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ON A CLASS OF ANALYTIC AND MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY AL-OBOUDI DIFFERENTIAL OPERATOR

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Abstract. In this paper, we introduce a new class of analytic functions defined by Al-Oboudi differential operator. For the functions belonging to this class, we obtain coefficient inequalities, Hadamard product, radii of close-to convexity, starlikeness and convexity, extreme points, the integral means inequalities for the fractional derivatives, and further we give distortion theorems using fractional calculus techniques.

1. Introduction

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$.

For $f \in \mathcal{A}$, Al-Oboudi [1] introduced the following operator:

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D^1 f(z) = (1-\delta) f(z) + \delta z f'(z) = D_\delta f(z), \quad \delta \geq 0 \tag{1.3}$$

$$D^n f(z) = D_\delta (D^{n-1} f(z)), \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots\}). \tag{1.4}$$

If f is given by (1.1), then from (1.3) and (1.4) we see that

$$D^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\delta]^n a_k z^k, \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}), \tag{1.5}$$

with $D^n f(0) = 0$.

When $\delta = 1$, we get Sălăgean's differential operator [4].

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Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad (p = 1, 2, \ldots) \tag{1.6}$$

which are analytic and p-valent in the open unit disk $\mathbb{U}$.

We can write the following equalities for the functions $f \in \mathcal{A}_p$:

$$D^0_{\delta,p} f(z) = f(z), \tag{1.7}$$

$$D^1_{\delta,p} f(z) = (1-\delta) f(z) + \frac{\delta}{p} z f'(z) = D_{\delta,p} f(z), \quad \delta \geq 0 \tag{1.8}$$

$$D^n_{\delta,p} f(z) = D_{\delta,p}(D^{n-1}_{\delta,p} f(z)), \quad (n \in \mathbb{N}). \tag{1.9}$$

If f is given by (1.6), then from (1.8) and (1.9) we see that

$$D^n_{\delta,p} f(z) = z^p + \sum_{k=p+1}^{\infty} \left[1 + \left(\frac{k}{p} - 1\right)\delta\right]^n a_k z^k, \quad (n \in \mathbb{N}_0). \tag{1.10}$$

Let $\mathcal{T}_p$ denote the subclass of $\mathcal{A}_p$ consisting of functions of the form

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k \qquad (a_k \geq 0). \tag{1.11}$$

If f is given by (1.11), then from (1.8) and (1.9) we see that

$$D^n_{\delta,p} f(z) = z^p - \sum_{k=p+1}^{\infty} \left[1 + \left(\frac{k}{p} - 1\right)\delta\right]^n a_k z^k, \quad (n \in \mathbb{N}_0). \tag{1.12}$$

Definition 1.1. A function $f \in \mathcal{T}_p$ is in $\mathcal{R}^n_p(\alpha, \beta, \gamma, \mu)$ if and only if

$$\left| \frac{\left(D^n_{\delta,p} f(z)\right)' - p z^{p-1}}{\alpha \left(D^n_{\delta,p} f(z)\right)' + (\beta - \gamma)} \right| < \mu, \quad (z \in \mathbb{U},\ n \in \mathbb{N}_0), \tag{1.13}$$

for $0 \leq \alpha < 1$, $0 \leq \gamma < 1$, $0 < \beta \leq 1$, $0 < \mu < 1$. Here $D^n_{\delta,p} f(z)$ is defined as in (1.12).

In this paper, basic properties of the class $\mathcal{R}^n_p(\alpha, \beta, \gamma, \mu)$ are studied, such as coefficient bounds, Hadamard product, radii of close-to convexity, starlikeness and convexity, extreme points, the integral means inequalities for the fractional derivatives, and further distortion theorems are given using fractional calculus techniques.

2. **Coefficient inequalities**

Theorem 2.1. *A function $f \in \mathcal{T}_p$ is in the class $\mathcal{R}_p^n(\alpha, \beta, \gamma, \mu)$ if and only if*

$$\sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n (1+\mu\alpha)a_k \leq \mu(\alpha p+\beta-\gamma). \tag{2.1}$$

The result is sharp for the function f given by

$$f(z) = z^p - \frac{\mu(\alpha p+\beta-\gamma)}{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n (1+\mu\alpha)} z^k \quad (k \geq p+1).$$

Proof. Suppose that $f \in \mathcal{R}_p^n(\alpha, \beta, \gamma, \mu)$. Then we have from (1.13)

$$\left|\frac{\sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n a_k z^{k-1}}{\alpha\left(pz^{p-1}-\sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n a_k z^{k-1}\right)+(\beta-\gamma)}\right| < \mu.$$

So, we obtain

$$\Re\left\{\frac{\sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n a_k z^{k-1}}{\alpha\left(pz^{p-1}-\sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n a_k z^{k-1}\right)+(\beta-\gamma)}\right\} < \mu.$$

