STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 4, December 2010

SUFFICIENT CONDITIONS FOR UNIVALENCE AND QUASICONFORMAL EXTENSIONS IN SEVERAL COMPLEX VARIABLES

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Abstract. The method of subordination chains is used to established an univalence criterion which contains as particular cases some univalence criteria for holomorphic mappings in the unit ball B of $\mathbb{C}^n$. We also obtain a sufficient condition for a normalized mapping $f \in \mathcal{H}(B)$ to be extended to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

1. Introduction and preliminaries

Pfaltzgraff [16] was the first who obtained an univalence criterion in the n-variable case. He [17] also initiated the study of quasiconformal extensions for quasiregular holomorphic mappings defined on the unit ball of $\mathbb{C}^n$.

The problems of univalence criteria and quasiconformal extensions for holomorphic mappings on the unit ball in $\mathbb{C}^n$ have been studied by P. Curt [3], [4], [5], [7], H. Hamada and G. Kohr [14], [15], P. Curt and G. Kohr [9], [10], [11], D. Răducanu [18].

In this work we generalize the results due to J.A. Pfaltzgraff [16], [17], P. Curt [3], [5], [7], D. Răducanu [18].

Let $\mathbb{C}^n$ denote the space of n-complex variables $z = (z_1, \ldots, z_n)$ with the usual inner product $\langle z, w\rangle = \sum_{i=1}^{n} z_i\overline{w}_i$ and Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$. Let B denote the open unit ball in $\mathbb{C}^n$.

Let $\mathcal{H}(B)$ be the set of holomorphic mappings from B into $\mathbb{C}^n$. Also, let $\mathcal{L}(\mathbb{C}^n)$ be the space of continuous linear mappings from $\mathbb{C}^n$ into $\mathbb{C}^n$ with the standard operator norm

$$\|A\| = \sup\{\|Az\| : \ \|z\| = 1\}.$$

Received by the editors: 30.06.2009.
2000 *Mathematics Subject Classification.* Primary 32H, Secondary 30C45.
Key words and phrases. Loewner chains, Loewner differential equation, quasiconformal mapping, quasiconformal extension.

By I we denote the identity in $\mathcal{L}(\mathbb{C}^n)$. A mapping $f \in \mathcal{H}(B)$ is said to be normalized if $f(0) = 0$ and $Df(0) = I$.

We say that a mapping $f \in \mathcal{H}(B)$ is K-quasiregular, $K \geq 1$, if

$$\|Df(z)\|^n \leq K|\det Df(z)|, \quad z \in B.$$

A mapping $f \in \mathcal{H}(B)$ is called quasiregular if is K-quasiregular for some $K \geq 1$. Every quasiregular holomorphic mapping is locally biholomorphic.

Let G and G' be domains in $\mathbb{R}^m$. A homeomorphism $f : G \to G'$ is said to be K-quasiconformal if it is differentiable a.e., ACL (absolutely continuous on lines) and

$$\|Df(x)\|^m \leq K|\det Df(x)| \text{ a.e. } , x \in G$$

where $Df(x)$ denotes the real Jacobian matrix of f and K is a constant.

If $f, g \in \mathcal{H}(B)$, we say that f is subordinate to g (and write $f \prec g$) if there exists a Schwarz mapping v (i.e. $v \in \mathcal{H}(B)$ and $\|v(z)\| \leq \|z\|$, $z \in B$) such that $f(z) = g(v(z))$, $z \in B$.

A mapping $L : B\times[0,\infty) \to \mathbb{C}^n$ is called a subordination chain if the following conditions hold:

(i) $L(0,t) = 0$ and $L(\cdot,t) \in \mathcal{H}(B)$ for $t \geq 0$;

(ii) $L(\cdot,s) \prec L(\cdot,t)$ for $0 \leq s \leq t < \infty$.

An important role in our discussion is played by the n-dimensional version of the class of holomorphic functions on the unit disc with positive real part

$$\mathcal{N} = \{h \in \mathcal{H}(B) : \ h(0) = 0, \ \text{Re } \langle h(z), z\rangle > 0, \ z \in B \setminus \{0\}\}$$

$$\mathcal{M} = \{h \in \mathcal{N}; \ Dh(0) = I\}.$$

It is known that normalized univalent subordination chains satisfy the generalized Loewner differential equation ([12], [8]).

