STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 4, December 2010

A NOTE ON THE EXISTENCE OF INFINITELY MANY SOLUTIONS FOR THE ONE DIMENSIONAL PRESCRIBED CURVATURE EQUATION

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Abstract. In the present paper we deal with the one dimensional prescribed curvature equation. We prove, under a suitable oscillatory behaviour at zero of the nonlinearity, the existence of infinitely many solutions. Our approach combines variational techniques with classical regularity results.

1. **Introduction**

In the present paper we deal with the one dimensional prescribed curvature problem

$$(P) \qquad \begin{cases} -\left(\dfrac{u'}{\sqrt{1+u'^2}}\right)' = h(t)f(u) & \text{in }]0,1[\\ \\ u(0)=u(1)=0 \end{cases}$$

where $h:[0,1]\to\mathbb{R}$ is a positive bounded function with $\operatorname{ess\,inf}_{[0,1]} h > 0$, $f:\mathbb{R}\to\mathbb{R}$ is a continuous function. The problem of existence and multiplicity results for such problem is one of the most investigated issue in calculus of variations and differential geometry. We focus here on the existence of infinitely many solutions in the same spirit of some recent papers of Obersnel and Omari who studied the problem under different sets of assumptions on the nonlinearity f.

A sequence of weak solutions (tending in the C^1 norm to zero) has been obtained in [3] in any space dimension N via the Lusternik-Schnirelmann theory, provided the nonlinearity is odd and its primitive is subquadratic at zero. The same thesis for the one dimensional autonomous equation has been achieved in [2] via the analysis of some generalized Fučík spectrum under different behaviour of the

Received by the editors: 20.09.2010.
2000 *Mathematics Subject Classification.* 34B15, 49Q20.
Key words and phrases. One dimensional prescribed curvature problem, infinitely many solutions, oscillatory nonlinearity.

nonlinearity. Namely if f is superlinear or sublinear at zero and satisfies, with its primitive, suitable conditions at $+\infty$ and $-\infty$, the authors proved the existence of infinitely many solutions which are possibly discontinuous at the points where they attain the value zero. Finally we mention the paper [4], where the method of sub and super solutions guarantees, in any space dimension N, the existence of a sequence of weak solutions tending in the C^1 norm to zero.

We will prove the existence of infinitely many solutions for the one dimensional prescribed curvature problem under suitable oscillatory assumptions at zero on the nonlinearity f. We propose a new approach without requiring symmetry or conditions at $+\infty$.

Following the variational approach of [3], we will apply a variational principle by Ricceri [5] to an elliptic regularized problem to obtain a sequence of pairwise distinct critical points for the energy functional associated and subsequently, by the means of classical regularity results, we will achieve the existence of infinitely many solutions for the original problem.

Throughout the sequel by a solution of (P) we mean a *weak solution*, that is a function $u \in W_0^{1,2}(]0,1[)$ such that

$$\int_0^1 \frac{u'(t)}{\sqrt{1+u(t)'^2}} v'(t)dt - \int_0^1 h(t)f(u(t))v(t)dt = 0$$

for every $v \in W_0^{1,2}(]0,1[)$.

Our main result is

Theorem 1.1. *Assume that*

i) there exist two sequences $\{a_k\}$ *and* $\{b_k\}$ *in* $]0,\infty[$ *with* $b_{k+1} < a_k < b_k$, $\lim_{k\to\infty} b_k = 0$ *and* $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$ *such that* $f(s) \leq 0$ *for every* $s \in [a_k, b_k]$;

ii) if $F(s) = \int_0^s f(t)dt$, *then*

$$-\infty < \liminf_{s\to 0^+} \frac{F(s)}{s^2} \leq \limsup_{s\to 0^+} \frac{F(s)}{s^2} = +\infty;$$

iii) $\limsup_{k\to\infty} \frac{\max_{[0,a_k]} F}{b_k^2} < \frac{7\sqrt{2}}{32} \frac{1}{\|h\|_{L^1(]0,1[)}}$.

Then, problem (P) *admits a sequence of non negative weak solutions* $\{u_k\} \subseteq C^1([0,1])$ *which satisfy* $\lim_{k\to\infty} \|u_k\|_{C^1([0,1])} = 0$.

2. **Proof of Theorem 1.1**

2.1. **Preliminaries.** Our main tool is the following variational principle by Ricceri which is a consequence of a more general result.

