STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 4, December 2010

A MULTIPLICITY RESULT FOR NONLOCAL PROBLEMS INVOLVING NONLINEARITIES WITH BOUNDED PRIMITIVE

BIAGIO RICCERI

Abstract. The aim of this paper is to provide the first application of Theorem 3 of [2] in a case where the dependence of the underlying equation from the real parameter is not of affine type. The simplest particular case of our result reads as follows:
Let $f : \mathbf{R} \to \mathbf{R}$ be a non-zero continuous function such that

$$\sup_{\xi \in \mathbf{R}} |F(\xi)| < +\infty$$

where $F(\xi) = \int_0^\xi f(s)ds$.
Moreover, let $k : [0, +\infty[\to \mathbf{R}$ and $h :] - \operatorname{osc}_{\mathbf{R}} F, \operatorname{osc}_{\mathbf{R}} F[\to \mathbf{R}$ be two continuous and non-decreasing functions, with $k(t) > 0$ for all $t > 0$ and $h^{-1}(0) = \{0\}$. Then, for each μ large enough, there exist an open interval $A \subseteq]\inf_{\mathbf{R}} F, \sup_{\mathbf{R}} F[$ and a number $\rho > 0$ such that, for every $\lambda \in A$, the problem

$$\begin{cases} -k\left(\int_0^1 |u'(t)|^2 dt\right) u'' = \mu h\left(\int_0^1 F(u(t))dt - \lambda\right) f(u) \text{ in } [0,1] \\ u(0) = u(1) = 0 \end{cases}$$

has at least three solutions whose norms in $H_0^1(0,1)$ are less than ρ.

In [2], we established the following result:

Theorem 1.1. *Let X be a separable and reflexive real Banach space, $I \subseteq \mathbf{R}$ an interval, and $\Psi : X \times I \to \mathbf{R}$ a continuous function satisfying the following conditions:*
(a_1) for each $x \in X$, the function $\Psi(x, \cdot)$ is concave;
(a_2) for each $\lambda \in I$, the function $\Psi(\cdot, \lambda)$ is C^1, sequentially weakly lower semicontinuous, coercive, and satisfies the Palais-Smale condition;
(a_3) there exists a continuous concave function $h : I \to \mathbf{R}$ such that

$$\sup_{\lambda \in I} \inf_{x \in X} (\Psi(x, \lambda) + h(\lambda)) < \inf_{x \in X} \sup_{\lambda \in I} (\Psi(x, \lambda) + h(\lambda)) .$$

Received by the editors: 20.09.2010.
2000 *Mathematics Subject Classification.* 35A15, 35J20, 35J25, 49K35.
Key words and phrases. Nonlocal problem, multiplicity of solutions, minimax inequality.

Then, there exist an open interval $A \subseteq I$ and a positive real number ρ, such that, for each $\lambda \in J$, the equation

$$\Psi'_x(x, \lambda) = 0$$

has at least three solutions in X whose norms are less than ρ. A consequence of Theorem 1.1 is as follows:

Theorem 1.2. *Let X be a separable and reflexive real Banach space; $\Phi : X \to \mathbf{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbf{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact; $I \subseteq \mathbf{R}$ an interval. Assume that*

$$\lim_{\|x\| \to +\infty} (\Phi(x) + \lambda\Psi(x)) = +\infty$$

for all $\lambda \in I$, and that there exists a continuous concave function $h : I \to \mathbf{R}$ such that

$$\sup_{\lambda \in I} \inf_{x \in X} (\Phi(x) + \lambda\Psi(x) + h(\lambda)) < \inf_{x \in X} \sup_{\lambda \in I} (\Phi(x) + \lambda\Psi(x) + h(\lambda)) \ .$$

Then, there exist an open interval $A \subseteq I$ and a positive real number ρ such that, for each $\lambda \in A$, the equation

$$\Phi'(x) + \lambda\Psi'(x) = 0$$

has at least three solutions in X whose norms are less than ρ.

In appraising the literature, it is quite surprising to realize that, while Theorem 1.2 has been proved itself to be one of the most frequently used abstract multiplicity results in the last decade, it seems that there is no article where Theorem 1.1 has been applied to some Ψ which does not depend on λ in an affine way. For an up-dated bibliographical account related to Theorem 1.2, we refer to [3].

The aim of this paper is to offer a first contribution to fill this gap.

To state our results, let us fix some notation.