By using an elementary change of variable, it is not difficult to reformulate the mentioned result in the case of nonnormalized subordination chains $L(z,t) = a(t)z + \dots$, where $a : [0,\infty) \to \mathbb{C}$, $a(\cdot) \in C^1([0,\infty))$, $a(0) = 1$ and $\lim_{t\to\infty} |a(t)| = \infty$.

Theorem 1.1. *Let $L(z,t) : B \times [0,\infty) \to \mathbb{C}^n$ be a Loewner chain such that $L(z,t) = a(t)z + \dots$, where $a \in C^1([0,\infty))$, $a(0) = 1$, and $\lim_{t\to\infty} |a(t)| = \infty$. Then there exists a mapping $h = h(z,t) : B \times [0,\infty) \to \mathbb{C}^n$ such that $h(\cdot,t) \in \mathcal{N}$ for $t \geq 0$, $h(z,\cdot)$ is measurable on $[0,\infty)$ for $z \in B$ and*

$$\frac{\partial L}{\partial z}(z,t) = DL(z,t)h(z,t), \ \text{a.e. } t \geq 0, \ \forall \ z \in B. \tag{1.1}$$

We shall use the following theorem to prove our results [6]. We mention that this result is a simplified version of Theorem 3 [3] due to Theorem 1.2 [12].

Theorem 1.2. *Let $L(z,t) = a(t)z + \dots$, be a function from $B \times [0,\infty)$ into $\mathbb{C}^n$ such that*

(i) $L(\cdot,t) \in \mathcal{H}(B)$, for each $t \geq 0$

(ii) $L(z,t)$ is absolutely continuous of t, locally uniformly with respect to B.

Let $h(z,t)$ be a function from $B \times [0,\infty)$ into $\mathbb{C}^n$ such that

(iii) $h(\cdot,t) \in \mathcal{N}$ for each $t \geq 0$

(iv) $h(z,\cdot)$ is measurable on $[0,\infty)$ for each $z \in B$.

Suppose $h(z,t)$ satisfies:

$$\frac{\partial L}{\partial t}(z,t) = DL(z,t)h(z,t) \text{ a.e. } t \geq 0,\ \forall\ z \in B.$$

Further, suppose

(a) $a(0) = 1$, $\lim\limits_{t\to\infty} |a(t)| = \infty$, $a(\cdot) \in C^1([0,\infty))$.

(b) There is a sequence $\{t_m\}_m$, $t_m > 0$, $t_m \to \infty$ such that

$$\lim_{m\to\infty} \frac{L(z,t)}{a(t_m)} = F(z) \tag{1.2}$$

locally uniformly in B, where $F \in \mathcal{H}(B)$.

Then for each $t \geq 0$, $L(\cdot,t)$ is univalent on B.

Also, we shall use the following result that was recently proved by P. Curt and G. Kohr [11].

Theorem 1.3. *Let $L(z,t) : B \times [0,\infty) \to \mathbb{C}^n$, $L(z,t) = a(t)z + \dots$, be a Loewner chain such that $a(\cdot) \in C^1[0,\infty)$, $a(0) = 1$ and $\lim\limits_{t\to\infty} |a(t)| = \infty$. Assume that the following conditions hold:*

(i) There exists $K > 0$ such that $L(\cdot,t)$ is K-quasiregular for each $t \geq 0$.

(ii) There exist some constants $M > 0$ and $\beta \in [0,1)$ such that

$$\|DL(z,t)\| \leq \frac{M|a(t)|}{(1-\|z\|)^\beta}, \quad z \in B,\ t \in [0,\infty) \tag{1.3}$$

(ii) There exists a sequence $\{t_m\}_{m\in\mathbb{N}}$, $t_m > 0$, $\lim\limits_{m\to\infty} t_m = \infty$, and a mapping $F \in \mathcal{H}(B)$ such that

$$\lim_{m\to\infty} \frac{L(z,t_m)}{a(t_m)} = F(z) \text{ locally uniformly on } B.$$

Further, assume that the mapping $h(z,t)$ defined by Theorem 1.1 satisfies the following conditions

(iv) There exists a constant $C > 0$ such that

$$C\|z\|^2 \leq \operatorname{Re} \langle h(z,t), z\rangle, \quad z \in B,\ t \in [0,\infty) \tag{1.4}$$

(v) There exists a constant $C_1 > 0$ such that

$$\|h(z,t)\| \leq C_1, \quad z \in B,\ t \in [0,\infty). \tag{1.5}$$

Then the function $f = L(\cdot, 0)$ extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

2. **Univalence criteria**

In this section, by using the Loewner chains method, we obtain some univalence criteria involving the first and second derivative of an holomorphic mapping in the unit ball B.

