$ and Note that the classes $UST(\gamma, 0)$ and $UST(0, 0)$ were first introduced in [23]. We also observe that $UST(\gamma, 0) \equiv T^*(\gamma)$ is well-known subclass of starlike functions of order γ.

Example 1.2. If $\mu = 1$ and $b = \nu$ with $\nu > -1$ then

$$\mathcal{J}_\nu^1(\gamma, k) \equiv B_\nu(\gamma, k) = \left\{f \in A : \text{Re}\ \left(\frac{z(J_\nu f(z))'}{J_\nu f(z)} - \gamma\right) > k\left|\frac{z(J_\nu f(z))'}{J_\nu f(z)} - 1\right|,\ z \in U\right\},$$

where J_ν is a Bernardi operator [3] defined by

$$J_\nu f(z) := \frac{\nu+1}{z^\nu}\int_0^z t^{\nu-1}f(t)dt.$$

Note that the operator J_1 was studied earlier by Libera [13] and Livingston [17]. Further, $TB_\nu(\gamma,k) = B_\nu(\gamma,k)\cap T$, where T is given by (1.14).

Example 1.3. If $\mu = \sigma$ and $b = 1$ with $\sigma > 0$ then

$$\mathcal{J}_1^\sigma(\gamma,k) \equiv \mathcal{I}^\sigma(\gamma,k) = \left\{ f \in A : \mathrm{Re}\ \left(\frac{z(\mathcal{I}^\sigma f(z))'}{\mathcal{I}^\sigma f(z)} - \gamma\right) > k \left|\frac{z(\mathcal{I}^\sigma f(z))'}{\mathcal{I}^\sigma f(z)} - 1\right|,\ z \in U \right\},$$

where $\mathcal{I}^\sigma$ is the Jung-Kim-Srivastava integral operator [11] defined by

$$\mathcal{I}^\sigma f(z) := z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1}\right)^{\sigma} a_n z^n.$$

Further, $T\mathcal{I}^\sigma(\gamma,k) = \mathcal{I}^\sigma(\gamma,k)\cap T$, where T is given by (1.14).

Remark 1.4. Observe that, specializing the parameters μ γ and k in the class $\mathcal{J}_b^\mu(\gamma,k)$, we obtain various classes introduced and studied by Goodman [9, 10], Kanas et.al., [12], Ma and Minda [18], Rønning [21, 22] and others.

The object of the present paper is to investigate the coefficient estimates,extremepoint.Further, we obtain the subordination results and integral means inequalities for the generalized class k- uniformly starlike functions. Some interesting consequences of our results are also pointed out.

2. **Coefficient Estimates**

We first mention a sufficient condition for function $f(z)$ of the form (1.1) to belong to the class $\mathcal{J}_b^\mu(\gamma,k)$,given by the following theorem which can be established easily on lines similar to Aouf and Murugusundaramoorthy [2] hence we omit the details.

Theorem 2.1. *A function $f(z)$ of the form (1.1) is in $\mathcal{J}_b^\mu(\gamma,k)$ if*

$$\sum_{n=2}^{\infty}[n(1+k)-(\gamma+k)]C_n(b,\mu)\ |a_n| \le 1-\gamma, \tag{2.1}$$

where $0 \le \gamma < 1$, $k \ge 0$, and $C_n(b,\mu)$ is given by (1.7).

Theorem 2.2. *Let $0 \le \gamma < 1$, $k \ge 0$ and a function f of the form (1.14) to be in the class $T\mathcal{J}_b^\mu(\gamma,k)$ if and only if*

$$\sum_{n=2}^{\infty}[n(1+k)-(\gamma+k)]C_n(b,\mu)\ |a_n| \le 1-\gamma, \tag{2.2}$$

where $C_n(b,\mu)$ is given by (1.7).

Corollary 2.3. *If $f \in \mathcal{T}\mathcal{J}_b^{\mu}(\gamma, k)$, then*

$$|a_n| \leq \frac{1-\gamma}{[n(1+k)-(\gamma+k)]C_n(b,\mu)}, \quad , \; 0 \leq \gamma < 1, k \geq 0, \tag{2.3}$$

where $C_n(b,\mu)$ is given by (1.7).
Equality holds for the function $f(z) = z - \frac{1-\gamma}{[n(1+k)-(\gamma+k)]C_n(b,\mu)} z^n$.