If we choose z real and let $z \to 1^-$, then we get

$$\sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n (1+\mu\alpha)a_k \leq \mu(\alpha p+\beta-\gamma).$$

Conversely, suppose that the inequality (2.1) holds true and let

$$z \in \partial\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}.$$

Then we find from (1.13) that

$$\begin{aligned}
& \left|\left(D^n_{\delta,p}f(z)\right)' - pz^{p-1}\right| - \mu\left|\alpha\left(D^n_{\delta,p}f(z)\right)' + (\beta-\gamma)\right| \\
= & \left|\sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n a_k z^{k-1}\right| \\
& -\mu\left|\alpha\left(pz^{p-1} - \sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n a_k z^{k-1}\right) + (\beta-\gamma)\right| \\
\leq & \sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n a_k |z|^{k-1} - \mu(\alpha p + \beta - \gamma) \\
& +\mu\alpha \sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n a_k |z|^{k-1} \\
= & \sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n (1+\mu\alpha)a_k - \mu(\alpha p + \beta - \gamma) \leq 0.
\end{aligned}$$

Hence, by the maximum modulus theorem, we have $f \in \mathcal{R}^n_p(\alpha,\beta,\gamma,\mu)$. □

Corollary 2.2. *If $f \in \mathcal{R}^n_p(\alpha,\beta,\gamma,\mu)$, then*

$$a_{p+1} \leq \frac{\mu(\alpha p + \beta - \gamma)p^n}{(p+1)(p+\delta)^n(1+\mu\alpha)}.$$

Theorem 2.3. *Let the functions*

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k \qquad (a_k \geq 0), \tag{2.2}$$

$$g(z) = z^p - \sum_{k=p+1}^{\infty} b_k z^k \qquad (b_k \geq 0) \tag{2.3}$$

be in the class $\mathcal{R}^n_p(\alpha,\beta,\gamma,\mu)$. Then for $0 \leq \lambda \leq 1$, the function h defined by

$$h(z) = (1-\lambda)f(z) + \lambda g(z) = z^p - \sum_{k=p+1}^{\infty} c_k z^k,$$

$$c_k := (1-\lambda)a_k + \lambda b_k \geq 0$$

is also in the class $\mathcal{R}^n_p(\alpha,\beta,\gamma,\mu)$.

Proof. Suppose that each of the functions f and g is in the class $\mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$. Then, making use of (2.1), we see that

$$\begin{aligned}\sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)c_k &= (1-\lambda)\sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)a_k\\ &\quad+\lambda\sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)b_k\\ &\le (1-\lambda)\mu(\alpha p+\beta-\gamma)+\lambda\mu(\alpha p+\beta-\gamma)\\ &= \mu(\alpha p+\beta-\gamma)\end{aligned}$$

which completes the proof of Theorem 2.3. □

3. **Hadamard product**

Next we define the modified Hadamard product of functions f and g, which are defined by (2.2) and (2.3), respectively, by

$$f * g(z) = z^p - \sum_{k=p+1}^{\infty} a_k b_k z^k \qquad (a_k \ge 0, b_k \ge 0).$$

Theorem 3.1. *If each of the functions f and g is in the class $\mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$, then*

$$f * g(z) \in \mathcal{R}_p^n(\alpha,\beta,\gamma,\eta),$$

where

$$\eta \ge \frac{\mu^2(\alpha p+\beta-\gamma)}{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)^2-\mu^2\alpha(\alpha p+\beta-\gamma)}.$$

Proof. From Theorem 2.1, we have

$$\sum_{k=p+1}^{\infty}\frac{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)}{\mu(\alpha p+\beta-\gamma)}a_k \le 1 \tag{3.1}$$

and

$$\sum_{k=p+1}^{\infty}\frac{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)}{\mu(\alpha p+\beta-\gamma)}b_k \le 1. \tag{3.2}$$

We need to find the smallest η such that

$$\sum_{k=p+1}^{\infty}\frac{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\eta\alpha)}{\eta(\alpha p+\beta-\gamma)}a_k b_k \le 1. \tag{3.3}$$

From (3.1) and (3.2) we find, by means of the Cauchy-Schwarz inequality, that

$$\sum_{k=p+1}^{\infty} \frac{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n (1+\mu\alpha)}{\mu(\alpha p+\beta-\gamma)} \sqrt{a_k b_k} \leq 1. \tag{3.4}$$