Theorem 2.1. *Let $f \in \mathcal{H}(B)$ be a normalized mapping (i.e. $f(0) = 0$ and $Df(0) = I$). Let $\beta \in \mathbb{R}$, $\beta \geq 2$ and α, c be complex numbers such that*

$$c \neq -1, \quad \alpha \neq 1 \quad \textit{and} \quad \left|\frac{c+\alpha}{1-\alpha}\right| \leq 1.$$

If the function $f(z) - \alpha z$, $z \in B$ is locally biholomorphic on B and if the following conditions hold

$$\left\|(Df(z) - \alpha I)^{-1}(cDf(z) + \alpha I) - \left(\frac{\beta}{2} - 1\right) I\right\| < \frac{\beta}{2}, \quad z \in B \tag{2.1}$$

and

$$\Big\|\|z\|^{\beta}(Df(z) - \alpha I)^{-1}(cDf(z) + \alpha I) \tag{2.2}$$

$$+(1 - \|z\|^{\beta})(Df(z) - \alpha I)^{-1}D^2 f(z)(z, \cdot) + \left(1 - \frac{\beta}{2}\right) I\Big\| < \frac{\beta}{2}, \quad z \in U$$

then the function f is univalent on B.

Proof. we will show that the relations (2.1) and (2.2) allow us to embed f as the initial element $f(z) = L(z, 0)$ of an appropriate subordination chain.

We define

$$L(z,t) = f(e^{-t}z) + \frac{1}{1+c}(e^{\beta t} - 1)e^{-t}[Df(e^{-t}z) - \alpha I](z), \ t \geq 0, \ z \in B. \tag{2.3}$$

Since

$$a(t) = e^{(\beta-1)t}\frac{1-\alpha}{1+c}\left(1 + e^{-t}\frac{c+\alpha}{1-\alpha}\right) \quad \text{and} \quad \left|\frac{c+\alpha}{1-\alpha}\right| \leq 1$$

we deduce that $a(t) \neq 0$, $a(0) = 1$, $\lim\limits_{t\to\infty} |a(t)| = \infty$ and $a(\cdot) \in C^1([0, \infty))$.

It can be easily verified that:

$$L(z,t) = a(t)z + (\text{holormohic term}) \quad \text{so} \quad \lim_{t\to\infty} \frac{L(z,t)}{a(t)} = z$$

locally uniformly with respect to $z \in B$, and thus (1.2) holds with $F(z) = z$.

It is obvious that L satisfies the absolute continuity requirements of Theorem 1.2.

From (2.3) we obtain:

$$DL(z,t) = \frac{1}{1+c}\frac{\beta}{2}e^{(\beta-1)t}[Df(e^{-t}z) - \alpha I]\Big[I - I \tag{2.4}$$

$$\frac{2}{\beta}e^{-\beta t}(c+1)(Df(e^{-t}z)-\alpha I)^{-1}Df(e^{-t}z)+\frac{2}{\beta}(1-e^{-\beta t})I$$
$$+\frac{2}{\beta}(1-e^{-\beta t})(Df(e^{-t}z)-\alpha I)^{-1}D^2f(e^{-t}z)(e^{-t}z,\cdot)\Big].$$

By using the obvious equality:

$$(c+1)[Df(e^{-t}z)-\alpha I]^{-1}Df(e^{-t}z)=Df(e^{-t}z)-\alpha I$$

the relation (2.4) becomes

$$DL(z,t)=\frac{1}{1+c}\frac{\beta}{2}e^{(\beta-1)t}[Df(e^{-t}z)-\alpha I]\Big\{I+\left(\frac{2}{\beta}-1\right)I \tag{2.5}$$
$$+\frac{2}{\beta}e^{-\beta t}[Df(e^{-t}z)-\alpha I]^{-1}[cDf(e^{-t}z)+\alpha I]$$
$$+\frac{2}{\beta}(1-e^{-\beta t})[Df(e^{-t}z)-\alpha I]^{-1}D^2f(e^{-t}z)(e^{-t}z,\cdot)\Big\}.$$

If we denote, for each fixed $(z,t)\in B\times[0,\infty)$, by $E(z,t)$ the linear operator

$$E(z,t)=-\frac{2}{\beta}e^{-\beta t}(Df(e^{-t}z)-\alpha I)^{-1}(cDf(e^{-t}z)+\alpha I) \tag{2.6}$$
$$-\frac{2}{\beta}(1-e^{-\beta t})(Df(e^{-t}z)-\alpha I)^{-1}D^2f(e^{-t}z)(e^{-t}z,\cdot)+\left(1-\frac{2}{\beta}\right)I,$$

then (2.5) becomes:

$$DL(z,t)=\frac{1}{1+c}\frac{\beta}{2}e^{(\beta-1)t}[Df(e^{-t}z)-\alpha I][I-E(z,t)]. \tag{2.7}$$

We will prove next that for each $z\in B$ and $t\in[0,\infty)$, $I-E(z,t)$ is an invertible operator.