Theorem 2.4. (Extreme Points) *Let*

$$f_1(z) = z \quad \text{and} \quad f_n(z) = z - \frac{1-\gamma}{[n(1+k)-(\gamma+k)]C_n(b,\mu)} z^n, \;\; n \geq 2,$$

for $0 \leq \gamma < 1$, $k \geq 0$, and $C_n(b,\mu)$ is given by (1.7). Then $f(z)$ is in the class $T\mathcal{J}_b^{\mu}(\gamma, k)$ if and only if it can be expressed in the form $f(z) = \sum_{n=1}^{\infty} \omega_n f_n(z)$,

where $\omega_n \geq 0$ and $\sum_{n=1}^{\infty} \omega_n = 1$.

3. **Subordination Results**

Before stating and proving our subordination theorem for the class $\mathcal{T}\mathcal{J}_b^{\mu}(\gamma, k)$, we need the following definitions and lemmas.

Definition 3.1. For analytic functions g and h with $g(0) = h(0)$, g is said to be subordinate to h, denoted by $g \prec h$, if there exists an analytic function w such that $w(0) = 0$, $|w(z)| < 1$ and $g(z) = h(w(z))$, for all $z \in U$.

Definition 3.2. A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating sequence if, whenever $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $a_1 = 1$ is regular, univalent and convex in U, we have

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z), \quad z \in U. \tag{3.1}$$

In 1961, Wilf [29] proved the following subordinating factor sequence.

Lemma 3.3. *The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$Re \left\{ 1 + 2\sum_{n=1}^{\infty} b_n z^n \right\} > 0, \quad z \in U. \tag{3.2}$$

Theorem 3.4. *Let $f \in \mathcal{T}\mathcal{J}_b^{\mu}(\gamma, k)$ and $g(z)$ be any function in the usual class of convex functions C, then*

$$\frac{(2+k-\gamma))C_2}{2[1-\gamma+(2+k-\gamma)C_2]}(f * g)(z) \prec g(z) \tag{3.3}$$

where $0 \leq \gamma < 1$; $k \geq 0$ with

$$C_2 = C_2(b,\mu) = \left(\frac{1+b}{2+b}\right)^{\mu} \tag{3.4}$$

and

$$Re\ \{f(z)\} > -\frac{[1-\gamma+(2+k-\gamma)C_2]}{(2+k-\gamma)C_2}, \quad z \in U. \tag{3.5}$$

The constant factor $\frac{(2+k-\gamma)C_2}{2[1-\gamma+(2+k-\gamma)C_2]}$ *in (3.3) cannot be replaced by a larger number.*

Proof. Let $f \in \mathcal{TJ}_b^\mu(\gamma, k)$ and suppose that $g(z) = z + \sum_{n=2}^{\infty} c_n z^n \in C$. Then

$$\frac{(2+k-\gamma)C_2}{2[1-\gamma+(2+k-\gamma)C_2]}(f*g)(z) \\ = \frac{(2+k-\gamma)C_2}{2[1-\gamma+(2+k-\gamma)C_2]}\left(z + \sum_{n=2}^{\infty} c_n a_n z^n\right). \tag{3.6}$$

Thus, by Definition 3.2, the subordination result holds true if

$$\left\{\frac{(2+k-\gamma)C_2}{2[1-\gamma+(2+k-\gamma)C_2]}\right\}_{n=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 3.3, this is equivalent to the following inequality

$$\text{Re}\left\{1 + \sum_{n=1}^{\infty} \frac{(2+k-\gamma)C_2}{[1-\gamma+(2+k-\gamma)C_2]} a_n z^n\right\} > 0, \quad z \in U. \tag{3.7}$$