Thus it is enough to show that

$$\frac{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n (1+\eta\alpha)}{\eta(\alpha p+\beta-\gamma)} a_k b_k \leq \frac{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n (1+\mu\alpha)}{\mu(\alpha p+\beta-\gamma)} \sqrt{a_k b_k},$$

that is

$$\sqrt{a_k b_k} \leq \frac{\eta(1+\mu\alpha)}{\mu(1+\eta\alpha)}. \tag{3.5}$$

On the other hand, from (3.4) we have

$$\sqrt{a_k b_k} \leq \frac{\mu(\alpha p+\beta-\gamma)}{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n (1+\mu\alpha)}. \tag{3.6}$$

Therefore in view of (3.5) and (3.6) it is enough to show that

$$\frac{\mu(\alpha p+\beta-\gamma)}{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n (1+\mu\alpha)} \leq \frac{\eta(1+\mu\alpha)}{\mu(1+\eta\alpha)}$$

which simplifies to

$$\eta \geq \frac{\mu^2(\alpha p+\beta-\gamma)}{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n (1+\mu\alpha)^2 - \mu^2\alpha(\alpha p+\beta-\gamma)}.$$

□

4. **Close-to convexity, starlikeness and convexity**

A function $f \in \mathcal{T}_p$ is said to be p-valently close-to convex of order ρ if it satisfies

$$\Re\{f'(z)\} > \rho$$

for some ρ $(0 \leq \rho < p)$ and for all $z \in \mathbb{U}$.

Also a function $f \in \mathcal{T}_p$ is said to be p-valently starlike of order ρ if it satisfies

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \rho$$

for some ρ $(0 \leq \rho < p)$ and for all $z \in \mathbb{U}$.

Further a function $f \in \mathcal{T}_p$ is said to be p-valently convex of order ρ if

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \rho$$

for some ρ $(0 \leq \rho < p)$ and for all $z \in \mathbb{U}$.

Theorem 4.1. *If $f \in \mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$, then f is p-valently close-to convex of order ρ in $|z| < r_1(\alpha,\beta,\gamma,\mu,\rho)$, where*

$$r_1(\alpha,\beta,\gamma,\mu,\rho) = \inf_k \left\{ \frac{\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n (1+\mu\alpha)(p-\rho)}{\mu(\alpha p+\beta-\gamma)} \right\}^{\frac{1}{k-p}}, \quad k \geq p+1.$$

Proof. It is sufficient to show that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| < p-\rho.$$

We have

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq \sum_{k=p+1}^{\infty} k a_k |z|^{k-p} < p-\rho \tag{4.1}$$

and

$$\sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n (1+\mu\alpha) a_k \leq \mu(\alpha p+\beta-\gamma). \tag{4.2}$$

Hence (4.1) is true if

$$\frac{k|z|^{k-p}}{p-\rho} \leq \frac{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n (1+\mu\alpha)}{\mu(\alpha p+\beta-\gamma)}. \tag{4.3}$$

Solving (4.3) for $|z|$, we obtain

$$|z| \leq \left\{ \frac{\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n (1+\mu\alpha)(p-\rho)}{\mu(\alpha p+\beta-\gamma)} \right\}^{\frac{1}{k-p}}.$$

□

Theorem 4.2. *If $f \in \mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$, then f is p-valently starlike of order ρ in $|z| < r_2(\alpha,\beta,\gamma,\mu,\rho)$, where*

$$r_2(\alpha,\beta,\gamma,\mu,\rho) = \inf_k \left\{ \frac{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n (1+\mu\alpha)(p-\rho)}{\mu(\alpha p+\beta-\gamma)(k-\rho)} \right\}^{\frac{1}{k-p}}, \quad k \geq p+1.$$

Proof. We need to show that

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p-\rho.$$

The inequality

$$\left|\frac{zf'(z)}{f(z)} - p\right| = \left|\frac{-\sum_{k=p+1}^{\infty}(k-p)a_k z^{k-p}}{1-\sum_{k=p+1}^{\infty} a_k z^{k-p}}\right| \leq \frac{\sum_{k=p+1}^{\infty}(k-p)a_k |z|^{k-p}}{1-\sum_{k=p+1}^{\infty} a_k |z|^{k-p}} < p-\rho$$

holds true if

$$\frac{(k-\rho)\,|z|^{k-p}}{p-\rho} \leq \frac{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^{n}(1+\mu\alpha)}{\mu(\alpha p+\beta-\gamma)}.$$

Then f is starlike of order ρ. □

Theorem 4.3. *If $f \in \mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$, then f is p-valently convex of order ρ in $|z| < r_3(\alpha,\beta,\gamma,\mu,\rho)$, where*

$$r_3(\alpha,\beta,\gamma,\mu,\rho) = \inf_k \left\{ \frac{\left[1+\left(\frac{k}{p}-1\right)\delta\right]^{n}(1+\mu\alpha)p(p-\rho)}{\mu(\alpha p+\beta-\gamma)(k-\rho)} \right\}^{\frac{1}{k-p}}, \quad k \geq p+1.$$