For $t=0$,

$$E(z,0)=-\frac{2}{\beta}\left[(Df(z)-\alpha I)^{-1}(cDf(z)+\alpha I)-\frac{\beta}{2}I+I\right].$$

By using the condition (2.1) we obtain that $\|E(z,0)\|<1$ and in consequence $I-E(z,0)$ is an invertible operator.

For $t>0$ since $E(,t):\overline{B}\to\mathcal{L}(\mathbb{C}^n)$ is holomorphic, by using the weak maximum modulus theorem we obtain that $\|E(z,t)\|$ can have no maximum in B unless $\|E(z,t)\|$ is of constant value throughout $\overline{B}$.

If $z=0$ and $t>0$, since $\beta\geq 2$, we have

$$\|E(0,t)\|=\frac{2}{\beta}\left|1+\frac{c+\alpha}{1-\alpha}e^{-\beta t}-\frac{\beta}{2}\right|<1. \tag{2.8}$$

Also, we have

$$\|E(z,t)\|\leq\max_{\|w\|=1}\|E(w,t)\|. \tag{2.9}$$

If we let now $u = e^{-t}w$, where $\|w\| = 1$, then $\|u\| = e^{-t}$ and so

$$E(w,t) = -\frac{2}{\beta}\|u\|^{\beta}[Df(u) - \alpha I]^{-1}(cDf(u) + \alpha I)$$

$$-\frac{2}{\beta}(1 - \|u\|^{\beta})(Df(u) - \alpha I)^{-1}D^2 f(u)(u,\cdot) - \frac{2}{\beta}\left(1 - \frac{\beta}{2}\right)I.$$

By using (2.2), (2.8) and the previous equality we obtain

$$\|E(z,t)\| < 1, \quad t > 0.$$

Hence for $t > 0$, $I - E(z,t)$ is an invertible operator, too.

Further computations show that:

$$\frac{\partial L}{\partial t}(z,t) = \frac{1}{1+c}e^{(\beta-1)t}\frac{\beta}{2}[Df(e^{-t} - z) - \alpha I]\Big[I + \left(1 - \frac{2}{\beta}\right)I$$

$$-\frac{2}{\beta}e^{-\beta t}(Df(e^{-t}z) - \alpha I)^{-1}(cDf(e^{-t}z) + \alpha I)$$

$$-\frac{2}{\beta}(1 - e^{-\beta t})[Df(e^{-t}z) - \alpha I]^{-1}D^2 f(e^{-t}z)(e^{-t}z,\cdot)\Big](z)$$

and

$$\frac{\partial L}{\partial z}(z,t) = \frac{1}{c}e^{(\beta-1)t}\frac{\beta}{2}[Df(e^{-t}z) - \alpha I][I + E(z,t)](z). \tag{2.10}$$

In conclusion, by using (2.7) and (2.10) we obtain

$$\frac{\partial L}{\partial t}(z,t) = DL(z,t)[I - E(z,t)]^{-1}[I + E(z,t)](z), \quad z \in B.$$

Hence $L(z,t)$ satisfies the differential equation (1.1) for all $z \in B$ and $t \geq 0$ where

$$h(z,t) = [I - E(z,t)]^{-1}[I + E(z,t)](z), \quad z \in B. \tag{2.11}$$

It remains to show that the function defined by (2.11) satisfies the conditions of Theorem 1.2. Clearly $h(z,t)$ satisfies the holomorphy and measurability requirements and $h(0,t) = 0$.

Furthermore, the inequality:

$$\|g(z,t) - z\| = \|E(z,t)(h(z,t) + z)\| \leq \|E(z,t)\| \cdot \|h(z,t) + z\| < \|h(z,t) + z\|$$

implies that Re $\langle h(z,t), z\rangle > 0$, $\forall\ z \in B \setminus \{0\}$, $t \geq 0$.