Theorem 2.1. *([5], Theorem 2.5) Let X be a Hilbert space, $\Phi, \Psi : X \to \mathbb{R}$ two sequentially weakly lower semicontinuous, continuously Gâteaux differentiable functionals. Assume that Ψ is strongly continuous and coercive. For each $\rho > \inf_X \Psi$, set*

$$\varphi(\rho) := \inf_{\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(u) - \inf_{\overline{\Psi^{-1}(]-\infty,\rho[)}^w} \Phi}{\rho - \Psi(u)}, \tag{2.1}$$

where $\Psi^{-1}(]-\infty,\rho[) := \{u \in X : \Psi(u) < \rho\}$ and $\overline{\Psi^{-1}(]-\infty,\rho[)}^w$ is its closure in the weak topology of X. Furthermore, set

$$\delta := \liminf_{\rho \to (\inf_X \Psi)^+} \varphi(\rho). \tag{2.2}$$

If $\delta < +\infty$ then, for every $\lambda > \delta$, either $\Phi + \lambda\Psi$ possesses a local minimum, which is also a global minimum of Ψ, or there is a sequence $\{u_k\}$ of pairwise distinct critical points of $\Phi + \lambda\Psi$, with $\lim_{k\to\infty} \Psi(u_k) = \inf_X \Psi$, weakly converging to a global minimum of Ψ.

2.2. **Proof.** In the present section we will give the proof of Theorem 1.1. Following an idea of Obersnel and Omari in [3], we apply Theorem 2.1 to a modified problem and then, by the means of a regularity result by Lieberman (see [1]) we prove that the critical points of the energy are actually solutions of the original problem.

We split the proof in several steps.

Step 1. A modified problem.

Notice first that assumptions $i)$ and $ii)$ imply that $f(0) = 0$. We truncate f as follows:

$$g(s) = \begin{cases} 0 & s < 0 \\ f(s) & 0 \le s < b_1 \\ f(b_1) & s \ge b_1 \end{cases}$$

where b_1 is from assumption (i). The function g is continuous and if $G : \mathbb{R} \to \mathbb{R}$ denotes its primitive, that is $G(s) = \int_0^s g(t)dt$, g and G satisfy the assumptions $(i) - (iii)$ of Theorem 1.1. Define also $a : [0, +\infty[\to]0, +\infty[$ by

$$a(s) = \begin{cases} \dfrac{1}{\sqrt{1+s}} & 0 \le s < 1 \\ \dfrac{\sqrt{2}}{16}(s-2)^2 + \dfrac{7\sqrt{2}}{16} & 1 \le s < 2 \\ \dfrac{7\sqrt{2}}{16} & s \ge 2. \end{cases}$$

The function a is of class $C^{1,1}([0,+\infty[)$ and, for every $s \geq 0$, satisfies $\frac{7\sqrt{2}}{16} \leq a(s) \leq 1$. Denote by A its primitive, that is $A(s) = \int_0^s a(t)dt$, verifying then

$$\frac{7\sqrt{2}}{16} s \leq A(s) \leq s. \tag{2.3}$$

We introduce now the auxiliary problem

$$(P') \qquad \begin{cases} -\left(a(|u'|^2)u'\right)' = h(t)g(u) & \text{in }]0,1[\\ \\ u(0) = u(1) = 0 \end{cases}$$

Denote by X the space $W_0^{1,2}(]0,1[)$, endowed with the norm $\|u\| = \left(\int_0^1 |u'(t)|^2 dt\right)^{1/2}$. It is well known that the space X is compactly embedded into $C^0([0,1])$ and $\|u\|_\infty \leq \|u\|$ where $\|u\|_\infty = \max_{t\in[0,1]} |u(t)|$. Let Ψ and $\Phi : X \to \mathbb{R}$ be the functionals defined by

$$\Psi(u) = \frac{1}{2}\int_0^1 A(|u'(t)|^2)dt, \quad \Phi(u) = -\int_0^1 h(t)G(u(t))dt, \quad u \in X.$$