Proof. We must show that

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right| < p-\rho.$$

Since

$$\begin{aligned} \left|1+\frac{zf''(z)}{f'(z)}-p\right| &= \left|\frac{-\sum_{k=p+1}^{\infty} k(k-p)a_k z^{k-p}}{p-\sum_{k=p+1}^{\infty} k a_k z^{k-p}}\right| \\ &\leq \frac{\sum_{k=p+1}^{\infty} k(k-p)a_k\,|z|^{k-p}}{p-\sum_{k=p+1}^{\infty} k a_k\,|z|^{k-p}} < p-\rho, \end{aligned}$$

if

$$\frac{k(k-\rho)\,|z|^{k-p}}{p(p-\rho)} \leq \frac{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^{n}(1+\mu\alpha)}{\mu(\alpha p+\beta-\gamma)}$$

then f is convex of order ρ. □

5. **Extreme points**

Theorem 5.1. *Let $f_p(z) = z^p$ and*

$$f_k(z) = z^p - \frac{\mu(\alpha p+\beta-\gamma)}{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^{n}(1+\mu\alpha)} z^k \quad (k \geq p+1).$$

Then $f \in \mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$ if and only if it can be expresses in the form

$$f(z) = \lambda_p f_p(z) + \sum_{k=p+1}^{\infty} \lambda_k f_k(z),$$

where $\lambda_k \geq 0$ and $\lambda_p = 1-\sum_{k=p+1}^{\infty}\lambda_k$.

Proof. Assume that

$$f(z)=\lambda_p f_p(z)+\sum_{k=p+1}^{\infty}\lambda_k f_k(z).$$

Then

$$\begin{aligned} f(z) &= \left(1-\sum_{k=p+1}^{\infty}\lambda_k\right)z^p+\sum_{k=p+1}^{\infty}\lambda_k\left(z^p-\frac{\mu(\alpha p+\beta-\gamma)}{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)}z^k\right) \\ &= z^p-\sum_{k=p+1}^{\infty}\lambda_k\frac{\mu(\alpha p+\beta-\gamma)}{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)}z^k. \end{aligned}$$

Thus

$$\begin{aligned} &\sum_{k=p+1}^{\infty}\left[k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)\right]\lambda_k\frac{\mu(\alpha p+\beta-\gamma)}{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)} \\ &= \mu(\alpha p+\beta-\gamma)\sum_{k=p+1}^{\infty}\lambda_k \\ &= \mu(\alpha p+\beta-\gamma)(1-\lambda_p) \\ &\leq \mu(\alpha p+\beta-\gamma). \end{aligned}$$

Therefore, we have $f\in\mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$.

Conversely, suppose that $f\in\mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$. Since

$$a_k\leq\frac{\mu(\alpha p+\beta-\gamma)}{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)}\quad(k\geq p+1),$$

we can set

$$\lambda_k=\frac{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)}{\mu(\alpha p+\beta-\gamma)}a_k\quad(k\geq p+1),$$

$$\lambda_p=1-\sum_{k=p+1}^{\infty}\lambda_k.$$

Then

$$\begin{aligned} f(z) &= z^p-\sum_{k=p+1}^{\infty}a_k z^k \\ &= \lambda_p z^p+\sum_{k=p+1}^{\infty}\lambda_k\left(z^p-\frac{\mu(\alpha p+\beta-\gamma)}{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)}z^k\right) \\ &= \lambda_p f_p(z)+\sum_{k=p+1}^{\infty}\lambda_k f_k(z). \end{aligned}$$

This completes the proof of Theorem 5.1. □

Corollary 5.2. *The extreme points of* $\mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$ *are given by*

$$f_p(z)=z^p, \qquad f_k(z)=z^p-\frac{\mu(\alpha p+\beta-\gamma)}{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)}z^k \quad (k\geq p+1).$$

6. **The main integral means inequalities for the fractional derivative**

We discuss the integral means inequalities for functions $f\in\mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$.

The following definitions of fractional derivatives by Owa [3] (also by Srivastava and Owa [5]) will be required in our investigation.

Definition 6.1. The fractional integral of order λ is defined by

$$D_z^{-\lambda}f(z)=\frac{1}{\Gamma(\lambda)}\int_0^z\frac{f(t)}{(z-t)^{1-\lambda}}dt \quad (\lambda>0),$$

where the function f is analytic in a simply connected region of the complex z-plane containing the origin and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t)>0$.

Definition 6.2. The fractional derivative of order λ is defined, for a function f, by

$$D_z^{\lambda}f(z)=\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z\frac{f(t)}{(z-t)^{\lambda}}dt \quad (0\leq\lambda<1), \tag{6.1}$$

where the function f is analytic in a simply connected region of the complex z-plane containing the origin and the multiplicity of $(z-t)^{-\lambda}$ is removed by requiring $\log(z-t)$ to be real when $(z-t)>0$.