Since $\frac{(n(1+k)-(\gamma+k))C_n(b,\mu)}{(1-\gamma)} \geq \frac{(2+k-\gamma)C_2}{(1-\gamma)} > 0$, for $n \geq 2$ we have

$$\begin{aligned}
&\text{Re}\left\{1 + \frac{(2+k-\gamma)C_2}{[1-\gamma+(2+k-\gamma)C_2]}\sum_{n=1}^{\infty} a_n z^n\right\} \\
&= \text{Re}\left\{1 + \frac{(2+k-\gamma)C_2}{[1-\gamma+(2+k-\gamma)C_2]} z + \frac{\sum_{n=2}^{\infty}(2+k-\gamma)C_2 a_n z^n}{[1-\gamma+(2+k-\gamma)C_2]}\right\} \\
&\geq 1 - \frac{(2+k-\gamma)C_2}{[1-\gamma+(2+k-\gamma)C_2]} r \\
&\qquad - \frac{1}{[1-\gamma+(2+k-\gamma)C_2]}\sum_{n=2}^{\infty} |[n(1+k)-(\gamma+k)(1+n\lambda-\lambda)]C_n(b,\mu)a_n|\, r^n \\
&\geq 1 - \frac{(2+k-\gamma)C_2}{[1-\gamma+(2+k-\gamma)C_2]} r - \frac{1-\gamma}{[1-\gamma+(2+k-\gamma)C_2]} r \\
&> 0, \quad |z| = r < 1,
\end{aligned}$$

where we have also made use of the assertion (2.1) of Theorem 2.1. This evidently proves the inequality (3.7) and hence the subordination result (3.3) asserted by Theorem 3.4. The inequality (3.5) follows from (3.3) by taking

$$g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n \in C.$$

Next we consider the function

$$F(z) := z - \frac{1-\gamma}{(2+k-\gamma)C_2} z^2$$

where $0 \leq \gamma < 1$, $k \geq 0$, and C_2 is given by (3.4). Clearly $F \in \mathcal{T}\mathcal{J}_b^{\mu}(\gamma, k)$. For this function ,(3.3)becomes

$$\frac{(2+k-\gamma)C_2}{2[1-\gamma+(2+k-\gamma)C_2]} F(z) \prec \frac{z}{1-z}.$$

It is easily verified that

$$\min\left\{\text{Re}\left(\frac{(2+k-\gamma)C_2}{2[1-\gamma+(2+k-\gamma)C_2]} F(z)\right)\right\} = -\frac{1}{2}, \quad z \in U.$$

This shows that the constant $\frac{(2+k-\gamma)C_2}{2[1-\gamma+(2+k-\gamma)C_2]}$ cannot be replaced by any larger one. □

By taking different choices of μ, γ and k in the above theorem and in view of Examples 1 and 2 in Section 1, we state the following corollaries for the subclasses defined in those examples.

Corollary 3.5. *If $f \in \mathbb{S}^*(\gamma, k)$, then*

$$\frac{2+k-\gamma}{2[3+k-\gamma]}(f * g)(z) \prec g(z), \tag{3.8}$$

where $0 \leq \gamma < 1$, , $k \geq 0$, $g \in C$ and

$$Re\{f(z)\} > -\frac{3+k-2\gamma}{2+k-\gamma}, \quad z \in U.$$

The constant factor

$$\frac{2+k-\gamma}{2[3+k-2\gamma]}$$

in (3.8) cannot be replaced by a larger one.

Remark 3.6. Corollary 3.5, yields the result obtained by Singh [26] when $\gamma = k = 0$.

Remark 3.7. Corollary 3.5 yields the results obtained by Frasin [7] for the special values of γ and k.

Corollary 3.8. *If $f \in B_\nu(\gamma, k)$, then*

$$\frac{(\nu+1)(2+k-\gamma)}{2[(\nu+2)(1-\gamma)+(\nu+1)(2+k-\gamma)]}(f*g)(z) \prec g(z), \tag{3.9}$$

where $0 \leq \gamma < 1,\ ,\ k \geq 0,\ \nu > -1,\ g \in C$ *and*

$$Re\{f(z)\} > -\frac{[(\nu+2)(1-\gamma)+(\nu+1)(2+k-\gamma)]}{(\nu+1)(2+k-\gamma)},\ \ z \in U.$$

The constant factor

$$\frac{(\nu+1)(2+k-\gamma)}{2[(\nu+2)(1-\gamma)+(\nu+1)(2+k-\gamma)]}$$

in (3.9) cannot be replaced by a larger one.

4. **Integral Means Inequalities**

Due to , Littlewood [14] we obtain integral means inequalities for the functions in the family$\mathcal{TJ}_b^\mu(\gamma, k)$. We also state the integral means inequalities for several known as well as new subclasses.