For a generic function $\psi : X \to \mathbf{R}$, we denote by $\operatorname{osc}_X \psi$ the (possibly infinite) number $\sup_X \psi - \inf_X \psi$.

In the sequel, $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary. We consider the space $H_0^1(\Omega)$ equipped with the norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^2 dx \right)^{\frac{1}{2}} \ .$$

If $I \subseteq \mathbf{R}$ is an interval, with $0 \in I$, and $g : \Omega \times I \to \mathbf{R}$ is a function such that $g(x, \cdot)$ is continuous in I for all $x \in \Omega$, we set

$$G(x, \xi) = \int_0^\xi g(x, t)dt$$

for all $(x, \xi) \in \Omega \times I$.

When $n \geq 2$, we denote by $\mathcal{A}$ the class of all Carathéodory functions $f : \Omega \times \mathbf{R} \to \mathbf{R}$ such that

$$\sup_{(x,\xi)\in\Omega\times\mathbf{R}} \frac{|f(x, \xi)|}{1 + |\xi|^q} < +\infty \ ,$$

for some q with $0 < q < \frac{n+2}{n-2}$ if $n \geq 3$ and $0 < q < +\infty$ if $n = 2$. When $n = 1$, we denote by $\mathcal{A}$ the class of all Carathéodory functions $f : \Omega \times \mathbf{R} \to \mathbf{R}$ such that, for each $r > 0$, the function $x \to \sup_{|t|\leq r} |f(x, t)|$ belongs to $L^1(\Omega)$.

If $f \in \mathcal{A}$, for each $u \in H_0^1(\Omega)$, we set

$$J_f(u) = \int_\Omega F(x, u(x))dx \ .$$

The functional J_f is C^1 and its derivative is compact. Moreover, we set

$$\alpha_f = \inf_{H_0^1(\Omega)} J_f \ ,$$

$$\beta_f = \sup_{H_0^1(\Omega)} J_f$$

and

$$\omega_f = \beta_f - \alpha_f \ .$$

Clearly, when f does not depend on x, we have

$$\alpha_f = \text{meas}(\Omega) \inf_{\mathbf{R}} F$$

and

$$\beta_f = \text{meas}(\Omega) \sup_{\mathbf{R}} F \ .$$

Our main result reads as follows:

Theorem 1.3. *Let $f, g \in \mathcal{A}$ be such that*

$$\sup_{(x,\xi)\in\Omega\times\mathbf{R}} \max\{|F(x, \xi)|, G(x, \xi)\} < +\infty$$

and

$$\sup_{u\in H_0^1(\Omega)} \left| \int_\Omega F(x, u(x))dx \right| > 0 \ .$$

Then, for every pair of continuous and non-decreasing functions $k : [0, +\infty[\to \mathbf{R}$ *and* $h :]-\omega_f, \omega_f[\to \mathbf{R}$*, with* $k(t) > 0$ *for all* $t > 0$ *and* $h^{-1}(0) = \{0\}$*, for which the number*

$$\theta^* = \inf \left\{ \frac{\frac{1}{2}K\left(\int_\Omega |\nabla u(x)|^2 dx\right) - \int_\Omega G(x, u(x))dx}{H\left(\int_\Omega F(x, u(x))dx\right)} : u \in H_0^1(\Omega), \int_\Omega F(x, u(x))dx \neq 0 \right\}$$

is non-negative, and for every $\mu > \theta^*$*, there exist an open interval* $A \subseteq]\alpha_f, \beta_f[$ *and a number* $\rho > 0$ *such that, for every* $\lambda \in A$*, the problem*

$$\begin{cases} -k\left(\displaystyle\int_\Omega |\nabla u(x)|^2 dx\right) \Delta u = \mu h\left(\int_\Omega F(x, u(x))dx - \lambda\right) f(x, u) + g(x, u) & in \quad \Omega \\ u = 0 & on \quad \partial\Omega \end{cases}$$

has at least three weak solutions whose norms in $H_0^1(\Omega)$ *are less than* ρ*.*

Clearly, a weak solution of the above problem problem is any $u \in H_0^1(\Omega)$ such that

$$k\left(\int_\Omega |\nabla u(x)|^2 dx\right) \int_\Omega \nabla u(x) \nabla v(x) dx =$$

$$= \mu h\left(\int_\Omega F(x, u(x))dx - \lambda\right) \int_\Omega f(x, u(x))v(x)dx + \int_\Omega g(x, u(x))v(x)dx$$

for all $v \in H_0^1(\Omega)$.