Definition 6.3. Under the hypothesis of Definition 6.2, the fractional derivative of order $p+\lambda$ is defined, for a function f, by

$$D_z^{p+\lambda}f(z)=\frac{d^p}{dz^p}D_z^{\lambda}f(z), \tag{6.2}$$

where $0\leq\lambda<1$ and $p\in\mathbb{N}_0$.

It readily follows from Definitions 6.1 and 6.2 that

$$D_z^{-\lambda}z^k=\frac{\Gamma(k+1)}{\Gamma(k+\lambda+1)}z^{k+\lambda} \quad (\lambda>0,\ k\in\mathbb{N}) \tag{6.3}$$

and

$$D_z^{\lambda}z^k=\frac{\Gamma(k+1)}{\Gamma(k-\lambda+1)}z^{k-\lambda} \quad (0\leq\lambda<1,\ k\in\mathbb{N}), \tag{6.4}$$

respectively.

We will also need the concept of subordination between analytic functions and a subordination theorem of Littlewood [2] in our investigation.

Definition 6.4. Given two functions f and g, which are analytic in $\mathbb{U}$, the function f is said to be subordinate to g in $\mathbb{U}$ if there exists a function w analytic in $\mathbb{U}$ with

$$w(0) = 0, \qquad |w(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

We denote this subordination by

$$f(z) \prec g(z).$$

Lemma 6.5. *If the functions f and g are analytic in $\mathbb{U}$ with*

$$f(z) \prec g(z),$$

then, for $\sigma > 0$ and $z = re^{i\theta}$ $\quad (0 < r < 1)$,

$$\int_0^{2\pi} |f(z)|^\sigma \, d\theta \leq \int_0^{2\pi} |g(z)|^\sigma \, d\theta.$$

Theorem 6.6. *Let $f \in \mathcal{R}_p^n(\alpha, \beta, \gamma, \mu)$ and suppose that*

$$\sum_{j=p+1}^{\infty} (j-q)_{q+1} a_j \leq \frac{\mu(\alpha p + \beta - \gamma)\Gamma(k+1)\Gamma(2+p-\lambda-q)}{k\left[1 + \left(\frac{k}{p} - 1\right)\delta\right]^n (1+\mu\alpha)\Gamma(k+1-\lambda-q)\Gamma(p+1-q)} \tag{6.5}$$

for $0 \leq q \leq j$, $0 \leq \lambda < 1$, where $(j-q)_{q+1}$ denotes the Pochhammer symbol defined by

$$(j-q)_{q+1} = (j-q)(j-q+1)\cdots j. \tag{6.6}$$

Also let the function f_k be defined by

$$f_k(z) = z^p - \frac{\mu(\alpha p + \beta - \gamma)}{k\left[1 + \left(\frac{k}{p} - 1\right)\delta\right]^n (1+\mu\alpha)} z^k \quad (k \geq p+1). \tag{6.7}$$

If there exists an analytic function w defined by

$$(w(z))^{k-p} = \frac{k\left[1 + \left(\frac{k}{p} - 1\right)\delta\right]^n (1+\mu\alpha)}{\mu(\alpha p + \beta - \gamma)} \frac{\Gamma(k+1-\lambda-q)}{\Gamma(k+1)} \sum_{j=p+1}^{\infty} (j-q)_{q+1} \Psi(j) a_j z^{j-p} \tag{6.8}$$

$(k \geq q)$, with

$$\Psi(j) = \frac{\Gamma(j-q)}{\Gamma(j+1-\lambda-q)}, \quad (0 \leq \lambda < 1, \ j \geq p+1), \tag{6.9}$$

then, for $\sigma > 0$ and $z = re^{i\theta}$ $\quad (0 < r < 1)$,

$$\int_0^{2\pi} \left|D_z^{q+\lambda} f(z)\right|^\sigma d\theta \leq \int_0^{2\pi} \left|D_z^{q+\lambda} f_k(z)\right|^\sigma d\theta, \quad (0 \leq \lambda < 1). \tag{6.10}$$

Proof. Let $f(z) = z^p - \sum_{j=p+1}^{\infty} a_j z^j$. By means of (6.4) and Definition 6.3, we have

$$\begin{aligned} D_z^{q+\lambda} f(z) &= \frac{\Gamma(p+1) z^{p-\lambda-q}}{\Gamma(p+1-\lambda-q)} \left[1 - \sum_{j=p+1}^{\infty} \frac{\Gamma(j+1)\Gamma(p+1-\lambda-q)}{\Gamma(p+1)\Gamma(j+1-\lambda-q)} a_j z^{j-p} \right] \\ &= \frac{\Gamma(p+1) z^{p-\lambda-q}}{\Gamma(p+1-\lambda-q)} \left[1 - \sum_{j=p+1}^{\infty} \frac{\Gamma(p+1-\lambda-q)}{\Gamma(p+1)} (j-q)_{q+1} \Psi(j) a_j z^{j-p} \right], \end{aligned}$$

where

$$\Psi(j) = \frac{\Gamma(j-q)}{\Gamma(j+1-\lambda-q)}, \quad (0 \le \lambda < 1,\ j \ge p+1).$$