Lemma 4.1. *If the functions f and g are analytic in U with $g \prec f$, then for $\eta > 0$, and $0 < r < 1$,*

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^\eta d\theta \leq \int_0^{2\pi} \left|f(re^{i\theta})\right|^\eta d\theta. \tag{4.1}$$

In [23], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality, conjectured in [24] and settled in [25], that

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^\eta d\theta \leq \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^\eta d\theta,$$

for all $f \in T$, $\eta > 0$ and $0 < r < 1$. In [25], he also proved his conjecture for the subclasses $T^*(\gamma)$ and $C(\gamma)$ of T.

Applying Lemma 4.1, Theorem 2.2 and Theorem 2.4, we obtain the following integral means inequalities for the functions in the family $\mathcal{TJ}_b^\mu(\gamma, k)$.

Theorem 4.2. *Suppose $f \in \mathcal{TJ}_b^\mu(\gamma, k)$, $\eta > 0$, $0 \leq \gamma < 1$, $k \geq 0$ and $f_2(z)$ is defined by*

$$f_2(z) = z - \frac{1-\gamma}{(2+k-\gamma)C_2}z^2,$$

where C_2 is given by (3.4). Then for $z = re^{i\theta}$, $0 < r < 1$, we have

$$\int_0^{2\pi} |f(z)|^\eta \, d\theta \leq \int_0^{2\pi} |f_2(z)|^\eta \, d\theta. \tag{4.2}$$

Proof. For $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$, (4.2) is equivalent to proving that

$$\int_0^{2\pi} \left| 1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \right|^\eta d\theta \leq \int_0^{2\pi} \left| 1 - \frac{(1-\gamma)}{(2+k-\gamma)C_2} z \right|^\eta d\theta.$$

By Lemma 4.1, it suffices to show that

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \prec 1 - \frac{1-\gamma}{(2+k-\gamma)C_2} z.$$

Setting

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} = 1 - \frac{1-\gamma}{(2+k-\gamma)C_2} w(z), \tag{4.3}$$

and using (2.2), we obtain

$$\begin{aligned} |w(z)| &= \left| \sum_{n=2}^{\infty} \frac{[n(1+k) - (\gamma+k)] C_n(b,\mu)}{1-\gamma} |a_n| z^{n-1} \right| \\ &\leq |z| \sum_{n=2}^{\infty} \frac{[n(1+k) - (\gamma+k)] C_n(b,\mu)}{1-\gamma} |a_n| \\ &\leq |z|, \end{aligned}$$

where $C_n(b,\mu)$ is given by (1.7). This completes the proof by Theorem 2.2. □

In view of the Examples 1 and 2 in Section 1 and Theorem 4.2, we can state the following corollaries without proof for the classes defined in those examples.

Corollary 4.3. *If $f \in T\mathbb{S}(\gamma, k)$, $0 \leq \gamma < 1$, $k \geq 0$ and $\eta > 0$, then the assertion (4.2) holds true where*

$$f_2(z) = z - \frac{1-\gamma}{[2+k-\gamma)]} z^2.$$

Remark 4.4. Fixing $k = 0$, Corollary 4.3 lead the integral means inequality for the class $T^*(\gamma)$ obtained in [25].

Corollary 4.5. *If $f \in TB_\nu(\gamma, k)$, $\nu > -1$, $0 \leq \gamma < 1$, $k \geq 0$ and $\eta > 0$, then the assertion (4.2) holds true where*

$$f_2(z) = z - \frac{(1-\gamma)(\nu+2)}{(\nu+1)[2+k-\gamma]} z^2 \ .$$

Concluding Remarks. The various results presented in this paper would provide interesting extensions and generalizations of those considered earlier for simpler analytic function classes. The details involved in the derivations of such specializations of the results presented in this paper are fairly straight- forward.

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References

[1] Alexander, J. W., *Functions which map the interior of the unit circle upon simple regions*, Ann. of Math., **17** (1915), 1222.

[2] Aouf, M. K., Murugusundaramoorthy, G., *On a subclass of uniformly convex functions defined by the Dziok-Srivastava Operator,* Austral. J. Math. Anal. Appl., **3** (2007).

[3] Bernardi, S.D., *Convex and starlike univalent functions,* Trans. Amer. Math. Soc., **135** (1969), 429–446.

[4] Choi, J., Srivastava, H. M., *Certain families of series associated with the Hurwitz-Lerch Zeta function*, Appl. Math. Comput., **170** (2005), 399-409.