Due to (2.3), Ψ is well defined on X, continuous and coercive. Moreover, by the convexity of the function $s \to A(s^2)$ in $\mathbb{R}$, Ψ is convex and then sequentially weakly lower semicontinuous. The functional Φ is well defined and sequentially weakly continuous. Moreover Ψ and Φ are continuously Gâteaux differentiable with derivative given by

$$\Psi'(u)(v) = \int_0^1 a(|u'(t)|^2)u'(t)v'(t)dt, \qquad \Phi'(u)(v) = -\int_0^1 h(t)g(u(t))v(t)dt,$$

for every $u, v \in X$. With these assumptions, the function φ from (2.1) reads as follows:

$$\varphi(\rho) = \inf_{\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(u) - \inf_{\Psi^{-1}(]-\infty,\rho])} \Phi}{\rho - \Psi(u)},$$

where $\Psi^{-1}(]-\infty,\rho]) = \{u \in X \ : \ \Psi(u) \leq \rho\}$.

Step 2. We claim that $\delta < 1$.

Recall that $\delta := \liminf_{\rho\to 0^+} \varphi(\rho)$ and clearly $\delta \geq 0$.

Notice that from $G(s) = 0$ for every $s \leq 0$ and *i)* it follows that

$$\max_{[-b_k,b_k]} G = \max_{[0,b_k]} G = \max_{[0,a_k]} G.$$

Let $\bar{s}_k \in [0,a_k]$ such that $G(\bar{s}_k) = \max_{[-b_k,b_k]} G$ and denote by $s_k = \dfrac{7\sqrt{2}b_k^2}{32}$.

We have that

$$\Psi^{-1}(]-\infty,s_k]) \subseteq \{v \in X : \|v\|_\infty \leq b_k\}.$$

Indeed, if $v \in X$ is such that $\Psi(v) \le s_k$, then by (2.3) we have

$$\frac{7\sqrt{2}}{32}\|v\|^2 \le \Psi(v) \le s_k,$$

and clearly

$$\|v\|_\infty^2 \le b_k^2,$$

which is our claim. Hence,

$$\sup_{\Psi^{-1}(]-\infty,s_k])}(-\Phi(v)) \le \max_{[-b_k,b_k]} G \int_0^1 h(t)dt = G(\bar{s}_k)\|h\|_{L^1(]0,1[)}. \tag{2.4}$$

By assumption (ii), $\liminf_{s\to 0^+} \frac{G(s)}{s^2} > -\infty$. It follows the existence of $\underline{M} > 0$ and $\tau \in]0, b_1[$ such that

$$G(s) > -\underline{M}s^2 \quad \text{for every } s \in]0,\tau[. \tag{2.5}$$

Choose now l such that

$$\limsup_{k\to\infty} \frac{\max_{[0,a_k]} G}{b_k^2} < l < \frac{7\sqrt{2}}{32}\frac{1}{\|h\|_{L^1(]0,1[)}}.$$

By $i)$ and $iii)$, for k big enough,

$$\frac{\max_{[0,a_k]} G}{b_k^2}\|h\|_{L^1(]0,1[)} + \left(\frac{1}{2}\underline{M}\|h\|_\infty + \frac{64\sqrt{2}}{7} l\|h\|_{L^1(]0,1[)}\right)\frac{a_k^2}{b_k^2} < l\|h\|_{L^1(]0,1[)}$$

which implies, as $\bar{s}_k \le a_k$

$$\frac{G(\bar{s}_k)}{s_k}\|h\|_{L^1(]0,1[)} + \left(\frac{1}{2}\underline{M}\|h\|_\infty + \frac{64\sqrt{2}}{7} l\|h\|_{L^1(]0,1[)}\right)\frac{\bar{s}_k^2}{s_k} < l\|h\|_{L^1(]0,1[)}. \tag{2.6}$$

Define

$$w_{\bar{s}_k}(t) = \begin{cases} 4\bar{s}_k t, & \text{if} \quad 0 \le t < \frac{1}{4} \\ \bar{s}_k, & \text{if} \quad \frac{1}{4} \le t < \frac{3}{4} \\ 4\bar{s}_k(1-t), & \text{if} \quad \frac{3}{4} \le t \le 1. \end{cases}$$

Clearly $w_{\bar{s}_k} \in X$ and

$$\Psi(w_{\bar{s}_k}) \le \frac{1}{2}\|w_{\bar{s}_k}\|^2 = 4\bar{s}_k^2. \tag{2.7}$$

So, using the definition of $w_{\bar{s}_k}$,

$$\begin{aligned} -\Phi(w_{\bar{s}_k}) &= \int_0^{1/4} h(t)G(w_{\bar{s}_k})dt + \int_{1/4}^{3/4} h(t)G(\bar{s}_k)dt + \int_{3/4}^1 h(t)G(w_{\bar{s}_k})dt \\ &> -\frac{\underline{M}\|h\|_\infty}{4}\bar{s}_k^2 + \frac{h_0}{2}G(\bar{s}_k) - \frac{\underline{M}\|h\|_\infty}{4}\bar{s}_k^2 \\ &> -\frac{\underline{M}\|h\|_\infty}{2}\bar{s}_k^2. \end{aligned} \tag{2.8}$$

where $h_0 = \operatorname{ess\,inf}_{[0,1]} h$.