So, the weak solutions of the problem are exactly the critical points in $H_0^1(\Omega)$ of the functional

$$u \to \frac{1}{2}K(\|u\|^2) - \int_\Omega G(x, u(x))dx - \mu H\left(\int_\Omega F(x, u(x))dx - \lambda\right) .$$

The problem that we are considering is a nonlocal one. We refer to the very recent paper [1] for a relevant discussion and an up-dated bibliography as well.

From what we said above, it is clear that our proof of Theorem 1.3 is based on the use of Theorem 1.1. This is made possible by the following proposition:

Proposition 1.4. *Let* X *be a non-empty set and let* $\gamma : X \to [0, +\infty[$, $J : X \to \mathbf{R}$ *be two functions such that* $\gamma(x_0) = J(x_0) = 0$ *for some* $x_0 \in X$*. Moreover, assume that* J *is bounded and takes at least four values. Finally, let* $\varphi :]-\mathrm{osc}_X J, \mathrm{osc}_X J[\to [0, +\infty[$ *be a continuous function such that*

$$\varphi^{-1}(0) = \{0\} \tag{1.1}$$

and

$$\min\left\{ \liminf_{t \to (-\mathrm{osc}_X J)^+} \varphi(t), \liminf_{t \to (\mathrm{osc}_X J)^-} \varphi(t) \right\} > 0 . \tag{1.2}$$

Put

$$\theta = \inf_{x \in J^{-1}(]\inf_X J, \sup_X J[\setminus\{0\})} \frac{\gamma(x)}{\varphi(J(x))} .$$

Then, for each $\mu > \theta$, we have

$$\sup_{\lambda \in]\inf_X J, \sup_X J[} \inf_{x \in X} (\gamma(x) - \mu\varphi(J(x) - \lambda)) < \inf_{x \in X} \sup_{\lambda \in]\inf_X J, \sup_X J[} (\gamma(x) - \mu\varphi(J(x) - \lambda)) .$$

Proof. First, we make some remarks on the definition of θ. Since J takes at least four values, the set $J^{-1}(]\inf_X J, \sup_X J[\setminus\{0\}$ is non-empty. So, if $x \in J^{-1}(]\inf_X J, \sup_X J[\setminus\{0\}$, we have $J(x) \in]-\operatorname{osc}_X J, \operatorname{osc}_X J[\setminus\{0\}$ (recall that $\inf_X J \leq 0 \leq \sup_X J$), and so $\varphi(J(x)) > 0$. Hence, θ is a well-defined non-negative real number. Now, fix $\mu > \theta$. Since φ is continuous, we have

$$\inf_{\lambda \in]\inf_X J, \sup_X J[} \varphi(J(x) - \lambda) = 0$$

for all $x \in X$. Hence

$$\begin{aligned}\inf_{x \in X} \sup_{\lambda \in]\inf_X J, \sup_X J[} (\gamma(x) - \mu\varphi(J(x) - \lambda)) &= \inf_{x \in X} \left(\gamma(x) - \mu \inf_{\lambda \in]\inf_X J, \sup_X J[} \varphi(J(x) - \lambda)\right) \\ &= \inf_X \gamma = 0 . \qquad (1.3)\end{aligned}$$

Now, since $\mu > \theta$, there is $x_1 \in X$ such that

$$\gamma(x_1) - \mu\varphi(J(x_1)) < 0 .$$

So, again by the continuity of φ, for $\epsilon, \delta > 0$ small enough, we have

$$\gamma(x_1) - \mu\varphi(J(x_1) - \lambda) < -\epsilon \qquad (1.4)$$

for all $\lambda \in [-\delta, \delta]$. On the other hand, (1.1) and (1.2) imply that

$$\nu := \inf_{\lambda \in]\inf_X J, \sup_X J[\setminus[-\delta,\delta]} \varphi(-\lambda) > 0 . \qquad (1.5)$$

From (1.4) and (1.5), recalling that $\gamma(x_0) = J(x_0) = 0$, it clearly follows

$$\sup_{\lambda \in]\inf_X J, \sup_X J[} \inf_{x \in X} (\gamma(x) - \mu\varphi(J(x) - \lambda)) \leq \max\{-\epsilon, -\mu\nu\} < 0$$

and so the conclusion follows in view of (1.3). □

Remark 1.5. It is clear that if a $\varphi :]-\operatorname{osc}_X J, \operatorname{osc}_X J[\to [0, +\infty[$ satisfies (1.1) and is convex, then it is continuous and satisfies (1.2) too.