Since Ψ is a decreasing function of j, we get

$$0 < \Psi(j) \le \Psi(p+1) = \frac{\Gamma(p+1-q)}{\Gamma(2+p-\lambda-q)}.$$

Similarly, from (6.7), (6.4), and Definition 6.3, we have

$$D_z^{q+\lambda} f_k(z)$$

$$= \frac{\Gamma(p+1) z^{p-\lambda-q}}{\Gamma(p+1-\lambda-q)} \left[1 - \frac{\mu(\alpha p + \beta - \gamma)\Gamma(k+1)\Gamma(p+1-\lambda-q)}{k\left[1 + \left(\frac{k}{p} - 1\right)\delta\right]^n (1+\mu\alpha)\Gamma(p+1)\Gamma(k+1-\lambda-q)} z^{k-p} \right].$$

For $\sigma > 0$ and $z = re^{i\theta} \quad (0 < r < 1)$, we must show that

$$\begin{aligned} &\int_0^{2\pi} \left| 1 - \sum_{j=p+1}^{\infty} \frac{\Gamma(p+1-\lambda-q)}{\Gamma(p+1)} (j-q)_{q+1} \Psi(j) a_j z^{j-p} \right|^{\sigma} d\theta \\ \le\ &\int_0^{2\pi} \left| 1 - \frac{\mu(\alpha p + \beta - \gamma)}{k\left[1 + \left(\frac{k}{p} - 1\right)\delta\right]^n (1+\mu\alpha)} \frac{\Gamma(k+1)\Gamma(p+1-\lambda-q)}{\Gamma(p+1)\Gamma(k+1-\lambda-q)} z^{k-p} \right|^{\sigma} d\theta. \end{aligned}$$

So, by applying Lemma 6.5, it is enough to show that

$$\begin{aligned} &1 - \sum_{j=p+1}^{\infty} \frac{\Gamma(p+1-\lambda-q)}{\Gamma(p+1)} (j-q)_{q+1} \Psi(j) a_j z^{j-p} \\ \prec\ &1 - \frac{\mu(\alpha p + \beta - \gamma)}{k\left[1 + \left(\frac{k}{p} - 1\right)\delta\right]^n (1+\mu\alpha)} \frac{\Gamma(k+1)\Gamma(p+1-\lambda-q)}{\Gamma(p+1)\Gamma(k+1-\lambda-q)} z^{k-p}. \end{aligned}$$

If the above subordination holds true, then we have an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ such that

$$\begin{aligned} & 1-\sum_{j=p+1}^{\infty}\frac{\Gamma(p+1-\lambda-q)}{\Gamma(p+1)}(j-q)_{q+1}\Psi(j)a_j z^{j-p} \\ = \quad & 1-\frac{\mu(\alpha p+\beta-\gamma)}{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)}\frac{\Gamma(k+1)\Gamma(p+1-\lambda-q)}{\Gamma(p+1)\Gamma(k+1-\lambda-q)}\left(w(z)\right)^{k-p}. \end{aligned}$$

By the condition of the theorem, we define the function w by

$$(w(z))^{k-p}=\frac{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)}{\mu(\alpha p+\beta-\gamma)}\frac{\Gamma(k+1-\lambda-q)}{\Gamma(k+1)}\sum_{j=p+1}^{\infty}(j-q)_{q+1}\Psi(j)a_j z^{j-p}$$

which readily yields $w(0) = 0$. For such a function w, we have

$$\begin{aligned} |w(z)|^{k-p} &\le \frac{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)}{\mu(\alpha p+\beta-\gamma)}\frac{\Gamma(k+1-\lambda-q)}{\Gamma(k+1)}\sum_{j=p+1}^{\infty}(j-q)_{q+1}\Psi(j)a_j\,|z|^{j-p} \\ &\le |z|\,\frac{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)}{\mu(\alpha p+\beta-\gamma)}\frac{\Gamma(k+1-\lambda-q)}{\Gamma(k+1)}\Psi(p+1)\sum_{j=p+1}^{\infty}(j-q)_{q+1}a_j \\ &= |z|\,\frac{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)\Gamma(k+1-\lambda-q)\Gamma(p+1-q)}{\mu(\alpha p+\beta-\gamma)\Gamma(k+1)\Gamma(2+p-\lambda-q)}\sum_{j=p+1}^{\infty}(j-q)_{q+1}a_j \\ &\le |z| < 1 \end{aligned}$$

by means of the hypothesis of the theorem.