[5] Ferreira, C., Lopez, J. L., *Asymptotic expansions of the Hurwitz-Lerch Zeta function*, J. Math. Anal. Appl., **298** (2004), 210-224.

[6] Flett, T. M., *The dual of an inequality of Hardy and Littlewood and some related inequalities*, J. Math. Anal. Appl., **38**(1972), 746765.

[7] Frasin, B. A., *Subordination results for a class of analytic functions defined by a linear operator,* J. Ineq. Pure Appl. Math., **7 (134)** (2006), no. 4, 1–7.

[8] Garg, M., Jain, K., Srivastava, H. M., *Some relationships between the generalized Apostol-Bernoulli polynomials and Hurwitz-Lerch Zeta functions*, Integral Transform. Spec. Funct., **17** (2006), 803-815.

[9] Goodman, A. W., *On uniformly convex functions,* Ann. Polon. Math., **56** (1991), 87–92.

[10] W. Goodman, A. W. *On uniformly starlike functions,* J. Math. Anal. Appl., **155** (1991), 364–370.

[11] Jung, I. B., Kim, Y. C., Srivastava, H. M., *The Hardy space of analytic functions associated with certain one-parameter families of integral operators*, J. Math. Anal. Appl., **176** (1993), 138147.

[12] Kanas, S., Srivastava, H. M., *Linear operators associated with $k-$uniformly convex functions,* Integral Transform Spec. Funct., **9** (2000), 121–132.

[13] Libera, R. J., *Some classes of regular univalent functions,* Proc. Amer. Math. Soc., **16** (1965), 755–758.

[14] Littlewood, J. E., *On inequalities in theory of functions,* Proc. London Math. Soc., **23** (1925), 481–519.

[15] Lin, S. D., Srivastava, H. M., *Some families of the Hurwitz-Lerch Zeta functions and associated fractional derivative and other integral representations*, Appl. Math. Comput., **154** (2004), 725-733.

[16] Lin, S. D., Srivastava, H. M., Wang, P. Y., *Some espansion formulas for a class of generalized Hurwitz-Lerch Zeta functions*, Integral Transform. Spec. Funct., **17** (2006), 817-827.

[17] Livingston, A. E., *On the radius of univalence of certain analytic functions,* Proc. Amer. Math. Soc., **17** (1966), 352–357.

[18] Ma, W. C., Minda, D., *Uniformly convex functions,* Annal. Polon. Math., **57** (1992), no. 2, 165–175.

[19] Prajapat, J. K., Goyal, S. P., *Applications of Srivastava-Attiya operator to the classes of strongly starlike and strongly convex functions*, J. Math. Inequal., **3** (2009), 129-137.

[20] Raducanu, D., Srivastava, H. M., *A new class of analytic functions defined by means of a convolution operator involving the Hurwitz-Lerch Zeta function*, Integral Transform. Spec. Funct., **18** (2007), 933-943.

[21] Rønning, F., *Uniformly convex functions and a corresponding class of starlike functions,* Proc. Amer. Math. Soc. **118** (1993), 189–196.

[22] Rønning, F., *On starlike functions associated with parabolic regions,* Ann. Univ. Mariae Curie-Sklodowska Sect. A, **45** (1991), 117–122.

[23] Silverman, H., *Univalent functions with negative coefficients,* Proc. Amer. Math. Soc., **51** (1975), 109–116.

[24] Silverman, H., *A survey with open problems on univalent functions whose coefficients are negative,* Rocky Mt. J. Math., **21** (1991), 1099–1125.

[25] Silverman, H., *Integral means for univalent functions with negative coefficients,* Houston J. Math., **23** (1997), 169–174.

[26] Singh, S., *A subordination theorem for spirallike functions,* Internat. J. Math. and Math. Sci., **24** (2000), no. 7, 433–435.

[27] Srivastava, H. M., Attiya, A. A., *An integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination*, Integral Transform. Spec. Funct., **18** (2007), 207–216.

[28] Srivastava, H. M., Choi, J., *Series associated with the Zeta and related functions*, Dordrecht, Boston, London: Kluwer Academic Publishers, 2001.

[29] Wilf, H. S., *Subordinating factor sequence for convex maps of the unit circle,* Proc. Amer. Math. Soc., **12** (1961), 689–693.

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