Putting together (2.4), (2.8), (2.6) and (2.7) we obtain

$$\begin{aligned}\sup_{\Psi-1(]-\infty,s_k])}(-\Phi(v))+\Phi(w_{\bar{s}_k}) &\leq G(\bar{s}_k)\|h\|_{L^1([0,1])}+\frac{M\|h\|_\infty}{2}\bar{s}_k^2\\ &< \frac{16\sqrt{2}}{7}l\|h\|_{L^1([0,1])}(s_k-4\bar{s}_k^2)\\ &\leq \frac{16\sqrt{2}}{7}l\|h\|_{L^1([0,1])}(s_k-\Psi(w_{\bar{s}_k})).\end{aligned}$$

Since $s_k \to 0$ as $k \to \infty$, we get

$$\delta \leq \liminf_k \frac{\Phi(w_{\bar{s}_k})-\inf_{\Psi^{-1}(]-\infty,s_k])}\Phi(v)}{s_k-\Psi(w_{\bar{s}_k})} \leq \frac{16\sqrt{2}}{7}l\|h\|_{L^1([0,1])} < 1.$$

Step 3. 0 is not a local minimum of $\Psi+\Phi$.

We will construct a sequence of functions in X tending in norm to zero where the energy attains negative value. By assumption (ii), $\limsup_{s\to 0^+}\frac{G(s)}{s^2}=+\infty$ and so if $\overline{M}>0$ is such that

$$\overline{M} > \frac{8+\|h\|_\infty \underline{M}}{h_0}, \tag{2.9}$$

(where $\underline{M}$ is as in Step 2), there exists a sequence $\{\tilde{s}_k\} \subset]0,\tau[$ converging to zero such that

$$G(\tilde{s}_k) > \overline{M}\tilde{s}_k^2. \tag{2.10}$$

Let $w_{\tilde{s}_k}$ defined as

$$w_{\tilde{s}_k}(t)=\begin{cases}4\tilde{s}_k t, & \text{if} \quad 0\leq t<\frac{1}{4}\\ \tilde{s}_k, & \text{if} \quad \frac{1}{4}\leq t<\frac{3}{4}\\ 4\tilde{s}_k(1-t), & \text{if} \quad \frac{3}{4}\leq t\leq 1.\end{cases}$$

It is clear that $\|w_{\tilde{s}_k}\| \to 0$ as $k \to \infty$. Let us prove that $\Psi(w_{\tilde{s}_k})+\Phi(w_{\tilde{s}_k})<0$. Indeed, by (2.5) and (2.10) we have

$$\begin{aligned}\Psi(w_{\tilde{s}_k})+\Phi(w_{\tilde{s}_k}) &\leq 4\tilde{s}_k^2-\int_0^{1/4}h(t)G(w_{\tilde{s}_k})dt-\int_{1/4}^{3/4}h(t)G(\tilde{s}_k)dt-\int_{3/4}^{1}h(t)G(w_{\tilde{s}_k})dt\\ &\leq 4\tilde{s}_k^2+\frac{\underline{M}\|h\|_\infty}{4}\tilde{s}_k^2-\frac{\overline{M}h_0}{2}\tilde{s}_k^2+\frac{\underline{M}\|h\|_\infty}{4}\tilde{s}_k^2\\ &= \tilde{s}_k^2\left(4+\frac{\underline{M}\|h\|_\infty}{2}-\frac{\overline{M}h_0}{2}\right)\\ &< 0=\Psi(0)+\Phi(0).\end{aligned}$$

Our claim is achieved.

Step 4. Existence of a sequence of critical points for $\Psi+\Phi$.