Since all the assumptions of Theorem 1.2 are satisfied, it follows that the functions $L(\cdot,t)$ $(t \geq 0)$ are univalent in B.

In particular $f = L(\cdot,0)$ is univalent in B.

Remark 2.2. If $\beta = 2$, $\alpha = 0$ and $c = 0$, then Theorem 2.1 becomes the n-dimensional version of Becker's univalence criterion [17].

If $\beta = 2$, $f = g$, then Theorem 2.1 becomes the n-dimensional version of Ahlfors and Becker's univalence criterion [3].

If $c = 0$ then Theorem 2.1 becomes Theorem 2 [4].
If $\alpha = 0$ and $c = 0$ then Theorem 2.1 becomes Theorem 2 [5].
If $\beta = 2$ then Theorem 2.1 becomes Theorem 2.1 [18].

3. **Quasiconformal extensions**

In this section we present a sufficient condition for a normalized holomorphic mapping on B to be extended to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

Theorem 3.1. *Let $f \in \mathcal{H}(B)$ be a normalized mapping (i.e. $f(0) = 0$, $Df(0) = I$) such that the mapping $f(z) - \alpha z$, $z \in B$, is quasiregular. Also let $\beta \geq 2$, and α, c be complex numbers such that*

$$c \neq -1, \quad \alpha \neq 1 \quad \text{and} \quad \left| \frac{c+\alpha}{1-\alpha} \right| \leq 1.$$

If there is $q \in [0,1)$ such that $1 - \frac{2}{\beta} \leq q < \frac{2}{\beta}$,

$$\frac{2}{\beta} \left\| (Df(z) - \alpha I)^{-1}(cDf(z) + \alpha I) - \left(\frac{\beta}{2} - 1 \right) I \right\| \leq q < 1, \; z \in B \tag{3.1}$$

and

$$\frac{2}{q} \Big\| \|z\|^{\beta} (Df(z) - \alpha I)^{-1}(cDf(z) + \alpha I) \tag{3.2}$$

$$+ (1 - \|z\|^{\beta})(Df(z) - \alpha I)^{-1} D^2 f(z)(z, \cdot) + \left(1 - \frac{\beta}{2} \right) I \Big\| \leq q < 1, \quad z \in B$$

then f extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

Proof. The conditions (3.1) and (3.2) enable us to embed f as the initial element $f(z) = L(z, 0)$ of the subordination chain defined by (2.3). In Theorem 2.1 we proved that L (defined by (2.3)) is a subordination chain which satisfies the generalized Loewner equation (1.1) where the mapping h is defined by (2.11) and the mapping $E : B \times [0, \infty) \to \mathcal{L}(\mathbb{C}^n)$ is defined by (2.6).

Next, we will show that $\|E(z,t)\| \leq q$ for all $(z,t) \in B \times [0, \infty)$. We have

$$\|E(z,0)\| = \frac{2}{\beta} \left\| (Df(z) - \alpha I)^{-1}(cDf(z) + \alpha I) - \left(\frac{\beta}{2} - 1 \right) I \right\| \leq q < 1,$$

$z \in B$, according to condition (3.1). Next, let $t \in (0, \infty)$.

In view of the maximum principle for holomorphic mappings into complex Banach spaces, by using the condition (3.2), we obtain:

$$\|E(z,t)\| \leq \max_{\|w\|=1} \|E(z,t)\|$$

$$= \frac{2}{\beta} \max_{\|w\|=1} \Big\| \|we^{-t}\|^{\beta} [Df(we^{-t}) - \alpha I]^{-1} [cDf(we^{-t}) + \alpha I]$$

$$+(1-\|we^{-t}\|)^{\beta}[Df(we^{-t})-\alpha I]^{-1}[D^2f(we^{-t})(we^{-t},\cdot)+I\left(1-\frac{\beta}{2}\right)\Big\|\leq q<1,$$

$z\in B$.

Therefore $\|E(z,t)\|\leq q<1$, $z\in B$, $t\in[0,\infty)$.