A joint application of Theorem 1.3 and Proposition 1.4 gives

Theorem 1.6. *Let X be a separable and reflexive real Banach space and let $\eta, J : X \to \mathbf{R}$ be two C^1 functionals with compact derivative and $\eta(0) = J(0) = 0$. Assume also that J is bounded and non-constant, and that η is bounded above.*

Then, for every sequentially weakly lower semicontinuous and coercive C^1 functional $\psi : X \to \mathbf{R}$ whose derivative admits a continuous inverse on X^* and *with* $\psi(0) = 0$*, for every convex* C^1 *function* $\varphi :]-\mathrm{osc}_X J, \mathrm{osc}_X J[\to [0, +\infty[$*, with* $\varphi^{-1}(0) = \{0\}$*, for which the number*

$$\hat{\theta} = \inf_{x \in J^{-1}(\mathbf{R} \setminus \{0\})} \frac{\psi(x) - \eta(x)}{\varphi(J(x))}$$

is non-negative, and for every $\mu > \hat{\theta}$ *there exist an open interval* $A \subseteq]\inf_X J, \sup_X J[$ *and a number* $\rho > 0$ *such that, for each* $\lambda \in A$*, the equation*

$$\psi'(x) = \mu\varphi'(J(x) - \lambda)J'(x) + \eta'(x)$$

has at least three solutions whose norms are less than ρ*.*

Proof. We apply Theorem 1.1 taking $I =]\inf_X J, \sup_X J[$ and

$$\Psi(x, \lambda) = \psi(x) - \eta(x) - \mu\varphi(J(x) - \lambda)$$

for all $(x, \lambda) \in X \times I$.

Clearly, Ψ is C^1 in X, continuous in $X \times I$ and concave in I. By Corollary 41.9 of [4], the functionals η, J are sequentially weakly continuous. Hence, for each $\lambda \in I$, the functional $\Psi(\cdot, \lambda)$ is sequentially weakly lower semicontinuous. Moreover, it is coercive, since ψ is so and $\sup_{x \in X} \max\{|J(x)|, \eta(x)\} < +\infty$. Moreover, it is clear that, for each $\lambda \in I$, the derivative of the functional $\eta(\cdot) + \varphi(J(\cdot) - \lambda)$ is compact (due to the assumptions on η and J and to the fact that φ' is bounded on the compact interval $[\inf_X J, \sup_X J] - \lambda)$, and so, by Example 38.25 of [4], the functional $\Psi(\cdot, \lambda)$ satisfies the Palais-Smale condition. Now, to realize that condition (a_3) is satisfied, we use Remark 1.5 and Proposition 1.4 with $\gamma = \psi - \eta$, observing that $\hat{\theta} = \theta$ since the range of J is an interval. Then, we see that all the assumptions of Theorem 1.1 are satisfied, and the conclusion follows in view of the chain rule. $\square$

It is worth noticing the following consequence of Theorem 1.6:

Theorem 1.7. *Let* X *be a separable and reflexive real Banach space, let* $J : X \to \mathbf{R}$ *be a non-constant bounded* C^1 *functional with compact derivative and* $J(0) = 0$*, and let* $\psi : X \to \mathbf{R}$ *be a sequentially weakly lower semicontinuous and coercive* C^1 *functional whose derivative admits a continuous inverse on* X^* *and with* $\psi(0) = 0$*. Assume that there exists* $\mu > 0$ *such that*

$$\inf_{x \in X}(\psi(x) - \mu(e^{J(x)} - 1)) < 0 \leq \inf_{x \in X}(\psi(x) - \mu J(x)) \ . \tag{1.6}$$

Then, there exist an open interval $A \subseteq]\mu e^{-\sup_X J}, \mu e^{-\inf_X J}[$ *and a number* $\rho > 0$ *such that, for each* $\lambda \in A$*, the equation*

$$\psi'(x) = \lambda e^{J(x)} J'(x)$$

has at least three solutions whose norms are less than ρ*.*

Proof. From (1.6), it clearly follows that

$$0 \leq \inf_{x \in J^{-1}(\mathbf{R} \setminus \{0\})} \frac{\psi(x) - \mu J(x)}{e^{J(x)} - J(x) - 1} < \mu \ .$$