We apply Theorem 2.1 to the functionals Ψ and Φ with $\lambda = 1$. One has that 0 is the global minimum of Ψ and by Step 3 is not a local minimum of $\Psi + \Phi$, hence there exists a sequence $\{u_k\}$ of pairwise distinct critical points of the energy such that $\lim_{k\to\infty} \Psi(u_k) = 0$. In particular,

$$\lim_{k\to\infty} \|u_k\|_\infty = 0. \tag{2.11}$$

Let us prove that the critical points of the energy are non negative. Assume that u is a critical point of $\Psi + \Phi$ and that the set $C = \{t \in [0,1] : u(t) < 0\}$ is non empty, i.e. has a positive measure. Then, the function $v = \min\{0, u\}$ still belongs to X and

$$\begin{aligned} 0 &= \Psi'(u)(v) + \Phi'(u)(v) = \int_0^1 a(|u'(t)|^2)u'(t)v'(t)dt - \int_0^1 h(t)g(u(t))v(t)dt \\ &= \int_0^1 a(|u'(t)|^2)u'(t)^2 dt \end{aligned}$$

which implies $u = 0$, a contradiction. Hence, by using (2.11), for k big enough, $0 \le u_k(t) \le b_1$ for every $t \in [0,1]$.

Step 5. Proof concluded.

If u_k is a critical point of $\Psi + \Phi$, then it is a weak solution of the auxiliary problem (P'), it is non negative and bounded from above by b_1, as proven in Step 4.

We are going to prove now that for k big enough, $\|u'_k\|_\infty \le 1$.

From [1], there exists $\alpha \in]0,1]$ and $c > 0$ such that $u_k \in C^{1,\alpha}([0,1])$ and

$$\|u_k\|_{C^{1,\alpha}([0,1])} \le c \qquad \text{for every } k \in \mathbb{N}. \tag{2.12}$$

Let us prove now that

$$\lim_{k\to\infty} \|u_k\|_{C^1([0,1])} = 0.$$

Indeed assume by contradiction that there exists a sequence $\{u_{k_h}\}$ such that $\lim_{h\to\infty} \|u_{k_h}\|_{C^1([0,1])} > 0$. Then, since $\lim_{k\to\infty} \|u_k\|_\infty = 0$ it must be

$$\lim_{h\to\infty} \|u'_{k_h}\|_\infty > 0. \tag{2.13}$$

From Ascoli Arzela' Theorem, there exists a subsequence still denoted by $\{u_{k_h}\}$ such that $\{u'_{k_h}\}$ is uniformly convergent to zero, in contradiction with (2.13).

In particular, for k big enough, we have that $\|u_k\|_{C^1([0,1])} \le 1$ and this implies at once that u_k is a weak solution of the original problem.

Remark 2.2. It is still an open question whether Theorem 1.1 is valid without assumption (iii).

Remark 2.3. We point out that our method works when the space dimension N is equal to 1. Indeed in our application of Theorem 2.1 it is crucial to embed the space X in $C^0([0,1])$. Notice however that the variational principle by Ricceri is valid for every space dimension, but it is still an open question how to apply it when there is no embedding into the space of continuous functions.

We conclude this note with an example of application of Theorem 1.1.

Example 2.4. Let $a_k = \frac{1}{k!k}$ and $b_k = \frac{1}{k!}$. Choose a constant $l \in]0, \frac{7\sqrt{2}}{32}[$.

Define $f : \mathbb{R} \to \mathbb{R}$ as follows

$$f(s) = \begin{cases} 4l(b_k^2 - b_{k+1}^2)\dfrac{(s-b_{k+1})}{(a_k - b_{k+1})^2}, & \text{if} \quad b_{k+1} \leq s \leq \frac{a_k+b_{k+1}}{2} \\ 4l(b_k^2 - b_{k+1}^2)\dfrac{(a_k - s)}{(a_k - b_{k+1})^2}, & \text{if} \quad \frac{a_k+b_{k+1}}{2} \leq s \leq a_k \\ 0, & \text{otherwise} \end{cases}$$

The function f is continuous and satisfies all the assumptions of Theorem 1.1. In particular, $\liminf_{s \to 0^+} \dfrac{F(s)}{s^2} = l$. Then, problem

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1+u'^2}}\right)' = f(u) & \text{in }]0,1[\\ \\ u(0) = u(1) = 0 \end{cases}$$

admits a sequence of non-negative weak solutions tending to zero in the C^1 norm.

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