From now on, for the simplicity of the notations, we will denote by g the function defined by $g(z)=f(z)-\alpha z$, $z\in B$. By taking into account the conditions (3.1) and (3.2) from the hypothesis, we deduce that

$$\begin{aligned}
&(1-\|z\|^{\beta})\|[Dg(z)]^{-1}D^2g(z)(z,\cdot)\|\\
&=(1-\|z\|^{\beta})\|[Df(z)-\alpha I]^{-1}D^2f(z)(z,\cdot)\|\\
&\leq q\frac{\beta}{2}+\Big\|\|z\|^{\beta}(Df(z)-\alpha I)^{-1}(cDf(z)-\alpha I)+\left(1-\frac{\beta}{2}\right)I\Big\|\\
&\leq q\frac{\beta}{2}\Big\|\|z\|^{\beta}\left\{(Df(z)-\alpha I)^{-1}(cDf(z)-\alpha I)-\left(\frac{\beta}{2}-1\right)I\right\}\\
&\quad+\left(1-\frac{\beta}{2}\right)(1-\|z\|^{\beta})I\Big\|\\
&\leq q\frac{\beta}{2}+\|z\|^{\beta}\cdot\frac{\beta}{2}\cdot q+\left(\frac{\beta}{2}-1\right)(1-\|z\|^{\beta})\\
&=\|z\|^{\beta}\left(q\frac{\beta}{2}-\frac{\beta}{2}+1\right)+q\frac{\beta}{2}+\frac{\beta}{2}-1\\
&\leq\max_{x\in[0,1]}\left\{x\left(q\frac{\beta}{2}-\frac{\beta}{2}+1\right)+q\frac{\beta}{2}+\frac{\beta}{2}-1\right\}\\
&=\max\left\{q\frac{\beta}{q}+\frac{\beta}{2}-1,q\beta\right\}=q\beta=2\gamma
\end{aligned}$$

where $\gamma=\frac{q\beta}{2}<1$.

Since $\beta\geq 2$, we deduce from the above relation that:

$$(1-\|z\|^2)\|[Dg(z)]^{-1}D^2g(z)(z,\cdot)\|\leq 2\gamma,\quad z\in U. \tag{3.3}$$

From the previous inequality, by using a similar argument with that used in the proof of Theorem 2.1 [17] we obtain that there exists $M>0$ such that

$$|\det Dg(z)|\leq\frac{M}{(1-\|z\|)^{n\gamma}},\quad z\in B \tag{3.4}$$

and hence

$$\|Dg(z)\|\leq\frac{L}{(1-\|z\|)^{\gamma}}\quad\text{where}\quad L=\sqrt[n]{MK}. \tag{3.5}$$

We prove now that the mappings $L(\cdot,t)$ are quasiregular. Since g is a quasiregular holomorphic mapping and the following inequality holds

$$1-q\leq\|I-E(z,t)\|\leq 1+q,\quad z\in B,\ t\geq 0,$$

by using (2.7) we easily obtain

$$\|DL(z,t)\| \leq \frac{\beta}{2}e^{(\beta-1)t}\frac{1}{|1+c|}\|Dg(ze^{-t})\|\|I-E(z,t)\| \tag{3.6}$$

$$\leq |a(t)| \cdot \frac{1+q}{1-q} \cdot \frac{1}{|1-\alpha|} \cdot \frac{L}{(1-\|z\|)^{\gamma}} = \frac{|a(t)|L^*}{(1-\|z\|)^{\gamma}}.$$

On the other hand, we have

$$\|DL(z,t)\|^n \leq \left(\frac{\beta}{2}\right)^n e^{n(\beta-1)t}\frac{1}{|1+c|^n}\|Dg(ze^{-t})\|^n(1+q)^n \tag{3.7}$$

$$\leq \left(\frac{\beta}{2}\right)^n e^{n(\beta-1)t}\frac{1}{|1+c|^n}|\det Dg(ze^{-t})|(1+q^n)$$

$$\leq \left(\frac{1+q}{1-q}\right)^n K|\det DL(z,t)|, \quad z \in B,\ t \geq 0.$$

By using Remark 2.2 from [9] we have that the function $h(z,t)$ satisfies the conditions (iv) and (v) of Theorem 1.3.

Since all the conditions of Theorem 1.3 are satisfied, it results that the function f admits a quasiconformal extension of $\mathbb{R}^{2n}$ onto itself.

Remark 3.2. If $\beta = 2$, $c = 0$ and $\alpha = 0$ in Theorem 3.1, we obtain the n-dimensional version of the quasiconformal extension result due to Becker [17].

If $\beta = 2$ and $\alpha = 0$ in Theorem 3.1 we obtain the n-dimensional version of the quasiconformal extension result due to Ahlfors and Becker [3].

Acknowledgement. This work is supported by UEFISCSU - CNCSIS, Grant PN II-ID-524/2007.

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