Thus theorem is proved. □

As a special case $q = 0$, we have following result from Theorem 6.6.

Corollary 6.7. *Let $f \in \mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$ and suppose that*

$$\sum_{j=p+1}^{\infty} j a_j \le \frac{\mu(\alpha p+\beta-\gamma)\Gamma(k+1)\Gamma(2+p-\lambda)}{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)\Gamma(k+1-\lambda)\Gamma(p+1)} \qquad (k \ge p+1).$$

If there exists an analytic function w defined by

$$(w(z))^{k-p}=\frac{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)}{\mu(\alpha p+\beta-\gamma)}\frac{\Gamma(k+1-\lambda)}{\Gamma(k+1)}\sum_{j=p+1}^{\infty} j\Psi(j)a_j z^{j-p},$$

with

$$\Psi(j)=\frac{\Gamma(j)}{\Gamma(j+1-\lambda)}, \quad (0\le\lambda<1,\ j\ge p+1),$$

then, for $\sigma > 0$ *and* $z = re^{i\theta} \quad (0 < r < 1)$,

$$\int_0^{2\pi} \left|D_z^{\lambda} f(z)\right|^{\sigma} d\theta \leq \int_0^{2\pi} \left|D_z^{\lambda} f_k(z)\right|^{\sigma} d\theta, \quad (0 \leq \lambda < 1).$$

Letting $q = 1$ in Theorem 6.6, we have the following.

Corollary 6.8. *Let* $f \in \mathcal{R}_p^n(\alpha, \beta, \gamma, \mu)$ *and suppose that*

$$\sum_{j=p+1}^{\infty} j(j-1)a_j \leq \frac{\mu(\alpha p + \beta - \gamma)\Gamma(k+1)\Gamma(1+p-\lambda)}{k\left[1 + \left(\frac{k}{p} - 1\right)\delta\right]^n (1+\mu\alpha)\Gamma(k-\lambda)\Gamma(p)}.$$

If there exists an analytic function w *defined by*

$$(w(z))^{k-p} = \frac{k\left[1 + \left(\frac{k}{p} - 1\right)\delta\right]^n (1+\mu\alpha)}{\mu(\alpha p + \beta - \gamma)} \frac{\Gamma(k-\lambda)}{\Gamma(k+1)} \sum_{j=p+1}^{\infty} j(j-1)\Psi(j)a_j z^{j-p}$$

with

$$\Psi(j) = \frac{\Gamma(j-1)}{\Gamma(j-\lambda)}, \quad (0 \leq \lambda < 1,\ j \geq p+1),$$

then, for $\sigma > 0$ *and* $z = re^{i\theta} \quad (0 < r < 1)$,

$$\int_0^{2\pi} \left|D_z^{1+\lambda} f(z)\right|^{\sigma} d\theta \leq \int_0^{2\pi} \left|D_z^{1+\lambda} f_k(z)\right|^{\sigma} d\theta, \quad (0 \leq \lambda < 1).$$

7. Distortion theorems involving operators of fractional calculus

Theorem 7.1. *If* $f \in \mathcal{R}_p^n(\alpha, \beta, \gamma, \mu)$, *then we have*

$$\left|D_z^{-\lambda} f(z)\right| \leq \frac{\Gamma(p+1)}{\Gamma(p+\lambda+1)} |z|^{p+\lambda} \left[1 + \frac{\mu(\alpha p + \beta - \gamma)p^n}{(p+\delta)^n(1+\mu\alpha)(p+\lambda+1)} |z|\right] \tag{7.1}$$

and

$$\left|D_z^{-\lambda} f(z)\right| \geq \frac{\Gamma(p+1)}{\Gamma(p+\lambda+1)} |z|^{p+\lambda} \left[1 - \frac{\mu(\alpha p + \beta - \gamma)p^n}{(p+\delta)^n(1+\mu\alpha)(p+\lambda+1)} |z|\right], \tag{7.2}$$

for $\lambda > 0$.