Consequently, we can apply Theorem 1.6 with $\eta = \mu J$ and $\varphi(t) = e^t - t - 1$, so that $\mu > \hat{\theta}$. Then, there exist an open interval $B \subseteq]\inf_X J, \sup_X J[$ and a number ρ such that, for each $\nu \in B$ the equation

$$\psi'(x) = \mu(e^{J(x)-\nu} - 1)J'(x) + \mu J'(x) = \mu e^{-\nu} e^{J(x)} J'(x)$$

has at least three solutions whose norms are less that ρ. Therefore, the conclusion follows taking

$$A = \{\mu e^{-\nu} : \nu \in B\} \ ,$$

and the proof is complete. □

Proof of Theorem 1.3. Let us apply Theorem 1.6 taking

$$X = H_0^1(\Omega) \ ,$$

$$J = J_f \ ,$$

$$\eta = J_g \ ,$$

$$\varphi = H$$

and

$$\psi(u) = \frac{1}{2} K(\|u\|^2)$$

for all $u \in X$.

Since $f, g \in \mathcal{A}$, the functionals J_f, J_g are C^1, with compact derivative. Since K is C^1, increasing and coercive, the functional ψ is sequentially weakly lower semicontinuous, C^1 and coercive. Let us show that ψ' has a continuous inverse on X^* (identified to X, since X is a real Hilbert space). To this end, note that the continuous function $t \to tk(t^2)$ is increasing in $[0, +\infty[$ and onto $[0, +\infty[$. Denote by σ its inverse and consider the operator $T : X \to X$ defined by

$$T(v) = \begin{cases} \dfrac{\sigma(\|v\|)}{\|v\|} v & \text{if} v \neq 0 \\ 0 & \text{if} \qquad v = 0. \end{cases}$$

Since σ is continuous and $\sigma(0) = 0$, the operator T is continuous in X. For each $u \in X \setminus \{0\}$, since $k(\|u\|^2) > 0$, we have

$$T(\psi'(u)) = T(k(\|u\|^2)u) = \frac{\sigma(k(\|u\|^2)\|u\|)}{k(\|u\|^2)\|u\|} k(\|u\|^2)u = \frac{\|u\|}{k(\|u\|^2)\|u\|} k(\|u\|^2)u = u \ ,$$

as desired. Clearly, the assumptions on h imply that φ is non-negative, convex, with $\varphi^{-1}(0) = \{0\}$. So, all the assumptions of Theorem 1.6 are satisfied, and the conclusion follows. $\square$

We conclude pointing out the following sample of application of Theorem 1.3 which is made possible by the fact that h is assumed to have the required properties on $]-\omega_f, \omega_f[$ only.

Example 1.8. Let $f : \mathbf{R} \to \mathbf{R}$ be a non-zero function belonging to $\mathcal{A}$, with $\sup_{\mathbf{R}} |F| < +\infty$ and let $k : [0, +\infty[\to \mathbf{R}$ be a continuous and non-decreasing function, with $k(t) > 0$ for all $t > 0$.

Then, for each μ large enough, there exist an open interval

$$A \subseteq]\mathrm{meas}(\Omega) \inf_{\mathbf{R}} F, \mathrm{meas}(\Omega) \sup_{\mathbf{R}} F[$$

and a number $\rho > 0$ such that, for every $\lambda \in A$, the problem

$$\begin{cases} -k\left(\int_\Omega |\nabla u(x)|^2 dx\right) \Delta u = \mu \dfrac{\int_\Omega F(u(x))dx - \lambda}{(\mathrm{meas}(\Omega)\mathrm{osc}_{\mathbf{R}} F)^2 - \left(\int_\Omega F(u(x))dx - \lambda\right)^2} f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least three weak solutions whose norms in $H^1_0(\Omega)$ are less than ρ.

References

[1] Fan, X. L., *On nonlocal $p(x)$-Laplacian Dirichlet problems*, Nonlinear Anal., **72** (2010), 3314-3323.

[2] Ricceri, B., *On a three critical points theorem*, Arch. Math. (Basel), **75** (2000), 220-226.

[3] Ricceri, B., *Nonlinear eigenvalue problems*, in Handbook of Nonconvex Analysis and Applications, D. Y. Gao and D. Motreanu eds., International Press, to appear.

[4] Zeidler, E., *Nonlinear functional analysis and its applications*, vol. III, Springer-Verlag, 1985.

Department of Mathematics
University of Catania
Viale A. Doria 6
95125 Catania, Italy
E-mail address: `ricceri@dmi.unict.it`