Proof. Suppose that $f \in \mathcal{R}_p^n(\alpha, \beta, \gamma, \mu)$. Using Theorem 2.1, we find that

$$(p+1)\left[1 + \frac{\delta}{p}\right]^n (1+\mu\alpha) \sum_{k=p+1}^{\infty} a_k \leq \mu(\alpha p + \beta - \gamma) \tag{7.3}$$

or

$$\sum_{k=p+1}^{\infty} a_k \leq \frac{\mu(\alpha p + \beta - \gamma)p^n}{(p+1)\,(p+\delta)^n\,(1+\mu\alpha)}. \tag{7.4}$$

From (6.3), we have

$$\begin{aligned}\frac{\Gamma(p+\lambda+1)}{\Gamma(p+1)}z^{-\lambda}D_z^{-\lambda}f(z) &= z^p-\sum_{k=p+1}^{\infty}\frac{\Gamma(k+1)\Gamma(p+\lambda+1)}{\Gamma(p+1)\Gamma(k+\lambda+1)}a_kz^k\\ &= z^p-\sum_{k=p+1}^{\infty}\Psi(k)a_kz^k, \qquad (7.5)\end{aligned}$$

where

$$\Psi(k)=\frac{\Gamma(k+1)\Gamma(p+\lambda+1)}{\Gamma(p+1)\Gamma(k+\lambda+1)}. \qquad (7.6)$$

Clearly, Ψ is a decreasing function of k and we get

$$0<\Psi(k)\le\Psi(p+1)=\frac{p+1}{p+\lambda+1}.$$

Using (7.4) – (7.6), we obtain

$$\begin{aligned}\left|\frac{\Gamma(p+\lambda+1)}{\Gamma(p+1)})z^{-\lambda}D_z^{-\lambda}f(z)\right| &\le |z|^p+\Psi(p+1)\,|z|^{p+1}\sum_{k=p+1}^{\infty}a_k\\ &\le |z|^p+\frac{\mu(\alpha p+\beta-\gamma)p^n}{(p+1)\,(p+\delta)^n\,(1+\mu\alpha)}\frac{p+1}{p+\lambda+1}\,|z|^{p+1}\end{aligned}$$

which is equivalent to (7.1) and

$$\begin{aligned}\left|\frac{\Gamma(p+\lambda+1)}{\Gamma(p+1)}z^{-\lambda}D_z^{-\lambda}f(z)\right| &\ge |z|^p-\Psi(p+1)\,|z|^{p+1}\sum_{k=p+1}^{\infty}a_k\\ &\ge |z|^p-\frac{\mu(\alpha p+\beta-\gamma)p^n}{(p+1)\,(p+\delta)^n\,(1+\mu\alpha)}\frac{p+1}{p+\lambda+1}\,|z|^{p+1}\end{aligned}$$

which is precisely the assertion (7.2). □

The proof of Theorem 7.2 below is similar to that of Theorem 7.1, which we have detailed above fairly fully. Indeed, instead of (6.3), we make use of (6.4) to prove Theorem 7.2.

Theorem 7.2. *If $f\in\mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$, then we have*

$$\left|D_z^{\lambda}f(z)\right|\le\frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)}\,|z|^{p-\lambda}\left[1+\frac{\mu(\alpha p+\beta-\gamma)p^n}{(p+\delta)^n(1+\mu\alpha)(p-\lambda+1)}\,|z|\right] \qquad (7.7)$$

and

$$\left|D_z^{\lambda}f(z)\right|\ge\frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)}\,|z|^{p-\lambda}\left[1-\frac{\mu(\alpha p+\beta-\gamma)p^n}{(p+\delta)^n(1+\mu\alpha)(p-\lambda+1)}\,|z|\right]. \qquad (7.8)$$

Corollary 7.3. *If $f\in\mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$, then we have*

$$|z|^p-\frac{\mu(\alpha p+\beta-\gamma)p^n}{(p+\delta)^n(1+\mu\alpha)(p+\lambda+1)}\,|z|^{p+1}\le|f(z)|$$

$$\leq |z|^p + \frac{\mu(\alpha p + \beta - \gamma)p^n}{(p+\delta)^n(1+\mu\alpha)(p+\lambda+1)} |z|^{p+1} . \tag{7.9}$$

Proof. From Definition 6.1, we have

$$\lim_{\lambda\to 0} D_z^{-\lambda} f(z) = f(z).$$

Therefore, letting $\lambda = 0$ in (7.1) and (7.2), we obtain (7.9). □

Corollary 7.4. *If $f \in \mathcal{R}_p^n(\alpha, \beta, \gamma, \mu)$, then we have*

$$p\,|z|^{p-1} - \frac{\mu(\alpha p + \beta - \gamma)p^n}{(p+\delta)^n(1+\mu\alpha)(p-\lambda+1)} |z|^p \leq |f'(z)|$$

$$\leq p\,|z|^{p-1} + \frac{\mu(\alpha p + \beta - \gamma)p^n}{(p+\delta)^n(1+\mu\alpha)(p-\lambda+1)} |z|^p . \tag{7.10}$$

Proof. From Definition 6.2, we have

$$\lim_{\lambda\to 1} D_z^{\lambda} f(z) = f'(z).$$

Therefore, letting $\lambda = 1$ in (7.7) and (7.8), we obtain (7.10). □

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